Springer Proceedings in Mathematics & Statistics

Tao Qian
Luigi G. Rodino *Editors*

Mathematical Analysis, Probability and Applications – Plenary Lectures

ISAAC 2015, Macau, China

Isaac

Springer

Springer Proceedings in Mathematics & Statistics

Volume 177

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at http://www.springer.com/series/10533

Tao Qian · Luigi G. Rodino
Editors

Mathematical Analysis, Probability and Applications – Plenary Lectures

ISAAC 2015, Macau, China

Springer

Editors
Tao Qian
Faculty of Science and Technology
University of Macao
Taipa
Macao

Luigi G. Rodino
Department of Mathematics
University of Turin
Turin
Italy

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-319-41943-5 ISBN 978-3-319-41945-9 (eBook)
DOI 10.1007/978-3-319-41945-9

Library of Congress Control Number: 2016945107

Mathematics Subject Classification (2010): 35QXX, 46EXX, 60GXX

This work is published under the auspices of the International Society of Analysis, its Applications and Computation (ISAAC).

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Preface

The present volume is a collection of papers devoted to current research topics in mathematical analysis, probability and applications, including the topics in mathematical physics and numerical analysis. It originates from plenary lectures given at the 10th International ISAAC Congress, held during 3–8 August 2015 at the University of Macau, China.

The papers, authored by eminent specialists, aim at presenting to a large audience some of the attractive and challenging themes of modern analysis:

- Partial differential equations of mathematical physics, including study of the equations of incompressible viscous flows, and of the Tricomi, Klein–Gordon and Einstein–de Sitter equations. Governing equations of fluid membranes are also considered in this volume.
- Fourier analysis and applications, in particular construction of Fourier and Mellin-type transform pairs for given planar domains, multiplication and composition operators for modulation spaces, harmonic analysis of first-order systems on Lipschitz domains.
- Reviews of results on probability, concerning in particular the bi-free extension of the free probability and a survey of Brownian motion based on the Langevin equation with white noise.
- Numerical analysis, in particular sparse approximation by greedy algorithms, and theory of reproducing kernels, with applications to analysis and numerical analysis.

The volume also includes a contribution on visual exploration of complex functions: the technique of domain colouring allows to represent complex functions as images, and it draws surprisingly mathematics near the modern arts.

Besides plenary talks, about 300 scientific communications were delivered during the Macau ISAAC Congress. Their texts are published in an independent volume. On the whole, the congress demonstrated, in particular, the increasing and major role of Asian countries in several research areas of mathematical analysis.

Taipa, Macao
Turin, Italy
March 2016

Tao Qian
Luigi G. Rodino

Contents

A Review of Brownian Motion Based Solely on the Langevin Equation with White Noise

L. Cohen

Abstract We give a historical and mathematical review of Brownian motion based solely on the Langevin equation. We derive the main statistical properties without bringing in external and subsidiary issues, such as temperature, Focker-Planck equations, the Maxwell–Boltzmann distribution, spectral analysis, the fluctuation-dissipation theorem, among many other topics that are typically introduced in discussions of the Langevin equation. The method we use is the formal solution approach, which was the standard method devised by the founders of the field. In addition, we give some relevant historical comments.

Keywords Brownian motion · Langevin equation · History · Einstein · White noise

1 Introduction

The two seemingly simple equations (as originally written)

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2} \tag{1}$$

and

$$m\frac{d^2x}{dt^2} = -6\pi\mu a\frac{dx}{dt} + X \tag{2}$$

revolutionized our understanding of the of the universe and ushered an incredible number of physical and mathematical ideas [11]. The first equation is due to Einstein [13], whose aim was to show that atoms exist; the second is due to Langevin [26], who brought forth a new perspective regarding both the physics and mathematics of Einstein's idea.

L. Cohen (✉)
City University of New York, 695 Park Ave., New York, NY 10065, USA
e-mail: leon.cohen@hunter.cuny.edu

T. Qian and L.G. Rodino (eds.), *Mathematical Analysis, Probability and Applications – Plenary Lectures*, Springer Proceedings in Mathematics & Statistics 177, DOI 10.1007/978-3-319-41945-9_1

It is often said that Brown [3] discovered Brownian motion, Einstein explained it, Langevin simplified it, and Perrin [31] proved it; this listing misses totally the motivations and wonderful history of the subject [11]. Brown did not discover Brownian motion, but did study it extensively. Einstein was not aware of Brownian motion; he *predicted* Brownian motion to obtain a macroscopic manifestation of atoms that could be measured. Equation (1) is the equation for the probability density for the Brownian particle at position x and time t. He derived the standard deviation of the visible Brownian particle that could be experimentally verified if indeed atoms exist. He solved explicitly for the standard deviation of position,

$$\lambda_x = \sqrt{\overline{x^2}} = \sqrt{2Dt} \tag{3}$$

and connected the parameters with the temperature of the medium, and the yet unnamed Avogadro number, a number that few believed in, and had never been measured or estimated at that time. Of course, Eq. (1) was known for 100 years before Einstein; it is the famous heat equation first derived by Fourier. However that is not relevant. What is important is that Einstein derived the probability density for position of the Brownian particle. Perrin had already been working on the issue of the existence of atoms, and his motivation was certainly heightened by Einstein's results. He experimentally verified Eq. (3), and hence verified the Einstein idea that the random "invisible" microscopic atoms can manifest a macroscopic effect which can be measured [31].

Equation (2) was the start of the field of random differential equation and is now called the Langevin equation. The way it stands, it is Newton's equation of motion where the left hand is mass times the acceleration, the first term on the right is the force of "friction" which is proportional to the velocity, and the second term, X, is an additional force. In Langevin's words: "X is indifferently positive and negative". X is what we now call the random force. Langevin's insight was to realize that to obtain the main result of Einstein, Eq. (3), one does not have to solve and derive the probability density, but one can obtain the second moment simply from Newton's equation and moreover that one can obtain it in a relatively simple manner. The Langevin equation has been applied to numerous fields and to a wide variety of physical situations. Random differential equations have become standard in many branches of science and has produced rich mathematics [7, 21–23, 27, 28, 35, 36, 39, 45].

1.1 The Aim of This Article

The author's involvement with Brownian motion [1, 2, 8] started with his attempt to understand and apply the theory of Chandrasekhar and von Neumann [4–6] regarding the random motion of stars, a subject that is fundamental in stellar dynamics, because it is the random motion that is important in the evolution of a collection of stars, such as globular clusters. In reading the review articles of that time I found considerable difficulty in that the articles mixed in a plethora of ideas which depended both on the interests of the author writing the article and relevant field of the article. Typically, historical and recent review articles and books mix together the Langevin equation with Fokker–Planck equations, temperature, mobility, master equations, the Maxwell–Boltzmann distribution, spectral analysis, Wiener processes, random walks, the fluctuation-dissipation theorem, white noise, Gaussian with noise, among many other topics. Of course, these are important to specific fields, but in my opinion, often detract from understanding the Langevin equation and its consequences as it stands. The aim of this article is to review and derive the relevant results of the Langevin equation without the encumbrance of other ideas. We give derivations of the main results based *solely* on the Langevin equation where the random force is taken to be white noise (not Gaussian white noise). Of course, most of the results we derive are known, but we hope that the presentation and derivations are of interest.

1.2 Notation

Expectation values of quantities that depend on time will be denoted in two equivalent ways

$$\langle x(t)\rangle = \langle x\rangle_t \tag{4}$$

Which notation is used is motivated by aiming at clarity of the equation and the historical usage.

We use the delta function, $\delta(t)$, routinely. The basic property is,

$$\int_a^b f(t)\delta(t-s)dt = f(s) \qquad a < s < b \tag{5}$$

If s is one of the end points then we will take half the value,

$$\int_a^b f(t)\delta(t-a)dt = \frac{1}{2}f(a) \qquad a < b \tag{6}$$

1.3 Deterministic and Random Initial Conditions

There is considerable variation in articles on Brownian motion regarding the initial conditions for the velocity $v(t)$, and position, $x(t)$. The two general approaches is to take them to be deterministic or random. When they are taken to be random, which is important in some fields, one very often averages over the initial conditions; this produces results which are seemingly different than if one takes them to be deterministic. We shall take them to be symbolically random, but we will not do any averaging over them. The deterministic case can be obtained from the random case in a manner that we now discuss. Our notation for the random initial conditions shall be, for example $\langle v(0)\rangle$ or $\langle v\rangle_0$. With this notation, to go to the deterministic case one just lets $\langle v(0)\rangle \to v_0$, where v_0 is the deterministic initial condition, etc. For $\left\langle v^2(0)\right\rangle \to v_0^2$ and for standard deviation of velocity $\sigma_v^2(0) \to 0$. Similarly for position.

2 The Langevin Equation with a White Noise Driving Force

In modern notation, the Langevin equation is generally written as

$$\frac{dv(t)}{dt} = -\beta v(t) + F(t) \tag{7}$$

It is a random differential equation for the velocity $v(t)$, but depending on the field, it could be any random variable that satisfies Eq. (7). The term $F(t)$ is called random force, and since it is random, the unknown $v(t)$ will also be random. The main issue is: given the statistical properties of $F(t)$, what are the statistical properties of $v(t)$. The standard statistical properties of F are taken to be

$$\langle F(t)\rangle = 0 \tag{8}$$

and

$$\left\langle F(t')F(t'')\right\rangle = 2D\delta(t' - t'') \tag{9}$$

The first indicates that the average at any one time of the random force is zero, and the second is that the force at two different times are uncorrelated except for equal times. Random processes that satisfy Eq. (9) are called white noise. Very often it is assumed that the statistical properties of the force are what is called Gaussian white noise. We will not assume so, and limit ourselves to results that follow only from Eqs. (7)–(9). We point out that a general view is that Eq. (9) implies a stationary process for the random force. That is not so. One can construct random process that satisfy Eq. (7) but are non-stationary.

How can one solve for $v(t)$? By solve we mean to obtain the statistical properties of $v(t)$. Historically, the first method was to solve the Langevin equation as if it were an ordinary differential equation, and then take appropriate expectation values. This method was implied by Langevin and developed by others, in particular, Ornstein, Uhlenbeck, Wang, and Chandrasekhar, among others. This is the procedure we will follow, and we discuss it further in Sect. 3.11. However, historically certain difficulties were pointed out and the first to do so was Doob [12].

Position. One also wants to study the statistical properties of position, $x(t)$, which is related to the velocity by

$$\frac{dx(t)}{dt} = v(t). \tag{10}$$

There are two approaches one can take. One is to first obtain the statistical properties of $v(t)$, and then consider Eq. (10) as a random differential equation for $x(t)$. Alternatively, one can combine Eqs. (7) and (10) to obtain a single differential equation for $x(t)$, namely

$$\frac{d^2x(t)}{dt^2} = -\beta\frac{dx(t)}{dt} + F(t) \tag{11}$$

and consider it a random differential equation with a driving force $F(t)$. Both approaches are interesting and are used.

It is important to appreciate that historically the calculation of $\left\langle x^2(t)\right\rangle$ was the focus, because it was the only measurable quantity of the Brownian particle.

3 Comments and Historical Notes

In this section we discuss some historical issues, motivations, and contributions of the many authors that developed the field of Brownian motion that was initiated by Einstein. This section may be skipped, as none of the results and discussions here are explicitly used in the subsequent derivations.

3.1 The Classic Review Articles

There are three classic historical review articles which are still the best review articles. In order of appearance, the first is by Uhlenbeck and Ornstein [38], titled “On the theory of Brownian motion” [38]; the second is the monumental article by Chandrasekhar [6], “Stochastic problems in Physics and Astronomy” [6]; and the third is that of Wang and Uhlenbeck [41], “On the theory of Brownian motion II” [41]. All these articles are much more than review articles, because they addressed new approaches and obtained new results.

Ornstein was among the first to solve the Langevin equation, and the paper by Uhlenbeck and him extended and simplified some of the results [38]. We discuss the Chandrasekhar paper below. The paper by Wang and Uhlenbeck, while having the same title as the paper by Uhlenbeck and Ornstein, is much more than a review of Brownian motion. It is a formulation of stochastic processes in general and a careful discussion of the Focker-Planck formulation.

The above three papers and three other important papers are collected in *Selected Papers on Noise and Stochastic Processes*, edited by Nelson Wax [42]. The three other papers are “Mathematical Analysis of Random Noise” by Rice [33]; “Random Walk and the Theory of Brownian Motion”, by Kac [24]; and “The Brownian Movement and Stochastic Equations” by Doob [12]. At one time, ownership of this book was mandatory for anyone interested in stochastic process. The book is still in print.

We also mention that Einstein’s five papers on Brownian motion were collected in a very short book, in 1926, edited by Furth [16]. The book is short because all of the Einstein articles are very short. The book was translated and published in English in 1956 and continues to be available. We also point out that Einstein published many papers on what we now call time series and stochastic processes. In fact, what is commonly called the Wiener-Khinchine theorem was first given by Einstein in 1914, in two papers entitled “A Method for the Statistical Use of Observations of Apparently Irregular, Quasiperiodic Process” and “Method for the Determination of Statistical Values of Observations Regarding Quantities Subject to Irregular Observations” [14].

3.2 Chandrasekhar

Chandrasekhar was one of the greatest scientists and astronomers of the last century, and received the Nobel Prize for the remarkable discovery of electron degeneracy in stars. He made monumental contributions to almost all fields of astronomy. He was one of the clearest scientific writers ever and while his famous article is often considered a review article, it is much more than that. It is perhaps one of the most remarkable articles written on the subject of stochastic process. The range is remarkable, ranging from the random walk to the recurrence theorem of Poincare.

Chandrasekhar wrote many articles on Brownian motion, but what is particularly important is that to the best of my knowledge, he was the first to *derive* the statistics of the random forces for the appropriate physical situation; in his case, the random force on a star. Subsequently, he and von Newman *derived* additional statistical properties for the random force, such as the two-time autocorrelation function.

While in his 1943 article, he derives the main results Brownian motion, most of the article concerns the issue of fluctuations, probability, and stochastic processes in general. Unlike previous works, he considers the three-dimensional case. He derives a number of new results regarding the transition from the Langevin equation to the problem of obtaining the probability densities. He obtains, in a very simple and elegant way the probability densities of position and velocity, and the equations of motion they satisfy. Moreover, he derives and discusses the joint position-velocity distribution, derives the partial differential equation that satisfies it, and gives a number of ways to solve it. Another part is a comprehensive discussion of the Langevin equation with an additional external force. Further, he makes connections between the Boltzmann equation, stochastic processes, Poincare cycles, and the fundamental deterministic equation of dynamics, the Liouville equation.

3.3 Smoluchowski

Smoluchowski developed the theory of Brownian motion and in fact he did considerably more than Einstein [37]. However he published his results in 1906, a year later than Einstein. He was the first to consider Brownian motion when there are external forces and in particular he considered Brownian motion under the influence of gravity. This was very important for the experimental procedures used by Perrin. He developed many of the mathematical issues. In the words of Chandrasekhar "The theory of density fluctuations as developed by Smoluchowski represents one of the most outstanding achievements..." [6].

3.4 White Noise, Gaussian White Noise, and Non-stationary White Noise

The power spectrum measures the intensity as a function of frequency. If the power spectrum is uniform, then it is called white noise. The power spectrum corresponding to Eq. (9) is indeed independent of frequency, and hence uniform [32]. White noise is called white because at one time it was thought that for the white light we perceive, the intensity as a function of frequency is more or less uniform. Of course that is not strictly the case, but the phrase has stuck.

If the statistics of the random force are Gaussian, then one says that we have Gaussian white noise. It is generally assumed that white noise is stationary. That is not necessarily the case. A process is stationary (more precisely, a second order stationary process) if the autocorrelation function depends on the difference of the two times,

$$\langle X(t_1)X(t_2)\rangle = R(t_2 - t_1) \qquad \text{(a function of } t_2 - t_1\text{)} \tag{12}$$

However white noise as defined by Eq. (12) is not necessarily a stationary process. Explicit methods for constructing non-stationary processes that are nonetheless white noise are given in [20].

3.5 The O-U Process

What has come to be called the Ornstein-Uhlenbeck process is the random process governed by the Langevin equation. The reason for the name is that Ornstein and later Ornstein and Uhlenbeck are the ones that derived the main results. However, we point out that depending on the field, the statistics of the standard driving force may be white noise, Gaussian white noise, or indeed, an arbitrary correlation function.

3.6 Brownian Motion: Stationary or Non-stationary?

The Langevin equation considered as a deterministic equations clearly produces a time dependent solution for $v(t)$ and $x(t)$, which of course depends on the time-dependence of the driving force. Considering it as a random differential equation also produces random quantities that evolve in time. In fact, Brownian motion is perhaps the most important and simplest example of a non-stationary random process. Nonetheless theorems which only apply to a stationary process, such as the Weiner Khinchine theorem, are often applied. The justification, often unstated, is that for large times, the autocorrelation function for the Brownian motion process does go to an autocorrelation that implies a stationary process. This will be further discussed in subsequent sections.

3.7 Spectral Properties

One of the major advances of noise theory is the work of Rice [33], who emphasized the spectral properties of a stochastic process. Wang and Uhlenbeck understood the importance of the spectral point of view, and calculated the power spectrum for velocity as given by the Langevin equation. Their often quoted result is that the power spectrum of velocity, $S(\omega)$, goes as [38]

$$S(\omega) \sim \frac{1}{\beta^2 + \omega^2} \tag{13}$$

Since $S(\omega)$ does not depend on time, the implication is that the process is stationary, but as we have discussed above Brownian motion is not stationary. Equation (13) is achieved by waiting an infinite amount of time, and this was achieved historically be starting the motion at minus infinity in time; hence for any finite time an infinite amount of time, will have already passed. Alternatively, the system is started at a finite time, and then one lets time go to infinity. This will be discussed further in Sect. 8, where we obtain the time-dependent power spectrum as defined by the Wigner distribution.

3.8 Brownian Motion in a Force Field

If in addition to the frictional force and the random force we have an external deterministic force which may be space and time dependent then the Langevin equation becomes [25, 37],

$$\frac{dv(t)}{dt} = -\beta v(t) + F(t) + K(x(t), t) \tag{14}$$

$$\frac{dx(t)}{dt} = v(t) \tag{15}$$

The equations are now coupled. We can no longer consider the velocity process by itself.

3.9 Focker-Planck Equations and Random Differential Equations

Focker-Planck equations are partial differential equations for evolution for the probability density [17, 34]. Einstein's equation, Eq. (1) is a Focker-Planck equation for position. The relation of the probability density to the Langevin equation is fundamental. A probability distribution is determined by its moments (except for some unusual circumstances), and hence if the Langevin equation can give us all the moments, say for the velocity, then indeed we could obtain the probability density of velocity. What moments can one obtain from the Langevin equation depends on the statistics of the random force. If one assumes white noise only, that is Eq. (7), then the probability distribution cannot be obtained, because only a few moments of velocity may be determined from the Langevin equation with white noise. However if one assumes that the random force is Gaussian white noise, then all the moments may be obtained, and hence so may the probability density.

3.10 Weiner and the Weiner Process

Weiner was not only a great mathematician, but also made major contributions to physics and engineering, and indeed is one the founders of noise theory and of modern electrical engineering. Weiner was a child prodigy, and as a young man aimed at making a contribution commensurate with his child prodigy status. He followed the major developments of his time both in mathematics and physics and was particularly interested in the then exciting development of Brownian motion and in so called pathological functions; functions that are continuous everywhere but differentiable nowhere. These functions were considered totally irrelevant to the real world. However, based on a hint by Perrin, he realized that the path of a Brownian particle may be a pathological function! So, he defined a mathematical idealization of Brownian motion based on measure theory [43]. To quote Wiener: "There were fundamental papers by Einstein and Smoluchowski that covered it, but whereas these papers concerned what was happening to any given particle at a specific time, or the long-time statistics of many particles, they did not concern themselves with the mathematical properties of the curve followed by a single particle. Here the literature was very scant, but it did include a telling comment by the French physicist Perrin in his book Les Atomes, where he said in effect that the very irregular curves followed by particles in the Brownian motion led one to think of the supposed continuous non-differentiable curves of the mathematicians. He called the motion continuous because the particles never jump over a gap, and non-differentiable because at no time do they seem to have a well-defined direction of movement."

What is currently called the "Wiener process", $W(t)$, is defined in various ways. Most commonly it is defined as a process governed by

$$\frac{dW(t)}{dt} = F(t) \tag{16}$$

where $F(t)$ is white noise, or sometimes Gaussian white noise. Hence we see that it is the Langevin equation without friction. One can also define it as process where the mean is zero and where the variance of $W(t) - W(t')$ is proportional to $t - t'$, and further that for $t_1 < t_2 \ldots < t_n$, then $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$, are mutually independent.

3.11 The Wiener–Khinchine Theorem

The so called the Wiener–Khinchine theorem, originally derived by Einstein in 1914, relates the autocorrelation to the power spectrum. The theorem applies only to a stationary processes. In particular, if $X(t)$ is a stationary random process where the autocorrelation function depends only on the difference in times

$$\langle X(t_1)X(t_2)\rangle = R(t_2 - t_1) \tag{17}$$

then the power spectrum is given by

$$S(\omega) = \frac{1}{2\pi}\int R(\tau)\, e^{-i\omega\tau}\, d\tau \tag{18}$$

Often this theorem is applied to the Langevin equation, but that is not strictly proper because in the case of Brownian motion we do not have a stationary process. However, for large times, it does become a stationary process. See Sect. 8.

3.12 The Formal Solution Approach

The fundamental method of solution, that is, finding the statistical properties of $v(t)$ from the Langevin equation is to pretend it is an ordinary deterministic equation, and after solving it as such, one takes expectation values of the appropriate quantities.

This method was implied in Langevin's paper, but was developed by Ornstein, and Wang and Ornstein, Chandrasekhar, and others. This will be the general approach that will be taken in subsequent sections.

3.13 Brownian Motion Applied to Extended Bodies and Random Partial differential Equations

Vibrations of structures due to random forces is of great practical and theoretical interest. For example the vibrations of structures due to wind, earthquakes, etc., are often modeled as the response to random forces. Historically, most of the work on Brownian motion was on the random motion of a single particle, or more generally, one degree of freedom. van Lear and Uhlenbeck extended the results of standard Brownian motion to the case of a string, where the random force acts on each point of the string [40]. One can view this as the beginning of random *partial* different equations. For example, for the string one has

$$\frac{\partial^2 s(x,t)}{\partial x^2} - \frac{\partial^2 s(x,t)}{\partial t^2} = F(x,t) \tag{19}$$

where the driving random force $F(x,t)$ is given statistically.

3.14 Generalized Langevin Equation

In the standard Langevin equation, the friction term $\beta v(t)$ is directly proportional to the velocity at time t only, that is, it has no memory. One way to generalize it, is to make the friction coefficient β a function of time $\beta(t)$, so that the past is taken into account,

$$\frac{dv(t)}{dt} = -\int_{-\infty}^{t} \beta(t-\tau)v(\tau)d\tau + F(t) \tag{20}$$

3.15 Derivations of the Langevin Equation

Of fundamental importance is the derivation of the Langevin equation from first principles. There have been many approaches, and perhaps the first was that of Ford et al. [15]. The general idea is to start with the most fundamental equations of motion for N coupled particles, and focus on one of them. The rest are considered as the "heat bath". One then averages over the particles in the heat bath and aims to obtain the equation of motion for the particle we are focused on [30]. This continues to be an active area of research for both the classical and quantum case.

4 Statistical Properties of Velocity

Considering

$$\frac{dv(t)}{dt} = -\beta v(t) + F(t) \tag{21}$$

as an ordinary differential equation, the formal solution is (See Appendix 1)

$$v(t) = e^{-\beta t} v(0) + e^{-\beta t} \int_0^t e^{\beta t'} F(t') dt' \tag{22}$$

where $v(0)$ is the initial velocity. We obtain the statistical properties of $v(t)$ by manipulating Eq. (22) and then taking averages.

4.1 Average Velocity

We take the mean of both sides of Eq. (22) and assume that the averaging operator can be brought inside the integral; then we have

$$\langle v \rangle_t = \langle v \rangle_0 \, e^{-\beta t} + e^{-\beta t} \int_0^t e^{\beta t'} \left\langle F(t') \right\rangle dt' \tag{23}$$

Using Eq. (8), we have that

$$\langle v \rangle_t = \langle v \rangle_0 \, e^{-\beta t} \tag{24}$$

The limits of $\langle v \rangle_t$ at zero and infinity are,

$$\langle v \rangle_{t \to \infty} \to 0 \tag{25}$$

$$\langle v \rangle_{t \to 0} \sim \langle v \rangle_0 \, (1 - \beta t) \tag{26}$$

4.2 Second Moment

To obtain the second moment, we square the deterministic solution, Eq. (22), to obtain

$$v^2(t) = e^{-2\beta t} v^2(0) + e^{-2\beta t} \int_0^t e^{\beta t'} F(t') dt' \int_0^t e^{\beta t''} F(t'') dt'' + 2e^{-2\beta t} v(0) \int_0^t e^{\beta s} F(t') dt' \tag{27}$$

and again take expectation values of both sides, which yields

$$\left\langle v^2 \right\rangle_t = e^{-2\beta t} v^2(0) + e^{-2\beta t} \int_0^t \int_0^t e^{\beta(t'+t'')} \left\langle F(t') F(t'') \right\rangle dt' dt'' + 2e^{-2\beta t} \int_0^t e^{\beta s} \left\langle v(0) F(t') \right\rangle dt' \tag{28}$$

Assuming that

$$\left\langle v(0)F(t')\right\rangle = 0 \tag{29}$$

then

$$\left\langle v^2\right\rangle_t = e^{-2\beta t}\left\langle v^2\right\rangle_0 + e^{-2\beta t}\int_0^t\int_0^t e^{\beta(t'+t'')}\left\langle F(t')F(t'')\right\rangle dt'dt'' \tag{30}$$

Evaluation of the second term is done in Appendix 2,

$$e^{-2\beta t}\int_0^t\int_0^t e^{\beta(t'+t'')}\left\langle F(t')F(t'')\right\rangle dt'dt'' = \frac{D}{\beta}(1-e^{-2\beta t}) \tag{31}$$

Substituting this result in Eq. (30) we have

$$\left\langle v^2\right\rangle_t = e^{-2\beta t}\left\langle v^2\right\rangle_0 + \frac{D}{\beta}(1-e^{-2\beta t}) \tag{32}$$

which can also be written as

$$\left\langle v^2\right\rangle_t = \frac{D}{\beta} + \left(\left\langle v^2\right\rangle_0 - \frac{D}{\beta}\right)e^{-2\beta t} \tag{33}$$

Limits. The limiting value for time going to infinity is

$$\left\langle v^2\right\rangle_{t\to\infty} \longrightarrow \frac{D}{\beta} \tag{34}$$

For small times we have

$$\left\langle v^2\right\rangle_{t\to 0} \to \frac{D}{\beta} + \left(\left\langle v^2\right\rangle_0 - \frac{D}{\beta}\right)(1-2\beta t) = v_0^2 - 2\beta t\left(\left\langle v^2\right\rangle_0 - \frac{D}{\beta}\right) \tag{35}$$

giving

$$\left\langle v^2\right\rangle_{t\to 0} \to \left\langle v^2\right\rangle_0(1-2\beta t) + 2Dt \tag{36}$$

Derivative of $\left\langle v^2\right\rangle_t$. We will see in Sect. 4.6 that the derivative of $\left\langle v^2\right\rangle_t$ plays an important role. We calculate it here. Differentiation of Eq. (33) gives

$$\frac{d}{dt}\left\langle v^2\right\rangle_t = -2\beta\left(\left\langle v^2\right\rangle_0 - \frac{D}{\beta}\right)e^{-2\beta t} \tag{37}$$

and we note that

$$\frac{d}{dt}\left\langle v^2\right\rangle_t \to 0 \qquad \text{as } t\to\infty \tag{38}$$

4.3 Standard Deviation

The standard deviation of velocity at time t is defined by

$$\sigma_v^2(t) = \left\langle (v(t) - \langle v \rangle_t)^2 \right\rangle = \left\langle v^2 \right\rangle_t - \langle v \rangle_t^2 \tag{39}$$

Using the values given by Eqs. (33) and (24), we have

$$\sigma_v^2(t) = \frac{D}{\beta} + \left(v_0^2 - \frac{D}{\beta} \right) e^{-2\beta t} - \langle v \rangle_0^2 \, e^{-2\beta t} \tag{40}$$

or

$$\sigma_v^2(t) = \frac{D}{\beta} \left(1 - e^{-2\beta t} \right) \tag{41}$$

Limits. The limit for infinite time is

$$\sigma_v^2(t) \to \frac{D}{\beta} \qquad\qquad t \to \infty \tag{42}$$

It is important to appreciate that indeed $\sigma_v^2(t)$ goes to a constant for infinite time. This is important because the standard deviation of velocity is proportional to temperature. The fact that $\sigma_v^2(t)$ goes to constant value is consistent with what is called the equipartition theorem. In this case, it implies that the Brownian particles achieve the same value as that of the atoms that are causing the movement of the Brownian particle.

For the small time limit we have

$$\sigma_v^2(t) \sim 2Dt \qquad\qquad t \to 0 \tag{43}$$

Deviation from initial velocity. It is also of interest to define the deviation from the initial velocity. We define it by way of

$$\lambda_{v_0}^2 = \left\langle (v(t) - \langle v \rangle_0)^2 \right\rangle = \left\langle v^2 \right\rangle_t - 2 \langle v \rangle_t \langle v \rangle_0 + \langle v \rangle_0^2 \tag{44}$$

Using Eqs. (33) and (24) we obtain

$$\lambda_{v_0}^2 = e^{-2\beta t} \left\langle v^2 \right\rangle_0 + \frac{D}{\beta}(1 - e^{-2\beta t}) - 2 \langle v \rangle_0 e^{-\beta t} + \langle v \rangle_0^2 \tag{45}$$

$$= e^{-2\beta t} \left\langle v^2 \right\rangle_0 + \langle v \rangle_0^2 \left(1 - 2e^{-\beta t} \right) + \frac{D}{\beta}(1 - e^{-2\beta t}) \tag{46}$$

and therefore

$$\lambda_{v_0}^2 = e^{-2\beta t} \left\langle v^2 \right\rangle_0 - \langle v \rangle_0^2 \left(2e^{-\beta t} - 1 \right) + \frac{D}{\beta}(1 - e^{-2\beta t}) \tag{47}$$

4.4 Correlation and Covariance of Velocity

If two random variables are independent, then the joint probability distribution is the product of the individual distributions for each of the variables. A cruder but more accessible measure is the expected value of the product of the two random variables, $\langle XY\rangle$, where X and Y are the random variables. If the probability distribution is a product of the two distributions, one says that the two variables are independent, in which case, $\langle XY\rangle = \langle X\rangle \langle Y\rangle$. Therefore, a measure of dependence is the excess of $\langle XY\rangle$ over $\langle X\rangle \langle Y\rangle$. This is called the covariance

$$\mathrm{Cov}(X, Y) = \langle XY\rangle - \langle X\rangle \langle Y\rangle \tag{48}$$

We point out that zero covariance does not necessarily imply that the two variables are independent, but the covariance does give a measure of the dependence of two variables.

For our case we take the two variables as the velocities at two different times, namely at time t and time s. We write

$$\mathrm{Cov}(v(t), v(s)) = \langle v(t)v(s)\rangle - \langle v\rangle_t \langle v\rangle_s \tag{49}$$

When we have a stochastic process such as the one we are considered here, quantities such as $\langle v(t)v(s)\rangle$ are called two-time autocorrelation functions.

Writing Eq. (22) for times t and s, and multiplying the two expressions we have

$$v(t)v(s) = \left(e^{-\beta t}v(0) + e^{-\beta t}\int_0^t e^{\beta t'}F(t')dt'\right)\left(e^{-\beta s}v(0) + e^{-\beta s}\int_0^s e^{\beta t''}F(t'')dt''\right) \tag{50}$$

Taking expectation values gives

$$\langle v(t)v(s)\rangle = \left\langle v^2\right\rangle_0 e^{-\beta(t+s)} + e^{-\beta(t+s)}\int_0^t\int_0^s e^{\beta t'}e^{\beta t''}\left\langle F(t')F(t'')\right\rangle dt'dt'' \tag{51}$$

In Appendix 3 we evaluate the second term of Eq. (51) to give

$$\int_0^t\int_0^s e^{\beta t'}e^{\beta t''}\left\langle F(t')F(t'')\right\rangle dt'dt'' = \frac{D}{\beta}\begin{cases}\left(e^{2\beta s}-1\right) & t > s\\ \left(e^{2\beta t}-1\right) & t < s\end{cases} \tag{52}$$

Therefore

$$\langle v(t)v(s)\rangle = e^{-\beta(t+s)}\left\langle v^2\right\rangle_0 + \frac{D}{\beta}e^{-\beta(t+s)}\begin{cases}\left(e^{2\beta s}-1\right) & t > s\\ \left(e^{2\beta t}-1\right) & t < s\end{cases} \tag{53}$$

$$= e^{-\beta(t+s)}\left\langle v^2\right\rangle_0 + \frac{D}{\beta}\begin{cases}\left(e^{-\beta(t-s)} - e^{-\beta(t+s)}\right) & t > s\\ \left(e^{-\beta(s-t)} - e^{-\beta(t+s)}\right) & t < s\end{cases} \tag{54}$$

This can be written as

$$\langle v(t)v(s)\rangle = e^{-\beta(t+s)}\left\langle v^2\right\rangle_0 + \frac{D}{\beta}\left(e^{-\beta(|t-s|} - e^{-\beta(t+s)}\right) \tag{55}$$

For t positive, we have

$$\langle v(t)v(0)\rangle = e^{-\beta t}\left\langle v^2\right\rangle_0 \tag{56}$$

It is sometimes useful to consider $\langle v(t)v(t+\tau)\rangle$ which can be obtained form Eq. (55) by setting

$$s = t + \tau \tag{57}$$

in which case we have

$$\langle v(t)v(t+\tau)\rangle = e^{-\beta(2t+\tau)}\left\langle v^2\right\rangle_0 + \frac{D}{\beta}\left(e^{-\beta|\tau|} - e^{-\beta(2t+\tau)}\right) \tag{58}$$

which may also be written as

$$\langle v(t)v(t+\tau)\rangle = e^{-\beta(2t+\tau)}\left\langle v^2\right\rangle_0 + \frac{D}{\beta}\begin{cases}\left(e^{\beta\tau} - e^{-\beta(2t+\tau)}\right) & \tau < 0\\ \left(e^{-\beta\tau} - e^{-\beta(2t+\tau)}\right) & \tau > 0\end{cases} \tag{59}$$

Limits. Consider the large time limit. Taking $t \to \infty$ in Eq. (59) we obtain

$$\langle v(t)v(t+\tau)\rangle_{t\to\infty} = \frac{D}{\beta}\begin{cases} e^{\beta\tau} & \tau < 0\\ e^{-\beta\tau} & \tau > 0\end{cases} \tag{60}$$

which shows that for large times, the autocorrelation function becomes independent of time.

The covariance. Using Eqs. (54) and (24) we have that

$$\mathrm{Cov}(v(t), v(s)) = \langle v(t)v(s)\rangle - \langle v\rangle_t \langle v\rangle_s \tag{61}$$

$$= e^{-\beta(t+s)}\left\langle v^2\right\rangle_0 + \frac{D}{\beta}\left(e^{-\beta(|t-s|} - e^{-\beta(t+s)}\right) - \langle v\rangle_0^2\, e^{-\beta(t+s)} \tag{62}$$

giving

$$\mathrm{Cov}(v(t), v(s)) = e^{-\beta(t+s)}\sigma_v^2(0) + \frac{D}{\beta}\left(e^{-\beta(|t-s|} - e^{-\beta(t+s)}\right) \tag{63}$$

where we have defined the standard deviation of velocity at time zero as

$$\sigma_v^2(0) = \left(\left\langle v^2\right\rangle_0 - \langle v\rangle_0^2\right) \tag{64}$$

We also have

$$\mathrm{Cov}(v(t), v(t+\tau)) = e^{-\beta(2t+\tau)}\sigma_v^2(0) + \frac{D}{\beta}\begin{cases} \left(e^{\beta\tau} - e^{-\beta(2t+\tau)}\right) & \tau < 0 \\ \left(e^{-\beta\tau} - e^{-\beta(2t+\tau)}\right) & \tau > 0 \end{cases} \tag{65}$$

For large times

$$\mathrm{Cov}(v(t), v(t+\tau))_{t\to\infty} = \frac{D}{\beta}\begin{cases} e^{\beta\tau} & \tau < 0 \\ e^{-\beta\tau} & \tau > 0 \end{cases} \tag{66}$$

4.5 Correlation of Velocity and Force

It is particularly interesting to evaluate the two-time cross correlation of force and velocity. Langevin implied that it is zero, but Manoliu and Kittel showed it is not [29]. Multiplying the velocity equation, Eq. (22), at time t, by the force at time t', we have

$$v(t)F(t') = e^{-\beta t}v(0)F(t') + e^{-\beta t}\int_0^t e^{\beta' t''}F(t'')F(t')dt'' \tag{67}$$

Taking expectation values we have

$$\langle v(t)F(t')\rangle = e^{-\beta t}\langle v(0)F(t')\rangle + e^{-\beta t}\int_0^t e^{\beta' t''}\langle F(t'')F(t')\rangle dt'' \tag{68}$$

Assuming that $\langle v(0)F(t')\rangle = 0$, we have

$$\langle v(t)F(t')\rangle = e^{-\beta t}\int_0^t e^{\beta' t''}\langle F(t'')F(t')\rangle dt'' \tag{69}$$

$$= 2De^{-\beta t}\int_0^t e^{\beta' t''}\delta(t'-t'')dt'' \tag{70}$$

Clearly

$$\int_0^t e^{\beta' t''}\delta(t'-t'')dt'' = e^{\beta' t'} \qquad 0 < t' < t \tag{71}$$

and therefore

$$\langle v(t)F(t')\rangle = \begin{cases} 2De^{-\beta(t-t')} & 0 < t' < t \\ 0 & \text{otherwise} \end{cases} \tag{72}$$

Consider now $\langle v(t)F(t+\tau)\rangle$ with τ positive. Accordingly

$$\langle v(t)F(t+\tau)\rangle = 0 \qquad \tau > 0 \tag{73}$$

This a reflection of causality in that a future force has no effect on the present velocity. Now consider

$$\langle v(t)F(t-\tau)\rangle = \begin{cases} 2De^{-\beta\tau} & 0<\tau<t \\ 0 & \text{otherwise} \end{cases} \tag{74}$$

which shows that as long as the force acts at a time earlier than t, the velocity is affected by it.

The equal time case: For the equal time case, it is better to redo the calculation. In Eq. (67), take $t = t'$,

$$v(t)F(t) = e^{-\beta t}v(0)F(t) + e^{-\beta t}\int_0^t e^{\beta t'}F(t')F(t)dt' \tag{75}$$

and therefore

$$\langle v(t)F(t)\rangle = e^{-\beta t}\int_0^t e^{\beta t'}\left\langle F(t')F(t)\right\rangle dt' \tag{76}$$

$$= 2De^{-\beta t}\int_0^t e^{\beta t'}\delta(t-t')dt' \tag{77}$$

We take

$$\int_0^t e^{\beta t'}\delta(t-t')dt' = \frac{1}{2}e^{\beta t} \tag{78}$$

to yield,

$$\langle v(t)F(t)\rangle = D \tag{79}$$

4.6 Energy Balance

For deterministic systems that obey Newton's law, one attempts to obtain a conservation law or an equation for energy flow by the following procedure, as applied to the Langevin equation. Multiply the Langevin equation by $v(t)$ to obtain

$$v(t)\frac{dv(t)}{dt} = -\beta v^2(t) + v(t)F(t) \tag{80}$$

and rewrite it as

$$\frac{1}{2}\frac{dv^2(t)}{dt} = -\beta v^2(t) + v(t)F(t) \tag{81}$$

The kinetic energy (per unit mass) is defined by

$$T = \frac{1}{2}v^2(t) \tag{82}$$

The change in T is therefore

$$\frac{d}{dt}T = -2\beta T + v(t)F(t) \tag{83}$$

Taking expectation values of both sides we have

$$\frac{d}{dt}\langle T\rangle = -2\beta\langle T\rangle + \langle v(t)F(t)\rangle \tag{84}$$

This shows that the change in $\langle T\rangle$ is governed by two terms. The term $\langle v(t)F(t)\rangle$ increases it and the term $-2\beta\langle T\rangle$ decreases it. One says that $\langle v(t)F(t)\rangle$ is the average work done and $-2\beta\langle T\rangle$ is the dissipation.

If we use Eq. (79)

$$\langle v(t)F(t)\rangle = D \tag{85}$$

then we have that

$$\frac{d}{dt}\langle T\rangle = -2\beta\langle T\rangle + D \tag{86}$$

Now if we assume that

$$\frac{d}{dt}\langle T\rangle = 0 \qquad t = \infty \tag{87}$$

then

$$\langle T\rangle = \frac{D}{2\beta} \qquad t = \infty \tag{88}$$

Equation (87) was *assumed* by Langevin. It is reasonable on statistical mechanics grounds. However, Eq. (87) follows directly from the Langevin equation as was seen in Sect. 4.2 and was first shown by Uhlenbeck and Ornstein.

5 Statistical Properties of Position

We now obtain the statistical properties of position, $x(t)$, that is governed by

$$\frac{dx(t)}{dt} = v(t) \tag{89}$$

There are three approaches one may take:

Approach 1. Solving symbolically Eq. (89) we have,

$$x(t) = x(0) + \int_0^t v(t')\,dt' \tag{90}$$

Since we have derived the statistical properties of $v(t)$, we can obtain the statistical properties of $x(t)$. Note that once we have the statistical properties of $v(t)$, we do not need the statistical properties of $F(t)$.

Approach 2. Combing Eqs. (22) and (89) we have the differential equation

$$\frac{d^2x(t)}{dt^2} = -\beta\frac{dx(t)}{dt} + F(t) \tag{91}$$

This a second order differential equation whose formal solution is given by (see Appendix 1)

$$x(t) = x(0) + \frac{v(0)}{\beta}(1 - e^{-\beta t}) + \frac{1}{\beta}\int_0^t \left(1 - e^{\beta(t'-t)}\right) F(t')dt' \tag{92}$$

We can now use the same methods we used for the Langevin equation for velocity but of course we need the statistical properties of $F(t')$.

Approach 3. It is also of interest to write $x(t)$ in terms of $v(t)$ directly. In Appendix A we show that

$$x(t) = x(0) - \frac{1}{\beta}(v(t) - v(0)) + \frac{1}{\beta}\int_0^t F(t') \tag{93}$$

This explicitly expresses $x(t)$ in terms of $v(t)$ and hence the statistical properties of the two can be directly related.

5.1 Average Position

Taking the expectation value of Eq. (90)

$$\langle x\rangle_t = \langle x\rangle_0 + \int_0^t \left\langle v(t')\right\rangle dt' \tag{94}$$

and using Eq. (24)

$$\langle v\rangle_t = \langle v\rangle_0 \, e^{-\beta t} \tag{95}$$

we have

$$\langle x\rangle_t = \langle x\rangle_0 + \langle v\rangle_0 \int_0^t e^{-\beta t'} dt' \tag{96}$$

giving the expectation value of position at time t,

$$\langle x\rangle_t = \langle x\rangle_0 + \frac{\langle v\rangle_0}{\beta}(1 - e^{-\beta t}) \tag{97}$$

Limits. The limits at infinity and zero are

$$\langle x\rangle_{t\to\infty} \to \langle x\rangle_0 + \frac{\langle v\rangle_0}{\beta} \tag{98}$$

$$\langle x\rangle_{t\to 0} \to \langle x\rangle_0 + \langle v\rangle_0\, t \tag{99}$$

5.2 Standard Deviation of Position

We shall first evaluate the standard deviation of position at time t, defined by

$$\sigma_x^2(t) = \left\langle (x(t) - \langle x\rangle_t)^2 \right\rangle \tag{100}$$

In Appendix 4 we show that

$$\sigma_x^2(t) = \sigma_x^2(0) + \int_0^t \int_0^t \frac{D}{\beta} \left(e^{-\beta|t'-t''|} - e^{-\beta(t'+t'')} \right) dt'dt'' \tag{101}$$

and we further show that

$$\int_0^t \int_0^t \frac{D}{\beta} \left(e^{-\beta(|t'-t''|} - e^{-\beta(t'+t'')} \right) dt'dt'' = \frac{2D}{\beta^2} t + \frac{D}{\beta^3} \left(4e^{-\beta t} - 3 - e^{-2\beta t} \right) \tag{102}$$

Therefore

$$\sigma_x^2(t) = \sigma_x^2(0) + \frac{2D}{\beta^2} t + \frac{D}{\beta^3} \left(4e^{-\beta t} - 3 - e^{-2\beta t} \right) \tag{103}$$

5.3 Second Moment of Position

Writing Eq. (103) explicitly

$$\left\langle x^2 \right\rangle_t - \langle x\rangle_t^2 = \left\langle x^2 \right\rangle_0 - \langle x\rangle_0^2 + \frac{2D}{\beta^2} t + \frac{D}{\beta^3} \left(4e^{-\beta t} - 3 - e^{-2\beta t} \right) \tag{104}$$

and squaring Eq. (97)

$$\langle x\rangle_t^2 = \langle x\rangle_0^2 + \frac{\langle v\rangle_0^2}{\beta^2} (1 - e^{-\beta t})^2 + 2\frac{\langle x\rangle_0 \langle v\rangle_0}{\beta} (1 - e^{-\beta t}) \tag{105}$$

we have that

$$\left\langle x^2\right\rangle_t = \left\langle x^2\right\rangle_0 - \frac{\langle v\rangle_0^2}{\beta^2}(1-e^{-\beta t})^2 - 2\frac{\langle x\rangle_0\,\langle v\rangle_0}{\beta}(1-e^{-\beta t}) \tag{106}$$

$$+\frac{2D}{\beta^2}t + \frac{D}{\beta^3}\left(4e^{-\beta t} - 3 - e^{-2\beta t}\right) \tag{107}$$

5.4 Limits

For the limits of Eq. (103) we have

$$\sigma_x^2(t) \sim \frac{2D}{\beta^2}t \qquad\qquad t\to\infty \tag{108}$$

which is Einstein' result. Also

$$\sigma_x(t) \sim \sqrt{\frac{2D}{\beta^2}t} \qquad\qquad t\to\infty \tag{109}$$

which is not differentiable at *zero,* but of course it does not hold at zero. Historically, Eq. (109) was derived by different methods and the fact that the derivative does not exist at zero was taken as a criticism. But of course Einstein was aware that his result only held for large times.

For small times one obtains that

$$\sigma_x^2(t\to 0) \sim \frac{D}{3}t^3 \tag{110}$$

which is differentiable at zero.

5.5 Deviation from Initial Position

It is also of interest to calculate the deviation from the initial position

$$\lambda_{x_0}^2(t) = \left\langle (x(t)-x(0))^2\right\rangle \tag{111}$$

Starting with Eq. (92)

$$x(t) = x(0) + \frac{v(0)}{\beta}(1-e^{-\beta t}) + \frac{1}{\beta}\int_0^t \left(1-e^{\beta(t'-t)}\right)F(t')dt' \tag{112}$$