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p-Laplace Equation in the Heisenberg Group

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Preface

This work is based on my master's thesis from the University of Bologna, written under the supervision of my advisors Juan Manfredi and Bruno Franchi, and is intended to present a self-contained introduction to the p-Laplace equation and related regularity theory in the Heisenberg group. We also obtain new regularity results in the nondegenerate case $1 < p < 2$. I thank Drs. Franchi and Manfredi for encouraging me to prepare this manuscript and to submit it to the BCAM Springer Briefs.

The regularity theory of nonlinear elliptic equations is quite well understood in the Euclidean case, but in many problems in science and engineering the most natural setup is subelliptic. This is the case in non-holonomic mechanics, nonlinear elasticity, robotic control theory, and certain models in the neurobiology of vision, to name but a few examples. In the subelliptic case the velocity fields are restricted to reflect non-holonomic constraints, and this leads to the study of non-commuting vector fields generating nilpotent Lie algebras. The simplest, yet very important, example is the Heisenberg group, which we describe in Chap. 2. The complications resulting from the lack of commutativity of the primary vector fields generate new challenges in regularity theory, especially in the nonlinear case but also in the linear case, where the key result was proved by Hormander.

When we minimize non-quadratic energy functionals, the resulting Euler equations are quasilinear, of p-Laplacian type, as described in Chap. 3. The expected regularity would be that solutions have Holder continuous derivatives. In Chap. 5 we present Zhong's Lipschitz continuity results for p-harmonic functions in the full range $1 < p < \infty$. In the nondegenerate case, where vanishing gradients do not present a difficulty, we would expect C^∞ regularity. This was known for the case $p \geq 2$. Our main new result presented in this manuscript is a proof of this fact valid for the full range $1 < p < \infty$.

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Contents

Acronyms

∇	Euclidean gradient
∇^2	Euclidean Hessian
$\|\cdot\|_E$	Euclidean norm
$\langle \cdot \rangle$	Euclidean scalar product
$B_r^E(x)$	Euclidean ball of radius r and center x
$\mathbb{H}$	Heisenberg group
$\mathfrak{g}$	Lie algebra associated to $\mathbb{H}$
$C_0^\infty(\Omega)$	Smooth functions with compact support in Ω
$\nabla_{\mathbb{H}}$	Horizontal gradient
$\mathrm{div}_{\mathbb{H}}$	Horizontal divergence
$\nabla_{\mathbb{H}}^2$	Horizontal Hessian
$HW^{1,p}(\Omega)$	Horizontal Sobolev space
$HW_{loc}^{1,p}(\Omega)$	Local horizontal Sobolev space
$HW_0^{1,p}(\Omega)$	Closure of $C_0^\infty(\Omega)$ in the horizontal Sobolev norm
d_{cc}, d	Carnot–Carathéodory distance
$B_r(x)$	Carnot–Carathéodory ball of radius r and center x
$\Gamma^\alpha(\Omega)$	Hölder functions with respect to d_{cc}
$\|\cdot\|_K$	Korányi norm
supp	Support of a function
$\Omega_r(x)$	$= \Omega \cap B_r(x)$
$M^{p,\lambda}(\Omega)$	Morrey space
$\mathcal{L}^{p,\lambda}(\Omega)$	Campanato space
$\mathcal{D}_p(u)$	p-Dirichlet functional
$\fint_\Omega$	Integral average over Ω
$\mathrm{Lie}(X)$	Lie algebra generated by the set of vector fields X