Springer Series in Solid and Structural Mechanics 7

Michel Frémond

Virtual Work and Shape Change in Solid Mechanics

Springer

Springer Series in Solid and Structural Mechanics

Volume 7

Series editors

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Springer

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ISSN 2195-3511 ISSN 2195-352X (electronic)
Springer Series in Solid and Structural Mechanics
ISBN 978-3-319-40681-7 ISBN 978-3-319-40682-4 (eBook)
DOI 10.1007/978-3-319-40682-4

Library of Congress Control Number: 2016945146

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Preface

Mechanics is the science of motion: It predicts the motion we see with our eyes. Motion is involved in most scientific activities and in most engineering work. The importance of this topic has resulted in an axiomatization of mechanics and to have mathematics together with experiments to be tools widely used.

There is a large variety of motions we see with our eyes: the motion of the floor of a room, the motion of the grains of a sand pile, the motion of a debris flow, the motion of galaxies, etc. The elements which are used by engineers and scientists to describe them may seem different.

But these theories have in common the concepts to describe the motion. The equations of motion are either simple or very sophisticated. The sophistication may result from the need to define and quantify precisely together the shape of the system which is considered and the evolution of this shape, i.e., the shape change or the deformation of the system, and the velocity of deformation.

A tool to introduce the mechanical effects of the evolution of the shape of a system is the principle of virtual work. It has a status which is perhaps too theoretical even if it is widely used in numerics with the so-called variational formulations [1]. We show it is actually related to observation and experiments. Its utilization is flexible and may be adapted to produce predictive theories of numerous phenomena.

Part I is devoted to relate the virtual work principle to what we see with our eyes.

Part II shows how flexible it is. A large number of examples are given.

The principle is applied in Part III to predict the motion of solids with large deformations. The principle requires the description of the deformations: the way the shape of solids changes. We know that there are a large variety of possibilities. The choice has to be as simple as possible, but it has to cope with the every day life actions. It results from observations that third-order derivatives with respect to space of the displacement are needed to have a coherent description of large deformations.

Once the principle has defined the internal forces and given the equations of motion, we have to face the derivation of the constitutive laws which describe how a material behaves. Equations of motion are general. Constitutive laws are peculiar to each material. Theory and observation intervene in the derivation of the constitutive laws. For what concerns theory, the Clausius–Duhem inequality is the useful tool. For what concerns observation, experiments guide the choice of the free energy and the choice of the pseudo-potential of dissipation.

Following the examples of Parts I and II, we identify an internal constraint on the elongation matrix velocity. Following the way Lagrange takes into account an internal constraint, une liaison parfaite in French and un vincolo perfetto in Italian, we introduce a reaction [2]. As usual in Lagrangian mechanics, this reaction is given by both the constitutive laws and the equations of motion. It is impossible to derive entirely the value of the reaction with a constitutive law. It depends on the whole solid and on the external actions.

These problems have been investigated at the Università degli Studi di Roma "Tor Vergata" in the Dipartimento di Ingegneria Civile e Ingegneria Informatica and in the framework of the Laboratorio Lagrange, bringing together Italian and French scientists. Some of the topics have been taught in lectures given at the Scuola di Ingegneria of the Università.

The author discussed his points of view with Profs. Franco Maceri, Olivier Maisonneuve, and Christian Licht. He thanks them for their watchful views. He has appreciated the professional and kind interest of Profs. Elena Bonetti, Pierluigi Colli, Mauro Fabrizio, Bernard Nayroles, and Claude Stolz. Doctor Daniele Bianchi was helpful to solve editing problems, and the students of the Università have provided numerical illustrations while attending the lectures. All of them are warmly thanked.

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Contents

Part I
The Principle of Virtual Work

Chapter 1
Introduction

Abstract Introduction of Part I which is devoted to the derivation of the principle of virtual work. The derivation is based on observation and simple experiments. The principle is extended by induction.

The derivation of the equations of motion of a system is sometimes straightforward and simple. But enhanced description of motion may intricate their derivation.

A useful tool to overcome such difficulties may be to derive the equations of motion with the principle of virtual work.

We give the basic ideas while describing the motion on a plane of a system made of two disks. We show that the velocities and the velocities of deformation of the systems are quantities which are seen, experimented and measured: thus we choose them as the basic elements of a mechanical predictive theory. The internal forces are by-products which are abstract quantities which may be experimented only through motion.

The theory based on the two disk system motion is given in Chaps. 2–10. At each step, the theoretical aspects are supported by observation. As any principle, the principle of virtual work, is based on some experiments and extended by induction. Its ability to predict the motion of a system, is justified by the quality of the results when applied.

M. Frémond, *Virtual Work and Shape Change in Solid Mechanics*,
Springer Series in Solid and Structural Mechanics 7,
DOI 10.1007/978-3-319-40682-4_1

Chapter 2
The System

Abstract Two disks on a plane, which can be connected by a spring, is the system of which we investigate the motion. One may think of two coins sliding on a smooth plane.

Let us consider two rigid disks moving on a plane. They are the system the motion of which we want to predict. The disks may interact: we assume they are connected by a spring which applies actions within the system. The material points we consider are those of the system, i.e., the points of the two disks. For the sake of simplicity, we assume the spring has no mechanical property besides applying interactions at a distance (its mass is negligible compared to the mass of each disk). We assume, again for the sake of simplicity, that the system does not interact with the plane. We have a $2D$ problem. One may think of two coins connected by a light spring sliding on a smooth plane.

The whole world is divided into two parts: the *interior* of the system made of the points of the system and the *exterior* of the system made of the points of the remaining part of the world. The mechanical actions may be either internal or external depending on their origin.

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DOI 10.1007/978-3-319-40682-4_2

Chapter 3
The Principle of Virtual Work

Abstract The principle of virtual work is based on experiments: when pulling on the disks, the external work we apply to the system is used to modify the velocities of the system and to modify the shape of the system: the relative position and the relative orientation of the disks.

When pulling on the coins with the hands, the external actions we apply to the system or the work we spend have two effects. They

- modify the velocity of the elements of the system: the velocities of the disks vary;
- modify the shape of the system: the distance and orientation of the disks vary. The concept of shape is not precise but it is clear that if two pictures of the system are taken by a camera fixed on one of the disks, at different times, the two pictures cannot be superposed because the distance and orientation of the two disks have changed.

To investigate this physical point, we decide to split the provided work between the two effects: modification of the velocity and modification of the shape. The work we give to the system, $\tilde{W}_{ext}$, is the sum of the work $\tilde{W}_{velo}$ which is used to change its velocities and of the work $\tilde{W}_{def}$ which is used to change its shape or to deform it

$$\tilde{W}_{ext} = \tilde{W}_{velo} + \tilde{W}_{def}. \tag{3.1}$$

We have to make more precise the three quantities: $\tilde{W}_{ext}$ the work provided by the external actions, $\tilde{W}_{velo}$ the work to modify the velocities of the system and $\tilde{W}_{def}$ the work to change the shape of the system.

The ideas we develop are that:

- the works, which are the integrals of powers with respect to time, are scalars which are not so difficult to measure or to relate to experiments. A way to measure them is to consider that in many cases one has to pay for or to get tired to produce the work needed to modify the velocities and the work to change the shape of a system;

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DOI 10.1007/978-3-319-40682-4_3

- velocities are rather easy to experiment: they are the velocities with respect to the plane. It is clear that we can add two velocities and that we can define a velocity which is twice an other one. Thus *velocities are elements of a linear space;*
- a work or a power is a scalar which depends on velocities. Experiments show that to have the disks (i.e., the system) to evolve twice faster, the applied work has to be roughly doubled. Thus we choose the works to be linear functions of the velocities. A linear function is defined with an element of some dual space. It is natural to call forces the elements of this dual space.

A force appears as an abstract quantity whereas a velocity is a quantity which can be experimented and seen. A force has to be investigated by its effects on the motion.

We think a force is not so easy to figure out. As an example, think of a solid lying on a table. Everybody agrees that the reaction of the table is a force which is normal to the table. If one wants to make this force to work, a motion is applied: if the solid is lifted, the reaction disappears and does not work; if the solid is slid on the table, the reaction is normal to the velocity and does not work! Thus it appears that the reaction of the table is not easily figured out and not easily understood. Why the reaction is a force? Why not something more sophisticated or something more simple? We answer this question in Chap. 16. Let us also remark that the classical stress tensor is not easily figured out: the only way to feel the stress $\boldsymbol{\sigma}$ is to use the basic relationship $\boldsymbol{\sigma}\vec{N} = \vec{g}$ and investigate force $\vec{g}$ through motions. Think also of strain gauges which are not stress gauges. Again, the only way to investigate stresses is through motions.

Remark 1 The basic experimental fact is that it is possible to add velocities and to multiply velocities by a scalar. This obvious property is not true for displacements. For instance, think of two displacements of a point moving on the surface of a sphere: displacements $\vec{x} - \vec{x}(0)$ and $\vec{y} - \vec{x}(0)$ ($\vec{x}$ and $\vec{y}$ are positions of the point which is at position $\vec{x}(0)$ at time 0). The sum of the two displacements, $\vec{x} + \vec{y} - 2\vec{x}(0)$ is not a displacement because point $\vec{x} + \vec{y} - 2\vec{x}(0)$ is not on the surface of the sphere. In the same way, the displacement $\vec{x} - \vec{x}(0)$ multiplied by 2 is not a displacement because point $2\,(\vec{x} - \vec{x}(0))$ is not on the surface of the sphere. But note that the velocities $(d\vec{x}/dt)(t)$ at position $\vec{x}(t)$ are elements of the tangent plane which is a linear space.

Chapter 4
What We See: The Velocities

Abstract The actual linear and angular velocities of the disks are easily experimented. Virtual velocities are velocities we may think of. Some virtual velocities cannot be actual velocities, for instance in case of impenetrability of the disks. The set of the virtual velocities is such that it is a linear space which contains the actual velocities.

4.1 The Actual Velocities

The velocities are defined on the actual position of the structure

$$\mathcal{B}(t) = \mathcal{B}_1(t) \cup \mathcal{B}_2(t),$$

where $\mathcal{B}_n(t) \subset \mathbb{R}^2$ are the domains occupied by the two rigid disks at time t. The velocities are measured in plane frame $(\vec{e}_1, \vec{e}_2)$. Frame $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is direct, where $\vec{e}_3$ is normal to the plane of the system. At point $\vec{x}_n \in \mathcal{B}_n(t)$, the actual velocity $\widetilde{U}_n(\vec{x}_n, t)$ is

$$\widetilde{U}_n(\vec{x}_n, t) = \overrightarrow{U}_n(t) + \Omega_n(t)\vec{e}_3 \times (\vec{x}_n - \vec{G}_n(t)),$$

where $\overrightarrow{U}_n(t)$ is the velocity of the center of mass $\vec{G}_n(t)$ of the rigid disk $\mathcal{B}_n(t)$ and $\Omega_n(t)$ is the angular velocity.

4.2 The Virtual Velocities

The virtual velocities are velocities we may think of. Among the velocities we may think of are the actual velocities which are, as already seen, elements of a linear space. Thus we require the virtual velocities are also elements of a linear space.

The actual velocities are often submitted to unilateral or bilateral constraints in the linear space. We may think of motions where the two disks interpenetrate (think of two disks made of Velcro textile material). We may also think of motions where the two

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discs do not interpenetrate. The impenetrability condition may also be conditional: think of a steel ball falling on a sheet of paper. The sheet is either transfixed or not by the steel ball. The limitations on the velocities are constitutive properties which do not intervene in the equations of motion which apply to any situation. Thus we choose not to impose to the virtual velocities the limitations which are satisfied by the actual velocities: the virtual velocities are all the elements of the linear space.

The choice of the linear space of the virtual velocities is based on physical observation and on the sophistication of the motion we intend to predict.

The main properties we require for the virtual velocities, the velocities we think of, are

- *they are defined by the elements of a linear space* $\mathcal{V}$;
- *they are not submitted to constraints*;
- *the actual velocities are virtual velocities.*

At time t and point $\vec{x} \in \mathcal{B}_n(t)$, the virtual velocities $\widetilde{V}(\vec{x}, t)$, are defined by $\widehat{V}_n(t)$, $\widehat{\omega}_n(t)$ which are virtual velocities of the centers of mass and angular velocities

$$\widetilde{V}(\vec{x}, t) = \widehat{V}_n(t) + \widehat{\omega}_n(t)\vec{e}_3 \times (\vec{x}_n - \vec{G}_n(t)), \ \vec{x} \in \mathcal{B}_n(t).$$

The virtual velocities are defined by the elements of linear space

$$\mathcal{V} = \left\{ V = \left(\widehat{V}_1, \widehat{\omega}_1, \widehat{V}_2, \widehat{\omega}_2\right)\right\} = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R},$$

which are also called virtual velocities, to avoid too many definitions. This space is equipped with the duality pairing (a scalar product in this example)

$$< f(t), V(t) >= \vec{f}_1(t) \cdot \widehat{V}_1(t) + C_1(t)\widehat{\omega}_1(t) + \vec{f}_2(t) \cdot \widehat{V}_2(t) + C_2(t)\widehat{\omega}_2(t),$$

with $f = \left(\vec{f}_1, C_1, \vec{f}_2, C_2\right) \in \mathcal{V}^*$ where $\mathcal{V}^*$ is the dual space of $\mathcal{V}$ which in this example is identified to $\mathcal{V}$ because $< f, V >$ is a scalar product.

With respect to time, the velocities are special bounded variation functions, [1–4]. More precisely

$$V \in SBV(t_1, t_2; \mathcal{V}) = \tilde{\mathcal{V}}, \ \widehat{V}_n \in SBV(t_1, t_2; \mathbb{R}^2), \ \widehat{\omega}_n \in SBV(t_1, t_2; \mathbb{R}),$$

where $t_1 < t_2$ are two times. The special bounded variation functions are such that their time differential measure have a Lebesgue part and an atomic part

$$d\widehat{V} = \left\{\frac{d\widehat{V}}{dt}\right\} dt + \sum_{t_i \in E(\widehat{V})} (\widehat{V}^+(t_i) - \widehat{V}^-(t_i))\delta(t - t_i),$$

where $E(\widehat{V})$ is the numerable set of times where $\widehat{V}$ is discontinuous with respect to time, and

$$\left\{\frac{d\widehat{V}}{dt}\right\}(t) = \frac{d\widehat{V}}{dt}(t),\ \ for\ t \notin E(\widehat{V}),$$

is the usual time derivative outside set $E(\widehat{V})$, the smooth part of differential measure and $\delta(t - t_i)$ is the Dirac measure, the discontinuity of $\widehat{V}$, being $\widehat{V}^+(t_i) - \widehat{V}^-(t_i)$ with right and left values of $\widehat{V}$ at time $t_{(i)}$,

$$\widehat{V}^+(t_i) = \lim_{\Delta t \to 0, \Delta t > 0} \frac{\widehat{V}(t_i + \Delta t) - \widehat{V}(t_i)}{\Delta t},$$
$$\widehat{V}^-(t_i) = \lim_{\Delta t \to 0, \Delta t > 0} \frac{\widehat{V}(t_i) - \widehat{V}(t_i - \Delta t)}{\Delta t}.$$

Remark 2 There is an other time measure, the Cantor measure. For this presentation, we exclude it because its physical meaning is not clear. The space $\mathcal{V}$ is often an Hilbert space with scalar product $< f, V >$, the power of the pair force f and velocity V.

At this point of the modelling, the constitutive properties of the system do not intervene. The equations of motions apply to any material. Thus any velocity of $\tilde{\mathcal{V}}$ is a possible actual velocity. For instance, the two disks may be galaxies which can interpenetrate: being considered as continuum media they can interpenetrate because at the microscopic level (microscopic level compared to the level where a galaxy is considered as a continuum) the voids between the stars are so important that collisions are unexpected even if the stars interact through interactions at a distance. Thus the galaxies can interpenetrate. this phenomenon actually occurs: it is said that big galaxies eat the small ones.

Remark 3 Some presentation of mechanics are such that the actual velocities are not virtual velocities. For instance when investigating the motion of a solid with respect to a moving obstacle, the actual velocities are elements of an affine space. Our point of view is to include the obstacle in the system. Then the velocities of the obstacle are elements of the theory. It results the velocities are element of a linear space.

4.3 The Abstract Setting

The velocities denoted V are elements of linear space $\tilde{\mathcal{V}} = SBV(t_1, t_2; \mathcal{V})$ where $\mathcal{V}$ is a linear space. The forces are elements of the dual space $\tilde{\mathcal{V}}^*$. This choice results from the fact that is seems easy to see, to detect, to perceive and to measure the velocities of a system whereas it is not so obvious to see, detect or measure forces.

Let us notice that forces, stresses are singled out and measured through velocities (any scale weights with some motion) or through velocities of deformations (strain gauges and so on). It is with power $< f, V >$ that we feel force f.

Remark 4 Word force is used with two meanings:

1. the classical force which is a vector, an element of the dual set, $\mathbb{R}^3$ of the velocities set, $\mathbb{R}^3$. It is denoted with an arrow $\vec{f}$;
2. the generalized force which is an element of the dual space $\mathcal{V}^*$ of the velocities space $\mathcal{V}$ or of the deformation velocities. In case the context does not exclude ambiguity, we precise the meaning by specifying classical force or generalized force. The generalized forces are not denoted with an arrow.

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Chapter 5
The Actions Which are Applied to the System: The Work of the External Forces

Abstract The external forces applied to the system are defined by their work which is a linear function of the virtual velocities. They have a density with respect to the Lebesgue measure, the classical forces and torques, and a density with respect to the atomic measure, the percussions and percussion torques.

The work of an abstract force with velocity V is the integral of the power of the force with respect to time. We assume the time measures are the Lebesgue measure and the atomic measure: the density of the force with respect to the Lebesgue measure is the classical force and the density of the force with respect to the atomic measure is called a percussion. The work is

$$\begin{aligned}\tilde{W}_{ext} = &\int_{t_1}^{t_2} \left\{ \vec{f}_1(\tau) \cdot \widehat{V}_1(\tau) + C_1(\tau)\widehat{\omega}_1(\tau) + \vec{f}_2(\tau) \cdot \widehat{V}_2(\tau) + C_2(\tau)\widehat{\omega}_2(\tau) \right\} d\tau \\ &+ \sum_{t_i \in E_{ext}} \left\{ \vec{P}_1(t_i) \cdot \frac{\widehat{V}_1^+(t_i) + \widehat{V}_1^-(t_i)}{2} + Q_1(t_i)\frac{\widehat{\omega}_1^+(t_i) + \widehat{\omega}_1^-(t_i)}{2} \right. \\ &\left. + \vec{P}_2(t_i) \cdot \frac{\widehat{V}_2^+(t_i) + \widehat{V}_2^-(t_i)}{2} + Q_2(t_i)\frac{\widehat{\omega}_2^+(t_i) + \widehat{\omega}_2^-(t_i)}{2} \right\} \\ = &\int_{t_1}^{t_2} < f, V > d\tau + \sum_{t_i \in E_{ext}} < F(t_i), \frac{V^+(t_i) + V^-(t_i)}{2} >,\end{aligned}$$

with

$$f = \left(\vec{f}_1, C_1, \vec{f}_2, C_2\right) \in \mathcal{V}^*, \ F = \left(\vec{P}_1, Q_1, \vec{P}_2, Q_2\right) \in \mathcal{V}^*,$$

where the $\vec{f}_n$ are forces, the C_n are torques, the $\vec{P}_n$are percussions and the Q_n are percussion torques. The numerable set E_{ext} contains the times $t_i \in \,]t_1, t_2[$where external percussions $F(t_i)$ are applied to the system.

We define the linear function, *the virtual work of the external forces*

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DOI 10.1007/978-3-319-40682-4_5

$$V \rightarrow W_{ext}(V) = \tilde{W}_{ext},$$

where velocity V belongs to space $\tilde{\mathcal{V}}$.

5.1 The Abstract Setting

The forces are elements of the dual space $\mathcal{V}^*$. The power of the external forces has density with respect to the Lebesgue measure

$$V \in \mathcal{V},\ f \in \mathcal{V}^*,\ \mathcal{P}_{ext}dt =< V, f > dt,$$

where $< V, f >$ is the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$ (a scalar product in this example) and density with respect to the atomic measure

$$V \in \mathcal{V},\ P_i(t_i) \in \mathcal{V}^*,$$
$$\mathcal{W}_{ext}(t_i)\delta(t - t_i) =< \frac{V^+(t_i) + V^-(t_i)}{2}, P_i(t_i) > \delta(t - t_i),\ t_i \in E_{ext},$$

where E_{ext} is the numerable set of times $t_i \in]t_1, t_2[$ where percussion exterior actions are applied. The work is the integral of the power

$$\tilde{W}_{ext} = \int_{t_1}^{t_2} \mathcal{P}_{ext}(\tau)d\tau + \sum_{t_i \in E_{ext}} \mathcal{W}_{ext}(t_i).$$

This formula defines the linear function, *the virtual work of the external forces*

$$V \rightarrow W_{ext}(V) = \tilde{W}_{ext},$$

where velocity V belongs to space $\tilde{\mathcal{V}}$. The work of the external forces $V \rightarrow W_{ext}(V)$ defines a linear function on $\tilde{\mathcal{V}}$, thus an element $f^* \in \tilde{\mathcal{V}}^*$. But f^* is an abstract quantity which is not a classical mechanical quantity, we do not use it in the sequel and prefer to use the f and $P_i(t_i)$ which have a clear mechanical meaning.

In engineering there is almost only one external body force, the gravity force. But the surface external forces are numerous and sometimes sophisticated with power or work which can be experimented. Based on this observation, Gianpietro Del Piero has developed a description of the motion founded on the expression of the power of the external forces [1].

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Chapter 6
What We See: The Velocities of Deformation

Abstract The velocities of deformation of the system describe how the shape of the system changes. Their choice is based on observation. Different choices are possible. The rigid system velocities are velocities which do not change the shape of the system.

There are different ways to measure how the shape of the system changes or how it deforms. Its shape changes when the distance of the two disks changes and when the two disk angular orientation changes.

We may choose as velocities of deformation the relative angular velocity

$$\mathfrak{D}_r(V) = \widehat{\omega}_1 - \widehat{\omega}_2, \tag{6.1}$$

and the velocity of center of mass $\vec{G}_1$ with respect to ball 2 and velocity of center of mass $\vec{G}_2$ with respect to ball 1

$$\mathfrak{D}_1(V) = \widehat{V}_1 - (\widehat{V}_2 + \widehat{\omega}_2\vec{e}_3 \times (\vec{G}_1 - \vec{G}_2)) = \widehat{V}_1 - \widehat{V}_2 - \widehat{\omega}_2\vec{e}_3 \times (\vec{G}_1 - \vec{G}_2),$$
$$\mathfrak{D}_2(V) = \widehat{V}_2 - (\widehat{V}_1 + \widehat{\omega}_1\vec{e}_3 \times (\vec{G}_2 - \vec{G}_1)) = \widehat{V}_2 - \widehat{V}_1 - \widehat{\omega}_1\vec{e}_3 \times (\vec{G}_2 - \vec{G}_1).$$

Note that we have

$$\mathfrak{D}_1(V) + \mathfrak{D}_2(V) = (\widehat{\omega}_2 - \widehat{\omega}_1)\vec{e}_3 \times (\vec{G}_2 - \vec{G}_1). \tag{6.2}$$

A possible set of velocities of deformation for the system is

$$\hat{\mathfrak{D}}(V) = (\mathfrak{D}_1(V), \mathfrak{D}_2(V), \mathfrak{D}_r(V)).$$

These velocities of deformation are not independent: due to (6.2) we have

$$\mathfrak{D}_1(V) + \mathfrak{D}_2(V) = (\widehat{\omega}_2 - \widehat{\omega}_1)\vec{e}_3 \times (\vec{G}_2 - \vec{G}_1) = -\mathfrak{D}_r(V)\vec{e}_3 \times (\vec{G}_2 - \vec{G}_1).$$

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Springer Series in Solid and Structural Mechanics 7,
DOI 10.1007/978-3-319-40682-4_6