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Jan von Plato

Saved from the Cellar

Gerhard Gentzen's Shorthand Notes on Logic and Foundations of Mathematics

Springer

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„Alles Unwichtige bzw. Übernommene aus 245 in den Keller. Hier nur noch einiges vielleicht Verwendbare.“

Gerhard Gentzen, addition to manuscript page **BTIZ** 245.10

„Seite 161–198.2 (außer 170–171, welches den Hauptwertbegriff behandelt) in den Keller, betrifft die Ausarbeitung zur Habilschrift.“

Gerhard Gentzen, addition to manuscript page **BTIZ** 160d

Preface

In the summer of 1939, at the time of finishing his great paper on initial segments of transfinite induction, Gerhard Gentzen started to write down ideas for a popular book on the foundations of mathematics. His somewhat unlikely motto for the planned book was "Spannend wie ein Kriminalroman!" ("Exciting like a detective story"). Nothing definitive came of this book idea, for Gentzen was drafted to military service by the end of September of the same year. His war ended in a nervous breakdown by the beginning of 1942, with nothing at all of his plans finished after 1939, but just a life lost in 1945, at the age of 35.

Two slim folders of stenographic materials in Gentzen's hand were found in 1984. They contain notes and developments Gentzen thought could still be of some use to him, from between 1931 and 1944. In the past ten years or so, these notes and some additional manuscripts I had the fortune to find have been the object of a detective work of mine, done alongside systematic studies on proof theory. The major part of the work has consisted in the control, word for word, of the contents of transcribed manuscripts against the stenographic originals, in their interpretation in English, and in thinking of their significance against the rest of Gentzen's results. Occasional forays into archives in Germany, Switzerland, and elsewhere have provided some additional excitement.

I worked on the shorthand notes initially on the basis of transcriptions prepared by Christian Thiel. The sources of the present collection were completed through transcriptions made by Gerlinde Bach on my commission. On occasion, I filled in some gaps myself, though very slowly. I am indebted to both transcribers for what they have accomplished. I also thank Gereon Wolters for his help with the source materials.

The letters of Paul Bernays are published through an agreement with Dr. Ludwig Bernays.

My special thanks go to Eckart Menzler: Our knowledge of the details of Gentzen's life is the result of the relentless efforts of his *amateur* historianship in the proper sense of the word. I dedicate this edition to him, in thankfulness for his invaluable services to scholarship.

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Part I: A sketch of Gentzen's life and work

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1. Overture

Gerhard Gentzen died on August 4, 1945, in a prison in Prague. His fellow prisoners were professors of the local German university, and there are accounts of his last days and how he was, rendered weak by lack of food, still pondering over the consistency problem of analysis. After the war, some attempts were made to find any manuscripts he might have left behind; a mythical suitcase one letter reports he had been carrying around, filled with papers with a near-proof of the consistency of analysis. Nothing was found, though, in Prague. In Göttingen, instead, there were manuscripts that were preliminary studies for published work, by the account of Arnold Schmidt. He wrote in 1948 to Gentzen's mother that the papers would be placed and kept together with Hilbert's papers; yet again, nothing has been found.

More than thirty years later, in 1984, two slim folders of stenographic notes by Gentzen surfaced as if by a miracle, one blue, the other violet in colour. They were given by Gentzen's sister Waltraut Student through the mediation of Prof. Hans Rohrbach to Prof. Christian Thiel of Erlangen University. Thiel had learned the "unified shorthand" as youth, like many Germans, and listed the contents, some three-hundred-odd pages, and started doing some transcriptions. For various reasons, the project slowed down gradually, and none of his transcriptions into German have reached the stage of publication.

It appears from a note in the violet folder that Gentzen had left the papers in a summer place in the village of Putbus, on the island of Rügen in the Baltic Sea close to his hometown Stralsund. That took place in the summer of 1944 and suggests that Gentzen bore no illusions about a future in Prague.

The Putbus folders contain, by the passages that take the place of a frontispiece in this book, those parts of his manuscripts he thought could still prove to be useful. Page 245 of the series **BTIZ** is actually a set of over 20 pages, with the added remark on page 245.10: „Alles Unwichtige bzw. Übernommene aus 245 in den Keller. Hier nur noch einiges vielleicht Verwendbare.“ (All that is unimportant resp. superseded from 245 in the cellar. Here just some things that could be usable.) On page 160d of the same series, he has added in October 1942: „Seite 161–198.2 (außer 170–171, welches den Hauptwertbegriff behandelt) in den Keller, betrifft die Ausarbeitung zur Habilschrift.“ (Page 161–198.2 (except for 170–171 that treats the concept of main value) in the cellar, concerns the elaboration for the habilitation thesis.)

Save for the pages that were preserved in the two folders, the Gentzen manuscripts got lost in some forgotten cellar, in the attic of the Göttingen mathematics department to be subsequently discarded, and the rest burnt.

The amount of notes that Gentzen wrote was enormous. They are usually divided into series that bear acronyms, such as **WA, WTZ,** and **BTIZ**. These stand for *Widerspruchsfreiheit Analysis, Widerspruchsfreiheit transfinite Zahlen, Beweistheorie der intuitionistischen Zahlentheorie* (Consistency analysis, Consistency transfinite numbers, Proof theory of intuitionistic number theory). **WA** runs to at least page 339, by a summary that has been preserved, whereas the last extant page is 254 of September 1943. **WTZ** is the series of notes in which ordinal proof theory was created, but nothing is left of it except references such as **WTZ** 150 and **WTZ** 210 in the other series. The remaining pages of **BTIZ**, instead, amount to some two hundred printed pages. That is less than half of what the page numbering, running to 275, would suggest for the whole of it.

My work with the manuscripts began in 2005 with a stroke of luck right at the beginning, during my first visit to Erlangen in February of that year, namely, the rediscovery of a handwritten version of Gentzen's thesis with a detailed proof of the normalization of derivations in intuitionistic natural deduction. I made a translation into English and published the proof in *The Bulletin of Symbolic Logic* in 2008, seventy-five years after Gentzen had written it down. The motive for my visit was that Dov Gabbay had generously invited me to write a chapter on Gentzen's logic to the fifth volume of the *Handbook of the History of Logic* (von Plato 2009), and I felt that it would be important to give the readers at least some idea of the contents of the stenographic manuscripts. I had been in contact with Thiel about the manuscripts some five years earlier and he had even sent me a few samples of originals and transcriptions in that connection.

My interest in the Gentzen papers comes mainly from a desire to understand better his published work. For example, there is an offhand remark in his published thesis that he uses in the proof of the cut elimination theorem a rule called *Mischung* (and bear with me for a paragraph and a half if this means little), rendered as mix or multicut today, "to make the proof easier." A responsible author of a scientific text indicates by such even the possession of a proof without the multicut rule. The problem is how to permute cut up if its right premiss has been derived by a rule of left contraction. I figured out a solution on my own in 1998 and then tried to find similar in the Gentzen papers. My proof, in von Plato (2001), reduces contraction on a formula to contractions on its subformulas, but there is no such method in use in any of Gentzen's work, published or otherwise preserved.

There was another, more likely candidate for an original proof of cut elimination, but I wasn't quite sure about it. So I read the published papers over and over again and found more and more things in them. For example, the thesis contains a sequent calculus in which half of the logical rules are replaced by "logical groundsequents" such as $A \& B \rightarrow A$ and $A \rightarrow A \vee B$ and $A, A \supset B \rightarrow B$. Cut elimination can be only partial with this calculus, but it happens that the remaining cuts are innocent in the sense that they can be arranged so as not to violate the subformula property, the crucial consequence to cut elimination by which all formulas in a cut-free derivation are found as parts in the assumptions or the claim to be proved. In cut elimination for the groundsequent calculus, the cut formula is never principal in both premisses of a cut. There was thus the promising possibility that the problem with multicut, namely, that the right premiss of cut has been derived by a contraction on the cut formula, would vanish in this treatment. Some of these questions are explored in von Plato (2012, section 12). A decisive final insight on the problem came through systematic work: If possible multiplicities of formulas in sequents are erased at once at their appearance in the conclusions of rules, a cut elimination procedure quite different from Gentzen's published one with multicut is needed. The details of this proof have been worked out in Negri and von Plato (2016): The main result is a cut elimination procedure in which cut on the original cut formula is first permuted up, followed by cuts on the components of that cut formula.

I found it fascinating to ponder what Gentzen may have accomplished without telling anyone else, and applied the "hypothetico-deductive method" to textual interpretation, something I learned from Harry Wolfson's *The Philosophy of the Kalam* (Harvard U.P. 1976; don't ask me why I started reading that book): If Gentzen solved problem X in a certain way Y, then there should be some thing Z somewhere in his texts. If Z is found, it acts as a confirmation of the presence of solution Y. For the specific example of cut elimination without multicut, the evidence that the kind of solution proposed in Negri and von Plato (2016) was known to Gentzen came from two sources, the first his 1938 proof of the consistency of arithmetic: It contains a peculiar "altitude line" construction and a related cut elimination procedure with a direct connection to our proof. There is even a letter of 1938 to Paul Bernays in which Gentzen writes: "How I have obtained the consistency proof from the methods of proof in my dissertation is, I believe, now somewhat easy to see in the new version." Secondly, the idea of modifying derivations in sequent calculus so that no multiplicities appear is found in Gentzen's thesis, in the section in which he proves the decidability of intuitionistic propositional logic (1934–35, IV §1). The way to a proof of cut elimination for Gentzen's calculus *LI* through this modification is straightforward, as detailed out in Negri and von Plato (2016).

Systematic logical work is often helpful in understanding Gentzen's accomplishments. There is behind the polished articles an enormous amount of detailed results and profound insights. Moreover, such historically motivated systematic work can lead to valuable new results on the proof systems of logic and on their application to arithmetic. It remains to be seen to what extent this turns out to be the case for the two unknowns of the stenographic notes, the big series **BTIZ** and **WA**.

The Putbus notes give an idea of how Gentzen worked. He would start with a theme with an absolute independence of mind. Thus, having decided to try his hands on the consistency problem of arithmetic and analysis early in 1932, he would first clarify the nature of mathematical reasoning as it appears in practice. This starting point led soon to the abandonment of the prevailing axiomatic logical tradition. By September, he had found what is now the standard system of natural deduction, mainly by trying out all the possibilities, and had the first ideas about a proof of normalization. At that point, his notes for a series called **D**, possibly for *Deduktion* or *Dissertation*, run to the eighties. Each page is actually a four-page sheet, numbered, say, 87.1 to 87.4.

In October 1932, the series **D** got rebaptized into **INH** that stands, in translation, for something like *The formal conception of the notion of contentful correctness in number theory. Relation to proof of consistency.* Two things emerge from this 36-page bound and covered manuscript, the only one of its kind among the Gentzen papers: The idea of the subformula property comes from an attempt at a semantical explanation of intuitionistic logic that should be, as one says now, compositional. The first idea for a consistency proof of arithmetic is an extension of the subformula property to a formal system of arithmetic.

Gentzen managed to prove the normalization theorem first for a fragment of predicate logic with conjunction, negation, and the universal quantifier. Success with the $\& \neg \forall$-fragment led soon, in January 1933, to the realization that the classical law of double negation elimination is redundant for that fragment if all atomic formulas are double-negated at the outset. It also led to the mentioned very detailed proof of normalization for the full system of intuitionistic natural deduction. The former result made it to a stage of proofs of an article in which classical arithmetic is reduced to the intuitionistic one (Gentzen 1933). Also the idea to prove the consistency of intuitionistic arithmetic by transfinite induction appears in **INH** but was put aside for the time being.

No description of Gentzen's work methods exists. From some sources, as in Menzler-Trott (2007, p. 30), it seems he could figure out things in his head, out of touch with the world. When finished with a train of thought, he would write it down. That is the impression one gets from many of the series that have a couple of pages of notes with intervals of one or two days. In one case, the series **WA**, there is a secondary series titled **WAV** in which the **V** stands for *Veröffentlichung,* publication, with detailed section titles and outlines of their contents. These notes, running to eighty pages and beyond, were then used for writing the manuscript of Gentzen's famous proof of the consistency of arithmetic, published as a 73-page article in 1936.

At their best, the Gentzen notes clarify the development of his consistency program as it appears from his published work, as with the normalization idea. Here is another example: The consistency proof of the 1936-article was preceded by another one that used at a crucial point, as Gödel was quick to point out in the fall of 1935, Brouwer's principle of bar induction. From **WAV**, we find a reference to a first such proof that used the intuitionistic sequent calculus *LI*, and an outline of a similar proof with the classical calculus *LK*, referred to as "the second proof." Thus, the proof in the 1935-manuscript within yet another logical calculus, preserved in the form of galley proofs, is a third proof, and the published one by transfinite induction a fourth.

It becomes clear from the manuscripts that many aspects of Gentzen's published papers stem from concerns about presentation rather than intrinsic logical reasons. For example, the 1936 paper uses classical logic even if it is not needed, the likely reason being that Gentzen did not want to give the impression that his result somehow depended on the acceptance of intuitionistic logic. Secondly, the paper uses a special notation for logical derivations, what is known as natural deduction in sequent calculus style, by which derivation trees are avoided. An inspection of the literature shows that such two-dimensional proof patterns were a novelty at the time. Third, it avoids the use of the classical sequent calculus *LK* with its multiple succedent sequents $\Gamma \rightarrow \Delta$ in which Δ stands for a finite number of cases under assumptions Γ instead of the traditional single conclusion of a mathematical claim, as in $\Gamma \rightarrow C$.

Finally, the paper introduces a special decimal notation for ordinals. When in 1938 Gentzen turned into using the standard notation for ordinals, he was very careful to point out that even if the notation comes from set theory, "all definitions and proofs in this section are completely 'finite' and in this respect even of a particularly elementary kind" (1938, pp. 37–38). Thus, with the decimal notation of 1936 he most likely wanted to avoid the impression that his consistency proof depended on set-theoretic concepts.

Except for an eleven-page summary of the consistency problem of analysis, dated February 1945 and preserved through Hans Rohrbach, and the longhand manuscript of Gentzen's thesis, the Putbus notes are the only ones that have remained. The pre-war notes kept in Göttingen have been discarded decades ago. There are in addition some letters by Gentzen, those to Paul Bernays and Arend Heyting with a detailed scientific content and therefore included in this collection. Many more can be found in Menzler-Trott's Gentzen biography.

The stenographic notes together with the thesis manuscript can be divided into three: The series **WA**, the series **BTIZ**, and the rest. Each of these is *grosso modo* about two hundred pages in standard print. The present collection contains the English translations of the third group of notes and the mentioned letters. The other two series come in parts, with a first attempt at the consistency of analysis in the second half of 1938, followed by a more extensive series in 1943 and a brief summary in February 1945.

A similar first attempt at the proof theory of intuitionistic arithmetic from the summer of 1939 is preserved, followed by a shorter period of thoughts in the spring of 1942 and a sustained attempt in 1943. There are gaps, for example, **WA** starts in August 1938 from page 77, and the series **BTIZ** in May 1939 from page 133.

The stenographic notes often complement the published papers, by offering alternatives. The more unified series are typically inconclusive, loose ends in Gentzen's work, such as the attempt at a semantical decision procedure for intuitionistic propositional logic in the series **AL**, dated October 1942.

One has to get used to the nature of the texts. They are for the most part clearly written, but meant for the writer's eyes, who would remember their context. Phrases can be incomplete, a stenographic mark unreadable, and there are abrupt starts, loose ends, and connections that have to be figured out by the reader, if at all possible, and so on. My attitude is to take what I can understand and hope to get more out with a next reading. Bits and pieces here and there help, from other texts, from the published papers, and from works by others.

The English translations have been produced in two stages, the first the deciphering of the stenographic original, the second a translation into English. In the latter, the idea has been that the thought behind should be decisive. My method is this: Having had my elementary school training in German, at the *Deutsche Schule Helsinki*, that language became the one that dominated my verbal thinking for a good number of decisive years of my childhood. Taking in the German text, I fancy seeing in my mind's eye *precisely* the thought behind the words, and try to reproduce it in English; it is good to keep in mind that the original German need not be anything like handsome writing and that turning complicated German sentence constructions around, into typical sentence structures in English, may somewhat distort the contents. With the correspondence included in this collection, there are no exact matches to the different forms of addressing and saluting a person. I have given literal translations.

German is full of little words that convey some meaning such as a grade of hesitation but that may give an exaggerated feeling if translated into English: *eben, doch, also,* etc. As the guiding principle has been to always give precedence to thought over words, I have not followed any uniform policy with such words.

A part of the transcriptions had been prepared by Thiel, but the greater part is work commissioned by myself from Gerlinde Bach, a retired academic secretary in Munich who is fluent with the stenography and who even knows the logical symbolism. The translations into English have been made with the transcriptions and the originals at hand. My own experience is that a sufficient amount of concentrated effort leads to the ability to read the stenographic script against its transcription, to control the correctness of the latter word for word. Errors remain, mainly because the originals are sometimes unclear. On the other hand, as the contents are logical and mathematical, it is instead usually clear what reading makes sense in a context.

The texts have many cancelled passages. Often these are just wrong beginnings, formulated at once in an improved form, and usually not indicated in the translations. At other times I have judged the contents of a cancelled passage to merit being translated. The guiding idea has always been to reproduce the train of thought of Gentzen and not distracted by cancelled false ends, for this is no complete German edition, such as may have been Thiel's objective.

2. Gentzen's years of study

Gerhard Karl Erich Gentzen was born on November 24, 1909, in the northern German town of Greifswald by the Baltic Sea. He seems to have been a quiet and reserved person with a unique talent for mathematics. Craig Smorynski (2007) describes in an appendix to Eckart Menzler's Gentzen-biography, *Logic's Lost Genius*, the first preserved expression of Gentzen's mathematical talent, namely a little theorem about triangles and intersection points in the Euclidean plane that he found at the age of thirteen.

Gentzen's university studies began in 1928 under the guidance of the mathematician Hellmuth Kneser in Greifswald, but as was the habit of the times, he shifted between different places. In the Summer Semester of 1929, he was following Hilbert's lectures titled *Mengenlehre* (set theory) in Göttingen. Beautifully finished notes of the course by Gentzen's friend Lothar Collatz have been preserved; the topics include ordinal numbers and their arithmetic, and the second number class (i.e., constructive ordinals). A final part discusses paradoxes, the language of first-order logic, and the problem of consistency of arithmetic.

In the fall of 1930, Gentzen was enrolled in Berlin, where Johann von Neumann gave a course on Hilbert's proof theory, titled *Beweistheorie*. He had heard in Königsberg in September about Gödel's discovery of the incompleteness of arithmetic and decided to explain this work in his course. Contemporary accounts tell of a tremendous excitement that these developments aroused among the Berlin mathematicians. Carl Hempel, later a very famous philosopher, was one of the participants, and his recollection (Hempel 2000, pp. 13–14), even evidenced by contemporary correspondence for which see Mancosu (1999), is as follows:

> I took a course there with von Neumann which dealt with Hilbert's attempt to prove the consistency of classical mathematics by finitary means. I recall that in the middle of the course von

> Neumann came in one day and announced that he had just received a paper from... Kurt Gödel who showed that the objectives which Hilbert had in mind and on which I had heard Hilbert's course in Göttingen could not be achieved at all. Von Neumann, therefore, dropped the pursuit of this subject and devoted the rest of the course to the presentation of Gödel's results. The finding evoked an enormous excitement.

One who was present is Jacques Herbrand, who wrote a short paper on incompleteness in the spring of 1931. There are in addition letters he wrote late in 1930 that contain the basic ideas of Gödel's proof, as well as von Neumann's independent discovery of the second incompleteness theorem, by which the consistency of arithmetic is among the unprovable propositions. Von Neumann got in fact a copy of Gödel's paper with the second theorem only in early 1931.

I think it likely that Gentzen followed von Neumann's lectures, as perhaps suggested on p. 143 below, or at least shared some of the atmosphere around these foundational matters. A letter to Hellmuth Kneser of 2 July 1930 clearly indicates his interest (Menzler-Trott, p. 24):

> I have studied the book *Theoretische Logik* by Hilbert and Ackermann, as I would like to come to a greater clarity on the foundations of mathematics. Now I will attempt to acquire the recent treatments of Hilbert on these matters.

What could Gentzen have learned from von Neumann, next to Gödel's two theorems? There is, first of all, von Neumann's great paper of 1927, *Zur Hilbertschen Beweistheorie*, (On Hilbertian proof theory), with the intriguing footnote (note 9, p. 38):

> The possibility lies close of substituting the logical axioms by the logical ways of inference that rely on the axioms. There would be in place of a single syllogistic rule of inference several ones, by which the (rather arbitrarily built) group of axioms I would disappear. We have refrained from such a construction, because it departs too radically from the usual one.

The axioms are those of classical propositional logic with implication and negation as the primitive notions, and the rule is implication elimination, so here is one possible origin of Gentzen's calculus of natural deduction. A rule system that corresponds to von Neumann's axioms for logic is studied to some extent in von Plato (2014). It turns out that if a system has only rules and no axioms, the addition of a rule is required that in terms of natural deduction is implication introduction. In terms of axiomatic logic, the additional rule is called "the deduction theorem."

Von Neumann's paper on Hilbert's proof theory is known for its proof of the consistency of arithmetic when the principle of induction is restricted to free-variable formulas. It is referred to by Gentzen in his dissertation that gives a similar result, with a completely different proof. More could be learned from von Neumann: A long paper of his of 1928 discusses transfinite induction and three versions of set theory, one of them intuitionistic set theory. Further, it is known that von Neumann knew well and had by 1929 even applied the bar theorem, a fundamental principle of intuitionistic mathematics.

By the summer of 1931, Gentzen had already done work of his own in logic: At the suggestion of Paul Bernays, he had studied the logical theory of "sentence systems" of Paul Hertz. This was in Göttingen, the leading center for mathematics and exact science worldwide at the time. In connection with the preparation of Gentzen's *Collected Papers*, Bernays suggested in a letter of 14 May 1968 to the editor Manfred Szabo that the work of Hertz be explained, and then added:

> It would be worth mentioning that Gentzen showed here his great mathematical ability, bringing to a complete resolution a difficult problem that had remained open in the theory of sentence systems.

During the fall of 1931, Gentzen worked on rather detailed logical questions, especially the translation of arithmetic into pure predicate logic: The arithmetical operations are translated into relations such as $\sigma(x, y, z)$ ("the sum of x and y is z"). This was the way in which Gentzen in his thesis was able to reduce the consistency of arithmetic without the full induction principle to the cut elimination theorem of pure predicate logic.

In early February 1932, Gentzen sent his work on Hertz systems to the *Mathematische Annalen*, then started a systematic work the central aim of which was to find consistency proofs for arithmetic and analysis.

The dean of mathematics in Göttingen was David Hilbert whose main interest then was in the foundations of mathematics. Hilbert's closest collaborator in that field was the *Professor extraordinarius* Bernays, a native of Switzerland. The latter was writing the volume *Grundlagen der Mathematik* (Foundations of mathematics) that was planned to contain a full statement of the famous program Hilbert had set up, namely, to secure the foundation of mathematics through the steps of formalization of the language of mathematics and of mathematical proofs, through a proof of the consistency and completeness of the formalization, as well as through a study of the *Entscheidungsproblem*, the question whether a mathematical theory accepts a method for deciding theoremhood. The high hopes of Hilbert's program were shattered by the incompleteness theorem of Kurt Gödel. Bernays, who soon was to become Gentzen's teacher, had to start over in 1931 the writing of the *magnum opus.*

Mathematical logic and the foundations of mathematics became the focus of Gentzen's studies, and he finished in May 1933 a thesis with the title *Untersuchungen über das logische Schliessen*, (Investigations into logical inference). Meanwhile, Gentzen's teacher Bernays had been fired in April as a "non-Aryan," on the basis of the racial laws of the Nazi government of Hitler. These events may have contributed to the haste with which Gentzen finished the doctorate. He sought financing for continued studies on the problem of the consistency of analysis, but he also took an exam that gave him the right to teach at high schools. It was a rather serious undertaking that took a lot of his time, from the summer of 1933 until November of that year. He had to hand in two written works, one the published paper on Hertz systems, another a manuscript, subsequently lost, on the application of the recently discovered quantum-mechanical tunneling effect to cosmic radiation.

After the doctorate, Gentzen lived on small scholarships in Stralsund. In the second part of 1935, he was nominated assistant to David Hilbert. Hilbert's years of research were bygone at this point, and Gentzen was able to concentrate on his own research. Even before the assistant's position, he had finished his original proof of the consistency of arithmetic, then changed it around the turn of the year 1935–36 into the well-known proof based on transfinite induction. After this success, published in 1936, Gentzen fell in a serious depression that required treatment, but regained his powers in stages in 1938. In the summer of 1939, he finished his *Habilitationsschrift*, a considerable span of time from his doctorate, in part caused by his health problems, in part by the general circumstances of a country headed for war.

3. Dr. Gentzen's arduous years in Nazi Germany, 1933–45

Some days before the state exam of November 1933, Gentzen applied for membership in the SA in Göttingen, the paramilitary Nazi troops, and explained this as something "urgently recommended from several quarters" in a letter to Bernays. In the exam, he had to sign a document by which he swore that "no circumstances are known to me that would justify the assumption that I stem from non-Aryan parents or grandparents; especially, none of my parents or grandparents have belonged to the Jewish religion at any time." Gentzen was at this time living in Stralsund and there is no record of any participation on his part in any of the Göttingen SA-activities, to the extent that his membership was unknown there in 1935. That year, he applied for the assistantship with Hilbert, and got it towards the end of the year, despite reports from the Nazi teachers' union by which he had contacts with someone in Jerusalem, clearly Abraham Fraenkel, and thereby "had shown his loyalty to the Chosen People." At this time and later, Hermann Weyl attempted to bring Gentzen to the Institute for Advanced Study in Princeton, but failed to get the financing needed. He had perhaps a special interest in the matter, because Gentzen had found a consistency proof of Weyl's system of predicative analysis in the 1918 book *Das Kontinuum.*

In 1937, Heinrich Scholz wanted to take Gentzen to the international philosophy conference in Paris, and it seems that the trip was possible only under a Nazi party membership (of a "helpless opportunist," as Menzler-Trott writes, p. 77). It turned out to be the last time Gentzen and Bernays met, but the two continued their correspondence until the war. For Gentzen, it meant contacts with the expelled Jewish professor Bernays, an obvious risk to his own conditions in Göttingen. Bernays in turn must have had some understanding for Gentzen's decisions. Contacts other than occasional letters were greatly limited as the 1930s was coming to its disastrous end; for example, Gentzen was invited together with Wilhelm Ackermann to a conference in Zurich in 1938 but neither managed to participate. In September 1939, when Hitler and Stalin began their war against the rest of Europe, Gentzen was called to military service, and what seems a last encounter with another logician took place in December 1939 when Gödel gave a guest lecture in Göttingen and Gentzen got a leave from his service station in nearby Brunswick to attend it.

Gentzen gives the impression of having been in practice blind towards the Nazi regime and the conditions it set on life in the 1930s, academic and otherwise. Thus, Menzler reports the following (p. 260): "According to Bernays, Gentzen is supposed to have dismissed his reports on rioting against Jewish colleagues with the words: The government wouldn't allow such a thing!" In a letter to Hellmuth Kneser that involved a possible job, of 16 February 1935, he writes about "the recommendation of Professor Bernays" among others and ends the letter with a "Heil Hitler!" (see Menzler-Trott, p. 55). There were strict rules about such salutes in public places: A failed Hitler salute could lead to the loss of one's university job (see Allert 2008). By 1941, his readiness to use such salutes seems to have come to an end. Kneser had written to him and the reply, from May 1941, begins with: "I thank you for your letter, which to me seemed almost like news from another, past world." Then he continued with his dreams of a post-war Germany with several chairs devoted to logic and foundations (Menzler-Trott, p. 135). His own "respected Professor Bernays," the central figure in foundational research, robbed of his position and chased out of Göttingen, had no role in these plans.[1] The same totally blind attitude is seen in his notes of October 1942, on completeness in intuitionistic propositional logic in the series **AL**

[1] Incidentally, I had the occasion to ask in 2008 Dr. Ludwig Bernays why Paul Bernays didn't request the restitution of his position after the war: "Oh, uncle Paul would never have done such a thing!"

(page 137), where the work of Wajsberg on Heyting's propositional logic is referred to with great emphasis. It does not seem to have crossed Gentzen's mind that in a world governed by the Nazis, no such praise could ever be published. – Poor Mordchaj Wajsberg who had already lost his life in some unknown Nazi concentration camp by the time of Gentzen's writing.

It is perhaps all too easy to exercise moral judgment decades later. Alexander Soifer writes in his *The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators* (p. 480) about Van der Waerden that "one's response to living under tyranny can only be to leave, to die, or to compromise." One who left was Peter Thullen, a Catholic youth by his own words "totally absorbed by my mathematical research" at the time of Hitler's rise to power, as reported in Reinhard Siegmund-Schultze's pioneering study of the fate of mathematicians in the Third Reich, *Mathematicians Fleeing from Nazi Germany* (p. 395). Thullen's diary contains the following from 27 May 1933, one day after Gentzen had handed in his doctoral thesis (p. 403):

> One could read in the papers today that a bank employee had been punished because he had refused to do the Hitler salute while the Horst-Wessel song was being sung. I guess this will happen to me sooner or later, too. Obliged to watch what is happening, yet refusing to go along, seeing the way noble ideals and all that is good are increasingly being replaced by brutality, meanness, vacuity and the cowardice of petty bourgeois, all this drains one's energy and generates a feeling of impotent rage.

Two months later, after a visit to Münster, he had come to his conclusion (diary entry of 23 July, p. 408):

> I must admit that in Münster even I was tempted to give up resistance, at least outwardly. However, I just could not bring myself even to raise my arm, which felt as if it were weighed down by lead, for the German salute or–worse still–to end my letters with the obligatory "Heil Hitler," in the name of the man who ruined our Germany. The research grant I obtained in 1932 to go to Rome now seemed providential. It would allow me to gain some distance and to observe events from outside. It was clear to me that I would not return to Germany as long as Hitler remained in power.

Whatever we now think of Gentzen's compromises, let us keep in mind that he seems not to have had a choice with things such as Hitler salutes and that he, too, applied repeatedly for a scholarship, to go to Princeton.

Eckart Menzler had the fortunate idea of writing already in the 1980s to people who had known Gentzen with the results reported in his biography.

One missive was Egon Mohr who reported back about his friend Gentzen among others that "it pleased him greatly when he could acquire a copy of the second edition of Oskar Perron's *Irrazionalzahlen* in 1943 in a Prague bookstore." The edition came out in 1939 and ridiculed the racist ideas of Ludwig Bieberbach about a German-Aryan mathematics (Menzler-Trott, pp. 241–242). Mohr himself got a death sentence for having listened to foreign radio stations in early 1944, but was saved because of delays in the execution.

Gentzen's *Habilitationsschrift* was finished in 1939, when he entered military service in Brunswick, not far from Göttingen, in the end of September. He was given the task to observe the British airplanes that flew over and to listen to their radio traffic; not anything one with a mere school training would be able to do. Perhaps he had practised English with people such as Saunders MacLane with whom he studied in Göttingen. Moreover, Gentzen's mother Melanie Gentzen (born Bilharz) was born and lived the first years of her childhood in St. Louis. The task saved him from front service. Even so, he was by the end of 1941 in a bad shape and suffered a nervous breakdown. After some months of convalescence, he took up again scientific work.

In 1943, Gentzen was called to teach at the German University of Prague, and to obtain the position, he had to deliver a lecture. One report about the lecture mentions the "slowness while lecturing of Dr. Gentzen caused by nervous disease" (Menzler-Trott, p. 237). The period in Prague ended with an arrest in May 1945 and imprisonment with the other German professors that had remained. The separate German university was created towards the end of the 19th century, through the division of the university founded in the 14th century into a Czech and a German unit. The position of the latter became by the 1930s a question of nationalistic contend by the Czech majority. The German occupation of 1938 led to the closing of the Czech university in November 1939. Gentzen, when accepting the position in Prague, must have been aware of this situation.

Gentzen figures as one of the mathematicians in Maximilian Pinl's article suite *Kollegen in einer dunklen Zeit*, Colleagues in a dark era (Pinl 1969–76). Pinl was a Czech and a professor of the German University of Prague, arrested for half a year by the Gestapo in 1939–40 and forbidden any teaching activity in Germany, including occupied Czechoslovakia. This is how his series begins:

> One generation has passed since the troopers of apocalypse roared around a great part of the world. The time has at last come to look back at the caesura [complete break] of the years 1933–45 in the development of German mathematical science and to recollect the colleagues of these dark years the innocent victims

> of which they were.

Gentzen appears in the fourth part among the German mathematicians in Prague. Pinl describes first briefly the history of the German university, then begins with (Pinl 1974, pp. 173–74):

> Members of the mathematical institute of the German University of Prague and of the mathematical teaching unit of the German technical university had to lament the loss of colleagues Peter G. Bergmann, Lipman Bers, Ludwig Berwald, Philipp Frank, Walter Fröhlich, Paul Georg Funk, Gerhard Gentzen, Paul Kohn, Paul Kuhn, Heinrich Löwig, Karl Löwner, Ernst Mohr, Georg Pick, Max Pinl, Artur Winternitz.

Each person is then described in detail. Of Gentzen, we read a general account of his career and then the following (p. 173):

> Against the advice of the author of this report, he did not want to leave his position voluntarily in the Easter of 1945. It was taken from him when the pendulum of bloody terror of the past years hit back. He was put in a post-war forced-labour camp together with everyone else employed at the university. He did not measure up to the physical hardship and died on 4 August 1945. He hoped until the end to be able to prove the consistency of analysis, after the successful proof of the consistency of number theory, and dreamt of the founding of an institute of mathematical logic and foundational research.

The actual end was ghastlier than Pinl's report would suggest: The prisoners had to repair the cobbled streets that had been damaged, and passers-by threw stones on them. One stone cut tendons in Gentzen's hand; unable to work, he was deprived of food and died of starvation in a prison cell.

4. The scientific accomplishments

Gentzen's scientific results begin with his paper on what are known as Hertz systems, with research conducted in 1931 and published in 1932. The last work published during his life was the 1943 work on transfinite induction. These are the two fixed points of his achievement, and there do not seem to be any great results hidden in the stenographic manuscripts. There is, instead, one result Gentzen left unpublished that stands on a par with his most remarkable achievements, namely the proof of normalization for intuitionistic natural deduction in the longhand thesis manuscript of 1932–33. For the rest, Gentzen's central achievements can be read from his publications, and the manuscript sources serve mainly to enrich the understanding of the published papers. In the account of his achievements that follows, I have used in part my earlier reports, especially the Handbook-article of 2009, and the review *Gentzen's proof systems* of 2012.

1. Hertz systems

Gentzen's first research accomplishment came in the summer of 1931, at the age of 21: Bernays had given him the task of studying an open problem in a logical theory another extraordinarius at Göttingen, the mathematical physicist turned into a logician Paul Hertz had developed. Hertz, otherwise known for his contribution to the foundations of statistical mechanics, put up in the 1920s a general theory of "systems of sentences" (Hertz 1923, 1929, identically titled). A *sentence* is an expression of the form $a_1, \dots, a_n \rightarrow b$. It has several interpretations: the circumstances given by $a_1, \dots, a_n$ bring about b, objects with the properties $a_1, \dots, a_n$ have also the property b, etc. The logical interpretation is that from the propositions $a_1, \dots, a_n$, proposition b follows. The arrow thus represents the notion of logical consequence.

Hertz systems have two forms of rules of inference, the first with zero premisses:

Table 1. The rules of Hertz systems.

$$a_1, \dots, a_n \rightarrow a_i \quad \text{for } 1 \leqslant i \leqslant n$$

$$\frac{a_{1_1}, \dots, a_{1_n} \rightarrow b_1 \ \dots \ a_{m_1}, \dots, a_{m_k} \rightarrow b_m \quad b_1, \dots, b_m, c_1, \dots, c_l \rightarrow c}{a_{1_1}, \dots, a_{1_n}, \dots, a_{m_1}, \dots, a_{m_k}, c_1, \dots, c_l \rightarrow c}$$

Hertz called the former kind of rules "immediate inferences" and the latter "syllogisms." If a collection of sentences S is closed with respect to these rules, it forms a *sentence system.* A collection of sentences T is an *axiom system* for a sentence system S if each sentence in S is derivable from the ones in T.

One of the principal questions for Hertz was the existence of independent axiom systems, with independence defined in the obvious way, as underivability by the rules. Bernays had followed and clearly also sustained the work of Hertz, and he suggested that Gentzen work with the problem. The result was Gentzen's first paper, titled, in English translation, *On the existence of independent axiom systems for infinitary sentence systems.* He showed by a counterexample that there is an infinite sentence system for which there is no axiom system with independent axioms.

Gentzen noticed in the course of his work, conducted in the summer of 1931, that the second form of Hertz' rules can be replaced by one in which there is just one term b as a middle term of syllogism:

Table 2. Gentzen's cut rule in Hertz systems.

$$\frac{a_1, \dots, a_m, \ \rightarrow b \quad b, c_1, \dots, c_n \rightarrow c}{a_1, \dots, a_m, c_1, \dots, c_n \rightarrow c}$$

He called this form of rule *cut* (Schnitt), and the first form of Hertz' rules *thinning* (Verdünnung). The "sentences" of the calculus of Hertz became the "sequents" of Gentzen's later work.

The calculus of Hertz was one of the two main sources of Gentzen's sequent calculus. The other source was his development of the calculus of natural deduction. The rules of the latter, when adapted to the notation of sequents, led to sequent calculus proper. The background of sequent calculus in the work of Hertz and in Gentzen's first paper is discussed at length in Bernays (1965).

Gentzen's work on Hertz systems can be seen as an early contribution to logic programming. The sentences are the same as program clauses, and derivations consist of cuts with the terms in the clauses. Gentzen's work is described from this point of view in Peter Schroeder-Heister's paper (2002). The main results contain a normal form for derivations that amounts, in modern terms, "to the completeness of propositional SLD-resolution in logic programming" (ibid., p. 246). Gentzen uses in his proof of the existence of a normal form a semantical notion of "consequence" (Folgerung), instead of a direct method in which a derivation is transformed into normal form through the permutation of cuts, with the motivation that it "gives important additional results, namely the soundness and completeness of our ways of inference," and just mentions the possibility of a direct proof (p. 337). The normal form and completeness proofs of Gentzen are studied in great detail in Moriconi (2015), with a very detailed direct proof of the normal form theorem in what is the deepest account of Gentzen's first work so far.

2. A logical formulation of the decision problem of arithmetic

A two-page stenographic note, the earliest of all and dated September 1931, contains a translation of elementary arithmetic with the standard operations of sum and product into a relational language in which three-place relations $\sigma(x, y, z)$ and $\pi(x, y, v)$ represent the sum z and product v, respectively, of x and y. The effect is that the whole of arithmetic is expressed in the language of pure predicate logic. On this basis, Gentzen clearly thought that one should not expect predicate logic to be decidable. This little piece begins the present collection of notes. Its title amounts to *Reduction of number-theoretic problems to the decision problem of the lower functional calculus* and bears the motto:

> A result in the small is worth more than no overview at large.

The earliest systematic series is from the same period, with the signum **VOR**, the second item in this collection. It contains work on the replacement of functions by relations, in the same way in which the previous short note treated sums and products. Gentzen used this formulation in his doctoral thesis, in the proof of the consistency of arithmetic without the induction principle.

3. Interpretation of Peano arithmetic in Heyting arithmetic

When intuitionistic logic was in its infancy in the late 1920s, a lively debate arose on its proper formalization. It was conducted on the pages of the *Bulletin* of the Royal Belgian Academy. Valeri Glivenko contributed to this discussion in 1929 by a result by which classically provable negative formulas of propositional logic are already intuitionistically provable. The origins of the idea go back to Andrei Kolmogorov's paper of 1925; In it, Kolmogorov gave a double-negation interpretation of classical logic in intuitionistic logic. For each formula A, let A^* stand for the formula obtained by prefixing each subformula of A with a double negation. If A is provable classically, then A^* is provable intuitionistically.

Kolmogorov's paper was written in Russian and remained unknown to logicians outside Moscow, except for its indirect influence through Glivenko (1929). Gödel found in 1932 an interpretation of classical Peano arithmetic in the intuitionistic arithmetic of Heyting. Gentzen arrived at a slightly differently formulated interpretation by early 1933. His translation was as follows:

Table 3. The Gödel-Gentzen translation.

1. $a = b^* \rightsquigarrow a = b$
2. $(\neg A)^* \rightsquigarrow \neg A^*$
3. $(A\&B)^* \rightsquigarrow A^*\&B^*$
4. $(A \supset B)^* \rightsquigarrow A^* \supset B^*$
5. $(\forall x A)^* \rightsquigarrow \forall x A^*$
6. $(A \vee B)^* \rightsquigarrow \neg(\neg A^*\&\neg B^*)$
7. $(\exists x A)^* \rightsquigarrow \neg\forall x\neg A^*$

Gentzen tells to add double-negations to equations if these contain variables. Gödel's translation removes also implications $A \supset B$, by translating them into $\neg(A^*\&\neg B^*)$. The crucial points of the translation are disjunction and existence. The former is translated into the intuitionistically weaker $\neg(\neg A^* \ \& \ \neg B^*)$, and the same for existence that is translated into $\neg\forall x\neg A^*$. Even though there is a difference in the translations Gödel and Gentzen defined, the translation theorem that they established was the same: A formula A is a theorem of classical Peano arithmetic if and only if A^* is a theorem of intuitionistic Heyting arithmetic.

Gentzen's work was planned to be a chapter in his doctoral thesis, but by February 1933, he had submitted it as a separate paper to the *Mathematische Annalen*, the leading journal of mathematics of the day. Gödel heard of Gentzen's discovery through Heyting, with Gödel writing in a letter of 16 May that he had found the translation and that "it should have been known in Göttingen since at least June 1932." As a reason for the latter, Gödel wrote that at that time he had "reported on it in the Menger colloquium, to wit, in the presence of O. Veblen who shortly afterwards went to Göttingen." (See the Gödel-Heyting correspondence in volume V of Gödel's *Collected Works*.) We know from Menger's later recollections, based on minutes of the meetings, that Veblen got very enthusiastic about the lecture, and invited Gödel to visit the newly established Institute for Advanced Study in Princeton. However, there is no reason to think that Gentzen would have known of Gödel's discovery. Gentzen's reaction to the simultaneous discovery was to withdraw his paper from publication. The paper was preserved in the form of galley proofs and appeared in an English translation in 1969, in Manfred Szabo's edition of Gentzen's *Collected Papers*, then in the original German in 1974.

The importance of the Gödel-Gentzen translation was that it clearly showed the consistency of arithmetic to be within the reach of intuitionistic arguments. This was the first goal Gentzen had set to himself early in 1932.

As a special case of the Gödel-Gentzen translation, with all atomic formulas double-negated, a translation of classical predicate logic into intuitionistic predicate logic is obtained.

Clarification of the role of negation in intuitionistic arithmetic: Gentzen's paper of 1933 has a section 6.2 that has been added in the end: He had sent the original paper to Heyting and responded in a letter of 25 February 1933 to Heyting's assessment, meant for the journal *Mathematische Annalen.* In that connection, he also explained the contents of his section 6.2, by which it could not have been included in the original version. Gentzen's point is that one can "avoid entirely the negation in intuitionistic arithmetic," as well as the rule of *falsity elimination*, by defining negation as $\neg A \supset\subset . A \supset 1 \neq 1$. Now the axiom $A \supset (B \supset A)$ gives the instance $1 \neq 1 \supset \neg B$, and in particular, $1 \neq 1 \supset \neg\neg x = y$. As Gentzen had earlier shown equalities to be decidable, we have $\neg\neg x = y \supset x = y$, and in consequence $1 \neq 1 \supset x = y$. From the derivability of falsity elimination for atomic formulas follows now easily the derivability of $1 \neq 1 \supset A$ for any A, by induction on the length of the formula A.

A three-page note of January 1933, item 7 below with the title *Reduction of classical to intuitionistic logic*, goes much further than the Gödel-Gentzen translation: It contains a translation from derivations in classical natural deduction for predicate logic to derivations in the $\supset, \neg, \forall$-fragment of what is today called minimal logic, i.e., natural deduction with these logical operations, but without the rule of falsity elimination.

One can see from the note how Gentzen arrived at the double negation translation: In his attempts to prove normalization for natural deduction, he treated first the fragment without $\vee$ and $\exists$. Looking at the derivations in the fragment, he realized the following (here ( ) is the universal quantifier and V rule $\bot E$, falsity elimination):

> A proposition of the classical predicate calculus can be written equivalently with only $\supset$, ( ), and $\mathcal{F}$, with the elementary propositions occurring only negated, i.e., with $\supset\ \mathcal{F}$. And this proposition is classically and intuitionistically of the same value (i.e., correct or incorrect in both systems). By the subformula theorem it is therefore provable, if at all, already with the inferences for $\supset$, ( ), and V.

Next he found that steps of indirect inference to a formula A in the fragment can be reduced to the immediate subformulas of A, so, in the end to atomic formulas, and if these are negated at the outset, even falsity elimination or rule V can be dispensed with.

Gentzen's last preserved letter to Bernays, of 16 June 1939, is a reply to a missing letter by Bernays. It contains cryptic remarks about a double-negation translation in which a formula and each of its subformulas are prefixed with $\neg\neg$. It is noted that number-theoretic axioms and even the comprehension axiom scheme remain correct under the interpretation. A letter from Bernays to Gödel of 28 September 1939 tells (Gödel 2003, p. 126):

> I wanted to inform you that Mr. Gentzen has recently found out that the method of interpretation of the classical propositional calculus within the intuitionistic one can be extended from the number-theoretic formalism easily to the simple theory of types.

He adds that no intuitionistic interpretation is achieved, because impredicative definitions remain. Gödel in his reply of 29 December writes that "the result seems to be of no special interest because the constructivity of the concepts used is problematic" (ibid., p. 130).

4. Natural deduction

After Gentzen had finished the preparation of his Hertz-paper, he turned in early 1932 into the problem of the consistency of arithmetic and analysis, and wrote in December 1932 to his professor in Greifswald Hellmuth Kneser:

> I have set as my specific task to find a proof of the consistency of logical deduction in arithmetic... The task becomes a purely mathematical problem through the formalization of logical deduction. The proof of consistency has been so far carried out only for special cases, for example, the arithmetic of the integers