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Lluís Alsedà i Soler
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Difference Equations, Discrete Dynamical Systems and Applications

ICDEA, Barcelona, Spain, July 2012

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Group picture of the participants of the ICDEA 2012 on the main stairs of Casa de Convalescència (venue of the meeting)

Preface

This volume contains the proceedings of the 18th International Conference on Difference Equations and Applications held in Barcelona (Catalonia, Spain) from July 23rd to 27th, 2012. The conference was organized by the Departament de Matemàtiques of the Universitat Autònoma de Barcelona (UAB), under the auspices of the International Society of Difference Equations (ISDE).

The purpose of the conference was to bring together experts and novices in the theory and applications of difference equations and discrete dynamical systems. The main theme of the meeting was the interplay between difference equations and dynamical systems.

The plenary speakers were experts chosen from many different areas of difference equations, *broadly defined*, and discrete dynamical systems and their interplay with nonlinear science.

There were 129 presentations which included 14 plenary talks (including the Special talk of the winner of the "Best JDEA paper" prize), 39 contributed talks, and 87 special sessions talks. The main topics of the meeting were represented by the six special sessions which were organized during the conference. They cover the theory of difference equations and discrete dynamical systems and their applications to biology and economics and are described below.

- *Combinatorial and Topological Dynamics* (14 talks organized by S. Kolyada and L'. Snoha);
- *Complex Dynamics* (23 talks organized by B. Devaney, N. Fagella and X. Jarque);
- *Applications of Difference Equations to Biology* (10 talks organized by J. Cushing, S. Elaydi and J. Li);
- *Asymptotic Behavior and Periodicity of Difference Equations* (20 talks organized by I. Győri and M. Pituk);
- *Chaotic Linear Dynamics* (9 talks organized by J.P. Bès, P. Oprocha and A. Peris); and
- *Economic Dynamics and Control* (11 talks organized by A.A. Pinto and A. Yannacopoulos).

There were 157 participants (120 male and 37 female) from Austria, Belarus, Belgium, Bosnia-Herzegovina, Brazil, Canada, Chile, Czech Republic, Denmark, Finland, France, Germany, Greece, Hungary, India, Iran, Ireland, Israel, Italy, Jamaica, Japan, Latvia, Mexico, Oman, Poland, Portugal, Russia, Serbia, Slovakia, Spain, Sweden, Ukraine, United Kingdom, and USA.

We would like to acknowledge the financial support of the following institutions: Centro Internacional de Matemática, Institut de Matemàtiques de la Universitat de Barcelona, Centre de Recerca Matemàtica, Ministerio de Economía y Competitividad, Generalitat de Catalunya, Grup de Sistemes Dinàmics de la Universitat Autònoma de Barcelona. Also, we would like to thank the organizing committee that ensured a good organization and the success of the conference as well as the scientific committee that took care of the high scientific standards and the quality of the conference.

We warmly thank all the speakers and participants of the meeting for their contributions and helping to create a wonderful, friendly, and fruitful atmosphere.

All participants of ICDEA2012 were invited to submit a contribution to these proceedings, and all papers that were accepted had to pass through a refereeing process appropriate for a mathematical research journal.

Cerdanyola del Vallès, Spain — Lluís Alsedà i Soler
Tucson, USA — Jim M. Cushing
San Antonio, USA — Saber Elaydi
Porto, Portugal — Alberto A. Pinto

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Contents

On the Second Order Rational Difference Equation $x_{n+1} = \beta + \frac{1}{x_n x_{n-1}}$

Aija Anisimova

Abstract The author investigates the local and global stability character, the periodic nature, and the boundedness of solutions of the second-order rational difference equation

$$x_{n+1} = \beta + \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, \ldots,$$

with parameter β and with arbitrary initial conditions such that the denominator is always positive. The main goal of the paper is to confirm Conjecture 8.1 and to solve Open Problem 8.2 stated by A.M. Amleh, E. Camouzis and G. Ladas in *On the Dynamics of a Rational Difference Equations I* (International Journal of Difference Equations, Volume 3, Number 1, 2008, pp.1–35).

Keywords Boundedness · Periodicity · Rational difference equations · Stability

AMS Subject Classifications 39A10 · 39A20 · 39A30

1 Introduction and Preliminaries

The author investigates the local and global stability character, the periodic nature, and the boundedness of solutions of the second-order rational difference equation in the form

$$x_{n+1} = \beta + \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1}$$

with parameter β and with arbitrary initial conditions such that the denominator is always positive.

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In the paper [1] have been given several open problems and conjectures about such equations:

Conjecture 8.1 ([1]) *Every positive solution of (1) has a finite limit.*

Open Problem 8.2 ([1]) Assume that β is a real number. Determine the set G of real initial values x_{-1}, x_0 for which the equation (1) is well defined for all $n \geq 0$, and investigate the character of solutions of (1) with $x_{-1}, x_0 \in G$.

In this paper the author investigates the local and global stability character of the difference equation (1), describes the periodic nature of the difference equation (1), investigates the boundedness of solutions of (1) and poses some ideas how to confirm Conjecture 8.1 and to solve Open Problem 8.2.

Equation (1) is a special case of the second-order quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, 2, \ldots \tag{2}$$

with non-negative parameters and with arbitrary non-negative initial conditions such that the denominator is always positive.

Related non-linear second order rational difference equations have been investigated in [1, 2, 5–9].

Now we give well-known results, which will be useful in the investigation of (1).

For the next results, we consider the difference equation defined by

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \tag{3}$$

Let I be some interval of real numbers and let

$$f : I \times I \to I$$

be a continuously differentiable function.

Then for every set of initial conditions $x_{-1}, x_0 \in I$, the difference equation (3) has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of (3) if

$$\overline{x} = f(\overline{x}, \overline{x})$$

that is,

$$x_n = \overline{x}, \quad \forall n \geq 0$$

is a solution of (3), or equivalently, $\overline{x}$ is a fixed point of f.

Let $Dom(f)$ be a domain of function f of (1). The Forbidden set (denote with F) of function f is a set such that:

$$F := \{(c, d) \in R^2 : \quad \exists n \in N : (x_0, x_{-1}) = (c, d), (x_k, x_{k-1}) \in Dom(f) \quad \forall k = 0, 1 \ldots, n,$$

and $\quad (x_n, x_{n+1}) \notin Dom(f)\}$.

Let $p = \frac{\partial f}{\partial u}(\overline{x}, \overline{x})$ and $q = \frac{\partial f}{\partial v}(\overline{x}, \overline{x})$ denote the partial derivatives of $f(u, v)$ evaluated at the equilibrium $\overline{x}$ of (3). Then the equation

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, \ldots \tag{4}$$

is called the linearized equation associated with (3) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^2 - p\lambda - q = 0 \tag{5}$$

called the characteristic equation of (4) about $\overline{x}$.

Theorem 1 ([7])

1. *If both roots of quadratic equation (5) lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of (3) is locally asymptotically stable.*
2. *If at least one of the roots of (5) has absolute value greater than one, then the equilibrium $\overline{x}$ of (3) is unstable.*
3. *A necessary and sufficient condition for both roots of (5) to lie in the open unit disk $|\lambda| < 1$, is*
$$|p| < 1 - q < 2. \tag{6}$$

In the next theorem we make use of the following notation associated with a function $f(z_1, z_2)$, which is monotonic in both arguments. For each pair of numbers (m, M) and for each $i \in \{1, 2\}$, define

$$M_i(m, M) = \begin{cases} M, & \text{if } f \text{ is increasing in } z_i \\ m, & \text{if } f \text{ is decreasing in } z_i \end{cases}$$

and

$$m_i(m, M) = M_i(M, m).$$

Theorem 2 ([4, 7]) *Let $[a, b]$ be a closed and bounded interval of real numbers and let $f \in C([a, b]^2, [a, b])$ satisfy the following conditions:*

1. *$f(z_1, z_2)$ is monotonic in each of its arguments.*
2. *If (m,M) is a solution of the system*

$$\begin{cases} M = f(M_1(m, M), M_2(m, M)) \\ m = f(m_1(m, M), m_2(m, M)) \end{cases}$$

then $M = m$.

Then the difference equation (3) has a unique equilibrium point $\overline{x} \in [a, b]$ *and every solution of (3), with initial conditions in* $[a, b]$*, converges to* $\overline{x}$.

A solution $\{x_n\}_{n=-1}^{\infty}$ of (3) is said to be periodic with period p if

$$x_{n+p} = x_n \quad for \quad all \quad n \geq -1. \tag{7}$$

A solution $\{x_n\}_{n=-1}^{\infty}$ of (3) is said to be periodic with prime period p, or a p-cycle if it is periodic with period p and p is the least positive integer for which (7) holds.

With the change of variables

$$x_n = \frac{1}{y_n\sqrt{\beta}} \tag{8}$$

difference equation (1) can be transformed to the difference equation in the form

$$y_{n+1} = \frac{\alpha}{1 + y_n y_{n-1}}, \quad n = 0, 1, \ldots, \tag{9}$$

where

$$x_n = \frac{1}{y_n\sqrt{\beta}}, \quad x_{n-1} = \frac{1}{y_{n-1}\sqrt{\beta}}, \quad x_{n+1} = \frac{1}{y_{n+1}\sqrt{\beta}}$$

and by these equalities we obtain that

$$\frac{1}{y_{n+1}\sqrt{\beta}} = \beta + y_n\sqrt{\beta}y_{n-1}\sqrt{\beta} \Rightarrow$$

$$\Rightarrow y_{n+1}\sqrt{\beta} = \frac{1}{\beta(1 + y_n y_{n-1})} \Rightarrow y_{n+1} = \frac{\frac{1}{\beta\sqrt{\beta}}}{1 + y_n y_{n-1}},$$

where

$$\alpha = \frac{1}{\beta\sqrt{\beta}}. \tag{10}$$

Equality (10) and transformation to the difference equation (9) are true for all $\beta > 0$ (or $\alpha > 0$).

In paper [1] had been proved that:

1. Every solution of (9) is bounded by positive constants, precisely

$$\frac{\alpha}{1 + \alpha^2} \leq y_{n+1} = \frac{\alpha}{1 + y_n y_{n-1}} \leq \alpha, \quad \forall n \geq 1; \tag{11}$$

2. Assume that

$$0 < \alpha \leq 2. \tag{12}$$

Then the positive equilibrium of (9) is globally asymptotically stable.
These characteristics of (9) can be useful in the investigation of (1).

2 Boundedness

In this section we investigate the boundedness of (1).

Theorem 3 *Every positive solution of (1) is bounded from above and below by positive constants.*

Proof Obviously we can estimate equation (1) from below by

$$x_{n+1} = \beta + \frac{1}{x_n x_{n-1}} \geq \beta \quad \forall n \geq 1.$$

Set that $\beta \leq x_n$ and in equation (1) replacing it with a smaller value, the fraction increases and we get the estimation from above

$$x_{n+1} = \beta + \frac{1}{x_n x_{n-1}} \leq \beta + \frac{1}{\beta^2}, \quad \forall n \geq 1.$$

Finally we obtain

$$\beta \leq x_{n+1} = \beta + \frac{1}{x_n x_{n-1}} \leq \beta + \frac{1}{\beta^2}, \quad \forall n \geq 1. \quad \square \tag{13}$$

3 Local and Global Asymptotic Stability

In this section we investigate the local and global stability of the solutions of the difference equation (1) where the parameter β and the initial values x_{-1}, x_0 are arbitrary real numbers, such that denominator is not equal to zero.

Assume $x_{-1}, x_0 \in I$, where I is the set of all real initial values x_{-1}, x_0 for which the difference equation (1) is well defined for all $n \geq 0$.

The Equilibrium equation of (1)

$$\overline{x}^3 - \beta\overline{x}^2 - 1 = 0 \tag{14}$$

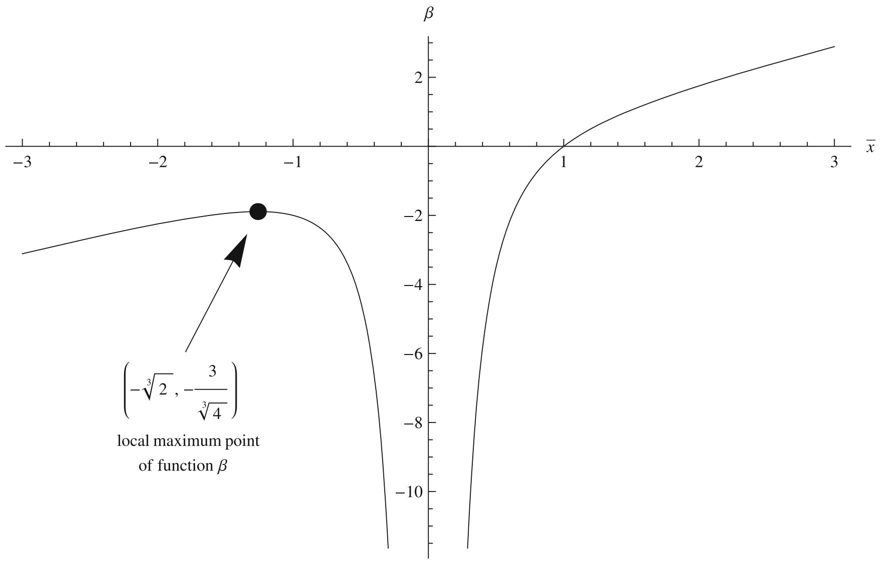

Fig. 1 Graph of function β

can be easily considered as a function of $\overline{x}$:

$$\beta = \frac{\overline{x}^3 - 1}{\overline{x}^2}. \tag{15}$$

Equilibrium equation (14) has three roots, the real roots can be obtained from the graph of the function (15) (see Fig. 1). From Fig. 1 we see that the parameter β is related to the equilibrium point and the function β has one local maximum point $(-\sqrt[3]{2} \approx -1.26; -\frac{3}{\sqrt[3]{4}} \approx -1.89)$, and behaviour of function β is as follows:

1. If $\beta > 0$ the function β is increasing and difference equation (1) has an unique real equilibrium point;
2. In the interval $-\frac{3}{\sqrt[3]{4}} < \beta < 0$ function is increasing. In this interval difference equation (1) has only one real equilibrium point;
3. In case $\beta < -\frac{3}{\sqrt[3]{4}}$ function β is decreasing and difference equation (1) has three real equilibrium points;
4. There exists vertical asymptote at zero ($\overline{x} = 0$).

We can make an assertion that the only local maximum point of function β is a very important point of reference when analysing character of solutions of difference equation (1), so further we will investigate behaviour of (1) depending on the different values of the parameter β.

Looking at many different examples and analysing the equilibrium equation for different values of the parameter β and investigating the function of β it can be concluded that the behaviour of solutions of (1) is very sensitive to different values of parameter β, and if we take arbitrary initial conditions such that the denominator is not equal to zero we can make the following hypotheses:

1. If $\beta > 0$, then the equilibrium equation has one real root and two complex conjugate roots. The solution of difference equation (1) converges to the unique real equilibrium point.
2. If $\beta = 0$, then the equilibrium equation has one real root and two complex conjugate roots. The solution of difference equation (1) is periodic with prime period 3.
3. If $-\frac{3}{\sqrt[3]{4}} < \beta < 0$, then the equilibrium equation has one real root and two complex conjugate roots, but solution of difference equation (1) is with oscillating character.
4. If $\beta = -\frac{3}{\sqrt[3]{4}}$, then the equilibrium equation has three real equilibrium points (two equal and one different). The solution of difference equation (1) converges to the double root $\overline{x}_2 = -\sqrt[3]{2}$ of the equilibrium equation.
5. If $\beta < -\frac{3}{\sqrt[3]{4}}$, then the equilibrium equation has three real roots. The solution of difference equation (1) converges to the root that is greatest by the absolute value.

In numerical calculations we have observed that in these four situations behaviour of solutions of difference equations (1) is similar with arbitrary initial values such that difference equation (1) is well defined.

In our investigation we have established the following results.

Theorem 4 *Assume $x_{-1}, x_0 \in I$, where I is the set of all real initial values x_{-1}, x_0 for which the difference equation (1) is well defined for all $n \geq 0$. If $\beta \in (-\infty; -\frac{3}{\sqrt[3]{4}}] \cup [\frac{1}{\sqrt[3]{4}}; \infty)$, then the solution of difference equation (1) is locally asymptomatically stable.*

Proof The proof is based on the theorem of linearized stability of second order difference equation (see Theorem 1). Let write the characteristic equation of (1) about $\overline{x}$:

$$f = \beta + \frac{1}{uv}$$

$$u := x_n, \quad v := x_{n-1}$$

$$f_u^{'} = (\beta + (uv)^{-1})_u^{'} = -\frac{v}{(uv)^2} \Rightarrow p = f_u^{'}(\overline{x}, \overline{x}) = -\frac{1}{\overline{x}^3}$$

$$f_v^{'} = (\beta + (uv)^{-1})_v^{'} = -\frac{u}{(uv)^2} \Rightarrow q = f_v^{'}(\overline{x}, \overline{x}) = -\frac{1}{\overline{x}^3}$$

$$p = q = -\frac{1}{\overline{x}^3}$$

$$z_{n+1} = p z_n + q z_{n-1}, \quad n = 0, 1, \ldots$$

$$z_{n+1} = -\frac{1}{\overline{x}^3} z_n - \frac{1}{\overline{x}^3} z_{n-1}. \tag{16}$$

Equation (16) is the linearized equation associated with (1) about the equilibrium point $\overline{x}$. The characteristic equation of (1) about $\overline{x}$ is in the form:

$$\lambda^2 + \frac{1}{\overline{x}^3}\lambda + \frac{1}{\overline{x}^3} = 0 \tag{17}$$

By Condition 1 of Theorem 1 if both roots of (17) lie in the open unit disk $|\lambda| < 1$ then the equilibrium $\overline{x}$ of (1) is locally asymptotically stable. Roots of (17) lie in the open unit disk if Condition 3 of Theorem 1 holds, that is,

$$\left| -\frac{1}{\overline{x}^3} \right| < 1 - \frac{1}{\overline{x}^3} < 2.$$

If $\overline{x} > 0$, we have

$$\begin{cases} \frac{1}{\overline{x}^3} < 1 - \frac{1}{\overline{x}^3} \\ \frac{1}{\overline{x}^3} > -1 + \frac{1}{\overline{x}^3} \\ 1 - \frac{1}{\overline{x}^3} < 2 \end{cases} \Rightarrow \overline{x} > \sqrt[3]{2}.$$

Since β can be expressed from (15) we obtain that

$$\beta = \overline{x} - \frac{1}{\overline{x}^2} > \sqrt[3]{2} - \frac{1}{\sqrt[3]{4}} = \frac{1}{\sqrt[3]{4}}.$$

We have established Condition 3 of Theorem 1 when $\beta > \frac{1}{\sqrt[3]{4}}$ and this means that in this case the equilibrium point $\overline{x}$ is locally asymptotically stable.
If $\overline{x} < 0$, we have

$$\begin{cases} \frac{1}{\overline{x}^3} < 1 - \frac{1}{\overline{x}^3} \\ \frac{1}{\overline{x}^3} > -1 + \frac{1}{\overline{x}^3} \\ 1 - \frac{1}{\overline{x}^3} < 2 \end{cases} \Rightarrow \overline{x} < -\sqrt[3]{2}$$

and by Eq. (15) we get that

$$\beta = \overline{x} - \frac{1}{\overline{x}^2} < -\sqrt[3]{2} - \frac{1}{\sqrt[3]{4}} = -\frac{3}{\sqrt[3]{4}}.$$

In this case we see that Condition 3 of Theorem 1 holds, that is, the equilibrium point $\overline{x}$ is locally asymptotically stable when $\beta < -\frac{3}{\sqrt[3]{4}}$. □

Theorem 5 *Assume $x_{-1}, x_0 \in I$, where I is the set of all real initial values x_{-1}, x_0 for which the difference equation (1) is well defined for all $n \geq 0$ and $\beta \in [\frac{1}{\sqrt[3]{4}}; \infty)$. Then every positive solution $\{x_n\}_{n=-1}^{\infty}$ of difference equation (1) has a finite limit.*

Proof With the change of variables $x_n = \frac{1}{y_n\sqrt{\beta}}$ difference equation can be transformed to the difference equation (1) in the form

$$y_{n+1} = \frac{\alpha}{1 + y_n y_{n-1}}, \quad n = 0, 1, \ldots,$$

where $x_n = \frac{1}{y_n\sqrt{\beta}}, \quad x_{n-1} = \frac{1}{y_{n-1}\sqrt{\beta}}, \quad x_{n+1} = \frac{1}{y_{n+1}\sqrt{\beta}}$ and $\alpha = \frac{1}{\beta\sqrt{\beta}}$.

In the paper [1] it has been shown that every positive solution of (9) converges to $\bar{y}$ for all values of $0 < \alpha \leq 2$. Since $\alpha = \frac{1}{\beta\sqrt{\beta}}$ then

$$0 < \frac{1}{\beta\sqrt{\beta}} \leq 2 \Rightarrow \infty > \beta\sqrt{\beta} \geq \frac{1}{2} \Rightarrow \beta^3 \geq \frac{1}{4} \Rightarrow \beta \geq \frac{1}{\sqrt[3]{4}}. \quad \square$$

4 Periodicity

In this section we discuss the periodicity of equation (1). We will show that difference equation (1) has no periodic solutions with period 2 and difference equation (1) has periodic solution with period 3 if and only if $\beta = 0$.

Assume $x_{-1}, x_0 \in I$, where I is the set of all real initial values x_{-1}, x_0 for which the difference equation (1) is well defined for all $n \geq 0$ then that next two results are true.

Theorem 6 *Difference equation (1) has no periodic solutions with period* 2.

Proof Assume that x_{-1}, x_0 are initial conditions such that difference equation (1) is well defined and solution of difference equation (1) is periodic with period 2 ($x_{-1} \neq x_0$), that is,

$$\ldots x_{-1}, x_0, x_{-1}, x_0, \ldots.$$

Then it must be that

$$x_1 = \beta + \frac{1}{x_0 x_{-1}} = x_{-1}$$

$$x_2 = \beta + \frac{1}{x_1 x_0} = \beta + \frac{1}{x_{-1} x_0} = x_{-1} \Rightarrow x_0 = x_{-1}$$

which is a contradiction from which follows that equation (1) has no periodic solutions with period 2. $\square$

Theorem 7 *Difference equation (1) has periodic solution with period* 3 *if and only if* $\beta = 0$.

Proof Assume that x_{-1}, x_0 are well defined initial conditions and the solution of difference Eq. (1) is periodic with period 3, that is,

$$\ldots x_{-1}, x_0, x_1, x_{-1}, x_0, x_1, \ldots .$$

1. Assume $\beta \neq 0$. Then it must be that

$$x_1 = \beta + \frac{1}{x_0 x_{-1}}$$

$$x_2 = \beta + \frac{1}{x_1 x_0} = x_{-1}$$

$$x_3 = \beta + \frac{1}{x_2 x_1} = x_0$$

$$x_4 = \beta + \frac{1}{x_3 x_2} = x_1$$

From which follows that

$$x_2 = \beta + \frac{1}{x_1 x_0} = \beta + \frac{1}{(\beta + \frac{1}{x_0 x_{-1}})x_0} = x_{-1} \Rightarrow$$

$$\Rightarrow \frac{\beta(\beta x_0 x_{-1} + 1) + x_{-1}}{\beta x_0 x_{-1} + 1} = x_{-1} \Leftrightarrow \beta = 0,$$

which is a contradiction from which follows that if $\beta \neq 0$ then (1) has no periodic solutions with period 3.

2. Assume $\beta = 0$. Then we can write that

$$x_1 = \frac{1}{x_0 x_{-1}}$$

$$x_2 = \frac{1}{x_1 x_0} = \frac{x_0 x_{-1}}{x_0} = x_{-1}$$

$$x_3 = \frac{1}{x_2 x_1} = \frac{x_{-1} x_0}{x_{-1}} = x_0$$

$$x_4 = \frac{1}{x_3 x_2} = \frac{1}{x_0 x_{-1}} = x_1$$

$$\vdots$$

$$x_{n-1} = \frac{1}{x_{n-2}x_{n-3}}$$

$$x_n = \frac{1}{x_{n-1}x_{n-2}} = \frac{x_{n-2}x_{n-3}}{x_{n-2}} = x_{n-3}$$

$$x_{n+1} = \frac{1}{x_n x_{n-1}} = \frac{x_{n-2}x_{n-3}}{x_{n-3}} = x_{n-2}$$

$$x_{n+2} = \frac{1}{x_{n+1}x_n} = \frac{1}{x_{n-2}x_{n-3}} = x_{n-1}$$

$$x_{n+3} = \frac{1}{x_{n+2}x_{n+1}} = \frac{1}{x_{n-1}x_{n-2}} = x_{n-3}$$

$$\vdots$$

From which follows that if $\beta = 0$, then (1) has a periodic solution with period 3. □

After these two last results we obtain the following conclusions.

Corollary 1 *1. If $\beta \geq \frac{1}{\sqrt[3]{4}}$ and $\beta \leq -\frac{3}{\sqrt[3]{4}}$ then (1) has no periodic solutions with period $p > 1$, because in these cases can be obtained that the solution of (1) is locally asymptotically stable.*

2. *If $\beta = 0$, $x_{-1}, x_0 > 0$, $x_{-1} \neq x_0$ then (1) has a periodic solution with period 3 and solution is bounded in interval $[\frac{1}{x_0 x_{-1}}; \max\{x_{-1}, x_0\}]$.*
3. *If $\beta = 0$, $x_{-1} = x_0 > 0$, $x_{-1} = x_0 \neq 1$ then the solution of (1) is periodic with period 3 and bounded in interval $[\frac{1}{x_{-1}^2}; x_{-1}]$. If $x_{-1} = x_0 = 1$ then the solution of (1) is $x_n = \{1\}_{n \geq -1}$ and it is periodic with period 1.*
4. *If $\beta = 0$, $x_{-1}, x_0 < 0$, $x_{-1} \neq x_0$ then the solution of (1) is periodic with period 3 and bounded from below by $\min\{x_{-1}, x_0\}$ and from above by $\frac{1}{x_0 x_{-1}}$.*
5. *If $\beta = 0$, $x_{-1} = x_0 < 0$ then the solution of (1) is periodic with period 3 and bounded in interval $[x_{-1}, \frac{1}{x_{-1}^2}]$.*
6. *If $\beta = 0$, $x_{-1} > 0, x_0 < 0$ or $x_{-1} < 0, x_0 > 0$ then the solution of (1) is periodic with period 3 and bounded from below by $\min\{x_{-1}, x_0\}$ and from above by $\max\{x_{-1}, x_0\}$.*
7. *If $\beta = 0$, then difference equation (1) can be written in the form*

$$x_{n+1} = \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, \ldots, \tag{18}$$

and the positive solution of difference equation (18) can be written in the form

$$x_n = e^{c_1 \sin\left(\frac{2\pi}{3} n\right) + c_2 \cos\left(\frac{2\pi}{3} n\right)}, \quad c_1, c_2 \in R \tag{19}$$

and it is periodic with prime period 3 for all positive well defined initial conditions.

5 Forbidden Set

Open Problem 8.2 ([1]) Assume that β is a real number. Determine the set I of real initial values x_{-1}, x_0 for which the difference equation (1) is well defined for all $n \geq 0$, and investigate the character of solutions of difference equation (1) with $x_{-1}, x_0 \in I$.

Let $Dom(f)$ be domain of function $f(x_n, x_{n-1}) = \beta + \frac{1}{x_n x_{n-1}}$ of difference equation (1).

If there exist initial values (x_0, x_{-1}) and such $n \in N$ for which difference equation (1) is well defined and in iteration $n + 1$ difference equation (1) becomes equal to zero then initial values (x_0, x_{-1}) belong to forbidden set (F). This holds if

$$\beta = -\frac{1}{x_n x_{n-1}}, \quad n = 0, 1, \ldots, \quad x_n, x_{n-1} \neq 0 \Rightarrow \quad \beta x_n x_{n-1} + 1 = 0. \tag{20}$$

If initial values (x_0, x_{-1}) belong to the forbidden set then holds one of these conditions:

1. $\beta \in R$ and x_{-1} or x_0 is equal to zero;
2. $\beta \in R$ and $x_{-1} = x_0 = 0$;
3. $\beta < -1$ and $x_n x_{n-1} \in (0; 1)$;
4. $\beta = -1$ and $x_n x_{n-1} = 1$;
5. $-1 < \beta < 0$ and $x_n x_{n-1} \in (1; +\infty)$;
6. $\beta = 0$ and $x_n x_{n-1} = 0$;
7. $0 < \beta < 1$ and $x_n x_{n-1} \in (-\infty; -1)$;
8. $\beta = 1$ and $x_n x_{n-1} = -1$;
9. $\beta > 1$ and $x_n x_{n-1} \in (-1; 0)$.

Let $\beta \neq 0, \quad x_{-1} x_0 \neq 0$. Now we determine the forbidden set in each iteration.

1. $n = 0: \quad x_1 = 0 \Leftrightarrow \beta x_0 x_{-1} + 1 = 0 \Rightarrow x_0 = -\frac{1}{\beta x_{-1}}$
   In this step initial conditions $(x_{-1}, -\frac{1}{\beta x_{-1}}) \in F$.
2. $n = 1: \quad x_2 = 0 \Leftrightarrow \beta^2 x_0 x_{-1} + \beta + x_{-1} = 0 \Rightarrow x_0 = \frac{-\beta - x_{-1}}{\beta^2 x_{-1}}$
   $(x_{-1}, \frac{-\beta - x_{-1}}{\beta^2 x_{-1}}) \in F$.
3. $n = 2: \quad x_3 = 0 \Leftrightarrow \beta^3 x_0 x_{-1} + \beta^2 + \beta x_{-1} + x_0 x_{-1} = 0 \Rightarrow x_0 = \frac{-\beta^2 - \beta x_{-1}}{\beta^3 x_{-1} + x_{-1}}$
   $(x_{-1}, \frac{-\beta^2 - \beta x_{-1}}{\beta^3 x_{-1} + x_{-1}}) \in F$.

4. $n = 3: \quad x_4 = 0 \Leftrightarrow \beta^4 x_0 x_{-1} + \beta^3 + \beta^2 x_{-1} + 2\beta x_0 x_{-1} + 1 = 0 \Rightarrow x_0 = \frac{-\beta^3-\beta^2 x_{-1}-1}{\beta^4 x_{-1}+2\beta x_{-1}}$
   $(x_{-1}, \frac{-\beta^3-\beta^2 x_{-1}-1}{\beta^4 x_{-1}+2\beta x_{-1}}) \in F.$
5. $n = 4: \quad x_5 = 0 \Leftrightarrow \beta^5 x_0 x_{-1} + \beta^4 + \beta^3 x_{-1} + 3\beta^2 x_0 x_{-1} + 2\beta + x_{-1} = 0 \Rightarrow x_0 = \frac{-\beta^4-\beta^3 x_{-1}-2\beta-x_{-1}}{\beta^5 x_{-1}+3\beta^2 x_{-1}}$
   $(x_{-1}, \frac{-\beta^4-\beta^3 x_{-1}-2\beta-x_{-1}}{\beta^5 x_{-1}+3\beta^2 x_{-1}}) \in F.$
6. $n = 5: \quad x_6 = 0 \Leftrightarrow \beta^6 x_0 x_{-1} + \beta^5 + \beta^4 x_{-1} + 4\beta^3 x_0 x_{-1} + 3\beta^2 + 2\beta x_{-1} + x_0 x_{-1} = 0 \Rightarrow x_0 = \frac{-\beta^5-\beta^4 x_{-1}-3\beta^2-2\beta x_{-1}}{\beta^6 x_{-1}+4\beta^3 x_{-1}+x_{-1}}$
   $(x_{-1}, \frac{-\beta^5-\beta^4 x_{-1}-3\beta^2-2\beta x_{-1}}{\beta^6 x_{-1}+4\beta^3 x_{-1}+x_{-1}}) \in F.$
7. $n = 6: \quad x_7 = 0 \Leftrightarrow \beta^7 x_0 x_{-1} + \beta^6 + \beta^5 x_{-1} + 5\beta^4 x_0 x_{-1} + 4\beta^3 + 3\beta^2 x_{-1} + 3\beta x_0 x_{-1} + 1 = 0 \Rightarrow x_0 = \frac{-\beta^6-\beta^5 x_{-1}-4\beta^3-3\beta^2 x_{-1}-1}{\beta^7 x_{-1}+5\beta^4 x_{-1}+3\beta x_{-1}}$
   $(x_{-1}, \frac{-\beta^6-\beta^5 x_{-1}-4\beta^3-3\beta^2 x_{-1}-1}{\beta^7 x_{-1}+5\beta^4 x_{-1}+3\beta x_{-1}}) \in F.$
8. $n = 7: \quad x_8 = 0 \Leftrightarrow \beta^8 x_0 x_{-1} + \beta^7 + \beta^6 x_{-1} + 6\beta^5 x_0 x_{-1} + 5\beta^4 + 4\beta^3 x_{-1} + 6\beta^2 x_0 x_{-1} + 3\beta + x_{-1} = 0 \Rightarrow x_0 = \frac{-\beta^7-\beta^6 x_{-1}-5\beta^4-4\beta^3 x_{-1}-3\beta-x_{-1}}{\beta^8 x_{-1}+6\beta^5 x_{-1}+6\beta^2 x_{-1}}$
   $(x_{-1}, \frac{-\beta^7-\beta^6 x_{-1}-5\beta^4-4\beta^3 x_{-1}-3\beta-x_{-1}}{\beta^8 x_{-1}+6\beta^5 x_{-1}+6\beta^2 x_{-1}}) \in F.$
9. $n = 8: \quad x_9 = 0 \Leftrightarrow \beta^9 x_0 x_{-1} + \beta^8 + \beta^7 x_{-1} + 7\beta^6 x_0 x_{-1} + 6\beta^5 + 5\beta^4 x_{-1} + 10\beta^3 x_0 x_{-1} + 6\beta^2 + 3\beta x_{-1} + x_0 x_{-1} = 0 \Rightarrow x_0 = \frac{-\beta^8-\beta^7 x_{-1}-6\beta^5-5\beta^4 x_{-1}-6\beta^2-3\beta x_{-1}}{\beta^9 x_{-1}+7\beta^6 x_{-1}+10\beta^3 x_{-1}+x_{-1}}$
   $(x_{-1}, \frac{-\beta^8-\beta^7 x_{-1}-6\beta^5-5\beta^4 x_{-1}-6\beta^2-3\beta x_{-1}}{\beta^9 x_{-1}+7\beta^6 x_{-1}+10\beta^3 x_{-1}+x_{-1}}) \in F.$
10. $\cdots$

The general case (when $n = k: \quad x_{k+1} = 0$) is still in investigation process.

Example 1 If $\beta = -\frac{3}{\sqrt[3]{4}}$, then with the change of variables $x_{n-1} = -\sqrt[3]{2} y_{n-1}$ equation (1) can be written in the form without irrationality

$$y_{n+1} = \frac{3}{2} - \frac{1}{2y_n y_{n-1}}, \quad n = 0, 1 \ldots. \tag{21}$$

$$\overline{y}_1 = -0.5, \quad \overline{y}_2 = \overline{y}_3 = 1$$

Let assume

$$y_{-1} = \frac{k}{k+1}, y_0 = \frac{k-1}{k}, \quad k \in N \Rightarrow y_k = \frac{1}{2}, y_{k+1} = 0.$$

As we can take k as big as we like initial points y_{-1}, y_0 will be close to the equilibrium point $\overline{y} = 1$, but after limited amount of iterations we get $y_{k+1} = 0$. Hence

$$(y_{-1}, y_0) = \left(\frac{k}{k+1}, \frac{k-1}{k}\right) \in F, \quad k \in N.$$

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Moment Vanishing of Piecewise Solutions of Linear ODEs

Dmitry Batenkov and Gal Binyamini

Abstract We consider the "moment vanishing problem" for a general class of piecewise-analytic functions which satisfy on each continuity interval a linear ODE with polynomial coefficients. This problem, which essentially asks how many zero first moments can such a (nonzero) function have, turns out to be related to several difficult questions in analytic theory of ODEs (Poincare's Center-Focus problem) as well as in Approximation Theory and Signal Processing ("Algebraic Sampling"). While the solution space of any particular ODE admits such a bound, it will in the most general situation depend on the coefficients of this ODE. We believe that a good understanding of this dependence may provide a clue for attacking the problems mentioned above. In this paper we undertake an approach to the moment vanishing problem which utilizes the fact that the moment sequences under consideration satisfy a recurrence relation of fixed length, whose coefficients are polynomials in the index. For any given operator, we prove a general bound for its moment vanishing index. We also provide uniform bounds for several operator families.

Keywords Moment vanishing · Holonomic ODEs · Recurrence relations · Generalised exponential sums

1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be a bounded piecewise-continuous function with points of discontinuity (of the first kind)

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$$a = \xi_0 < \xi_1 < \cdots < \xi_p < \xi_{p+1} = b,$$

satisfying on each continuity interval $[\xi_j, \xi_{j+1}]$ a linear homogeneous ODE

$$\mathfrak{D} f \equiv 0, \tag{1}$$

where $\mathfrak{D}$ is a linear differential operator of order n with polynomial coefficients:

$$\mathfrak{D} = p_n(x)\,\partial^n + \cdots + p_1(x)\,\partial + p_0(x)\,\mathrm{I}, \qquad \partial = \frac{\mathrm{d}}{\mathrm{d}x}, \ \deg\ p_j \le d_j. \tag{2}$$

We say that such f belongs to the class $\mathscr{PD}(\mathfrak{D}, p)$. The union of all such $\mathscr{PD}(\mathfrak{D}, p)$ is the class $\mathscr{PD}$ of *piecewise D-finite functions*, which was first studied in [2].

Any $f \in \mathscr{PD}$ has finite moments of all orders:

$$m_k(f) = \int_a^b x^k f(x)\,\mathrm{d}x, \qquad k = 0, 1, 2, \ldots \tag{3}$$

We consider the following questions.

Problem 1 Given $\mathfrak{D}$ and p, determine the *moment vanishing index* of $\mathscr{PD}(\mathfrak{D}, p)$, defined as

$$\sigma(\mathfrak{D}, p) \stackrel{\text{def}}{=} \sup_{f \in \mathscr{PD}(\mathfrak{D}, p),\ f \not\equiv 0} \{k : m_0(f) = \cdots = m_k(f) = 0\} + 1.$$

In Theorem 3 below we shall prove that the moment vanishing index is always finite. Consequently, the following problem becomes meaningful.

Problem 2 Find natural families $\mathscr{F} \subset \mathscr{PD}$ which admit a uniform bound on the moment vanishing index, i.e. for which

$$\sigma(\mathscr{F}) = \sup_{\mathscr{PD}(\mathfrak{D}, p) \subset \mathscr{F}} \sigma(\mathfrak{D}, p) < +\infty.$$

Our main results, presented in Sect. 4, provide a general bound for $\sigma(\mathfrak{D}, p)$ in terms of $\mathfrak{D}$. As a result, several examples of families $\mathscr{F}$ admitting uniform bound as above are given. The main technical tool is the recurrence relation satisfied by the moment sequence, established previously in [2].

Our main application is the problem of reconstructing functions $f \in \mathscr{PD}$ from a finite number of their moments. Inverse moment problems appear in some areas of mathematical physics, for instance heat conduction and inverse potential theory [1, 11], as well as in statistics. One particular reconstruction technique, introduced in [2] and further extended to two-dimensional setting in [7], can be regarded as a prototype for numerous "algebraic" reconstruction methods in signal processing, such as finite rate of innovation [17] and piecewise Fourier inversion [3, 5]. These methods, being essentially nonlinear, promise to achieve better reconstruction accuracy in some cases (as demonstrated recently in [3, 5]), and therefore we believe their study to be important. In Sect. 2 below we show that an answer to Problem 2 would in turn provide a bound on the minimal number of moments (measurements) required for unique reconstruction of any $f \in \mathscr{F}$. In essence, the results of this paper can be regarded as a step towards understanding the range of applicability of the piecewise D-finite reconstruction method to general signals in $\mathscr{PD}$. See Sect. 2 for further details.

Given a family $\mathscr{F} \subset \mathscr{PD}$, consider the corresponding family of moment generating functions $\left\{I_f(z)\right\}_{f \in \mathscr{F}}$, where $I_f(z) = \sum_{k=0}^{\infty} m_k(f) z^{-k-1}$. Obtaining information on the moment vanishing index is in fact an essential step towards studying the analytic properties of I_f, in particular a bound on its number of zeros near infinity (as provided by the notion of "Taylor Domination", see [4, 6]), as well as conditions for its identical vanishing. In turn, these questions play a central role in studies of the Center-Focus and Smale–Pugh problems for the Abel differential equation, see [8, 9, 16] and references therein.

The moment vanishing problem has been previously studied in the complex setting by V. Kisunko [14]. He showed that a uniform bound $\sigma(\mathscr{F})$ exists for families $\mathscr{F}$ consisting of non-singular operators, by using properties of Cauchy type integrals. In contrast, in this paper we consider the real setting only, while proving uniform bounds for some singular (as well as regular) operator families. Our method is based on the linear recurrence relation satisfied by the moment sequence. Using this method, in Sect. 5 we provide an alternative proof of Kisunko's result, stating that the moment generating function $I_f(z)$ of some $f \in \mathscr{PD}(\mathfrak{D}, p)$ satisfies a non-homogeneous ODE

$$\mathfrak{D} I_f(z) = R_f(z)$$

for a very special rational function $R_f(z)$, which depends on $\mathfrak{D}$ and on the values of f at the discontinuities.

In Sect. 6 we provide an interpretation of our main result in the language of Fuchsian theory of ODE.

2 Moment Reconstruction

We start by defining some preliminary notions.

Definition 1 The Pochhammer symbol $(i)_j$ denotes the falling factorial

$$(i)_j \stackrel{\text{def}}{=} i(i-1)\cdot\dots\cdot(i-j+1), \qquad i \in \mathbb{R},\ j \in \mathbb{N}$$

and the expression $(i)_j$ is defined to be zero for $i < j$.

Definition 2 Given $\mathfrak{D}$ of the form (2), the *bilinear concomitant* [13, p. 211] is the homogeneous bilinear form, defined for any pair of sufficiently smooth functions $u(x), v(x)$ as follows (all symbols depend on x):

$$\begin{aligned} P_{\mathfrak{D}}(u,v) \stackrel{\text{def}}{=} & \, u\left\{p_1 v - \partial(p_2 v) + \cdots + (-1)^{n-1}\partial^{n-1}(p_n v)\right\} \\ & + u'\left\{p_2 v - \partial(p_3 v) + \cdots + (-1)^{n-2}\partial^{n-2}(p_n v)\right\} \\ & + \ldots \\ & + u^{(n-1)} \cdot (p_n v). \end{aligned} \tag{4}$$

Proposition 1 (Green's formula, [13]) *Given $\mathfrak{D}$ of the form (2), let the formal adjoint operator be defined by*

$$\mathfrak{D}^*\{\cdot\} \stackrel{def}{=} \sum_{j=0}^{n} (-1)^j \partial^j \left\{p_j(x)\cdot\right\}.$$

Then for any pair of sufficiently smooth functions $u(x), v(x)$ the following identity holds:

$$\int_a^b v(x)(\mathfrak{D}u)(x)\,\mathrm{d}x - \int_a^b u(x)\left(\mathfrak{D}^* v\right)(x)\,\mathrm{d}x = P_{\mathfrak{D}}(u,v)(b) - P_{\mathfrak{D}}(u,v)(a). \tag{5}$$

Theorem 1 ([2]) *Let $f \in \mathscr{PD}(\mathfrak{D}, p)$ with $\mathfrak{D}$ of the form (2). Denote the discontinuities of f by $a = \xi_0 < \xi_1 < \cdots < \xi_p < \xi_{p+1} = b$. Then the moments $m_k = \int_a^b f(x)\,\mathrm{d}x$ satisfy*[1] *the recurrence relation*

$$\sum_{j=0}^{n}\sum_{i=0}^{d_j} a_{i,j}(-1)^j (i+k)_j\, m_{i-j+k} = \varepsilon_k, \quad k = 0, 1, \ldots, \tag{6}$$

where

$$\varepsilon_k = -\sum_{j=0}^{p}\left\{P_{\mathfrak{D}}\left(f, x^k\right)\left(\xi_{j+1}^-\right) - P_{\mathfrak{D}}\left(f, x^k\right)\left(\xi_j^+\right)\right\}. \tag{7}$$

Proof Apply Green's formula (5) to the identity

$$\int_{\xi_j}^{\xi_{j+1}} x^k (\mathfrak{D}f)(x)\,\mathrm{d}x \equiv 0$$

[1] For consistency of notation, the sequence $\{m_k\}$ is understood to be extended with zeros for negative k.

for each $j = 0, \ldots, p$ and sum up. The result is

$$\sum_{j=0}^{p} \int_{\xi_j}^{\xi_j+1} f(x)\, \mathfrak{D}^* \left\{x^k\right\} \mathrm{d}x = -\sum_{j=0}^{p} \left\{ P_{\mathfrak{D}} \left(f, x^k\right) \left(\xi_{j+1}^-\right) - P_{\mathfrak{D}} \left(f, x^k\right) \left(\xi_j^+\right) \right\}$$

$$\int_a^b f(x)\, \mathfrak{D}^* \left\{x^k\right\} \mathrm{d}x = \varepsilon_k$$

The left-hand side of the last formula is precisely the linear combination of the moments given by the left-hand side of (6). This finishes the proof. □

Now consider the problem of recovering $f \in \mathscr{PD}(\mathfrak{D}, p) \subset \mathscr{PD}$ from the moments $\{m_0(f), \ldots, m_N(f)\}$ (the operator $\mathfrak{D}$ is assumed unknown in the most general setting). Based on the recurrence relation (6), we demonstrate in [2] that an exact recovery is possible, provided that the number $N \in \mathbb{N}$ is sufficiently large. However, the question of obtaining an upper bound for N turns out to be non-trivial, as we now demonstrate.

Definition 3 Given $\mathfrak{D}$ and p, the *moment uniqueness index* $\tau(\mathfrak{D}, p)$ is defined by

$$\tau(\mathfrak{D}, p) \stackrel{\text{def}}{=} \sup_{f,g \in \mathscr{PD}(\mathfrak{D},p),\ f \not\equiv g} \left\{k : m_j(f) = m_j(g),\ 0 \leqslant j \leqslant k\right\} + 1.$$

In other words, given $\mathfrak{D}$ and p, at least $\tau(\mathfrak{D}, p)$ first moments of $f \in \mathscr{PD}(\mathfrak{D}, p)$ are necessary for unique reconstruction of f.

Recalling boundedness of $\sigma(\mathfrak{D}, p)$ (see Theorem 3 below), we immediately obtain the following conclusion.

Lemma 1 *For any operator* $\mathfrak{D}$ *and any* p

$$\tau(\mathfrak{D}, p) \leq \sigma(\mathfrak{D}, 2p).$$

Proof Let $N = \sigma(\mathfrak{D}, 2p)$. Take f_1, f_2 having p jump points each, satisfying $\mathfrak{D} f_1 \equiv 0$, $\mathfrak{D} f_2 \equiv 0$ on each continuity interval such that

$$\begin{aligned} m_0(f_1) &= m_0(f_2) \\ &\ldots \\ m_N(f_1) &= m_N(f_2). \end{aligned}$$

The function $g = f_1 - f_2$ has at most $2p$ jumps, and it satisfies $\mathfrak{D}g \equiv 0$ on each continuity interval. The first N moments of g are zero, therefore $g \equiv 0$ and thus $f_1 \equiv f_2$. Therefore $\tau(\mathfrak{D}, p) \leq N$. □