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Ishan Sharma

Shapes and Dynamics of Granular Minor Planets

The Dynamics of Deformable Bodies Applied to Granular Objects in the Solar System

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Ishan Sharma
Mechanics and Applied Mathematics Group
Department of Mechanical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

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आकृष्टशक्तिश्च मही तया यत् खस्थं गुरुं स्वाभिमुखं स्वशक्त्या ।
आकृष्यते तत् पततीव भाति समे समन्तात् क्व पतत्वियं खे ॥

— **भास्कराचार्य** (१२ शताब्दी)

The earth possesses the power of attraction;
Through its own power, whatever heavy substance
in the space it attracts towards it, seem to fall [towards it].
This happens from all sides simultaneously;
[So much so] where would this [Earth] fall in space?

— **Bhaskaracarya** (12th CE)

This book is dedicated to Indian Astronomy,
whose science continues to surprise.

Foreword

This book was born when advances in astronomical techniques permitted, for example, the shapes and spins of asteroids to be determined using radar (see, e.g., Ostro et al. 2002), just as interest in the mechanics and physics of granular materials was being renewed in the mid-1980s. The much-publicized tidal fragmentation of comet Shoemaker–Levy in 1994 as it passed Jupiter stimulated the development of numerical simulations. Asteroids, by their sheer number and variety, provide a natural laboratory in which the translational and rotational motion of deformable solid bodies can be investigated. Relatively high-resolution dynamic imaging made it clear that at least some and, perhaps, many near-Earth asteroids were not monolithic rocks, but likely consisted of discrete solid elements held together by their mutual gravity. The advances in the mechanics of granular materials provided a means to treat the collisional interactions between the elements that transferred momentum and dissipated the energy associated with them. The advances in both subjects made it possible for the two of us to develop our common interest in their dynamics.

Ishan Sharma, a gifted graduate student in the Department of Theoretical and Applied Mechanics at Cornell University, was the intellectual agent of this development. He shared our interest and enthusiasm for the subject and we benefited from his intelligence and energy. Starting with his doctoral work with us, Sharma restricted attention to affine deformations of extended bodies and employed a volume-averaged approach to determine their equations of motion. In doing this, he followed Chandrasekhar (1969), who introduced this technique in his famous work on the equilibrium shapes of spinning fluid ellipsoids. In a series of research papers in Icarus initially with us and, later, independently, Sharma also adopted and made more transparent elements of the dynamics of deforming bodies introduced by Cohen and Muncaster (1988). Sharma (2004) first determined the equilibrium and failure of a spinning asteroid and placed existing results by Holsapple (2001) in a dynamical context. Sharma also phrased and numerically solved the equations that describe planetary fly-bys of asteroids of a less tightly packed granular aggregate in which the elements interacted through collisions, so as to obtain results similar to those of the molecular dynamics simulations of Richardson et al. (1998) and others.

Since completing his dissertation (2004), Sharma has extended the results on to the equilibrium and failure of an exhaustive list of asteroids and satellites. He has also made important steps in characterizing the stability of their equilibrium states. Finally, he has completed a refined analysis of planetary fly-bys. These elements are collected in this volume. However, the resulting volume is much more than this. It is, also, a compact introduction to continuum mechanics of deformable bodies and, further, a rather complete treatment of the dynamics of self-gravitating deformable bodies, when they are treated, in first approximation, as having uniform material properties and deforming homogeneously. This makes the volume, on the one hand, a valuable general introduction to the dynamics of deformable bodies and, on the other hand, a detailed treatment of the multitude of objects in the solar system for which dynamics is likely to be coupled with deformation. We are proud to have been involved in the beginning of the scholarly activity that led to this manuscript. We believe it to be a worthy successor to the classic work of Chandrasekhar (1969).

Ithaca, NY, USA Jim Jenkins
September 2015 Joe Burns

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Preface

Starting about twenty years ago, astronomers gradually realized that many of the small bodies in the solar system (asteroids, comets and satellites) are rubble piles, i.e., granular aggregates. The first unequivocal evidence for this came when the comet Shoemaker–Levy nine broke apart, apparently by tides, into dozens of pieces as it passed close to Jupiter. Numerical simulations of self-gravitating granular aggregates were developed and they exhibited such fragmentation during close planetary encounters. Around the same time, researchers recognized that very few asteroids were found with spin periods of less than a few hours, and that this could be understood simply as the consequence of the fragility of fast-spinning bodies to centrifugal breakup. Furthermore, other asteroids and a few close-on satellites of the giant planets were observed in radar "images" and spacecraft images, respectively, to have smooth elongated shapes, suggesting rotational and tidal distortion.

Around the same time, the masses of dozens of asteroids began to be measured, usually by observing the orbital periods of binary asteroids, or by mutual gravitational perturbations of distant asteroids on Mars or another asteroid, or by spacecraft flybys. For those asteroids, comets and a few satellites that had known sizes, their densities were immediately available. More often than not, these measured densities were remarkably low, sometimes less than 0.5 g/cm^3 for comets and small satellites, or often 1–2 g/cm^3 for asteroids and satellites. Because the likely constituents of these bodies (water, ice and rock) have greater densities, the low bulk densities required significant pore space and, accordingly, implied granular aggregates held together primarily by self-gravity, rather than monolithic rocks. Such a loose character is not unexpected for objects that accreted gravitationally in a cold environment.

This book investigates the equilibrium, stability and dynamics of these rubble solar-system bodies. It is clear that any careful investigation will need to consider these bodies as objects with finite extent, and not as mere point masses, and as a granular medium, distinct in its constitutive response from solids and fluids. This book pays particular attention to these aspects.

In this book, we develop a framework for analyzing the dynamics of rotating complex materials; this is predicated on the systematic approximations of the system's kinematics. We apply the method to investigate rotating and self-gravitating granular aggregates in space. Here, we limit the kinematic approximation to, at most, an affine deformation, so that the most general shape that an object can take is that of an ellipsoid. Necessary governing equations may be obtained by a variety of methods, but we prefer to follow the virial method, or volume-averaging, employed by Chandrasekhar (1969). We do this primarily for historical continuity and for greater general familiarity with that method, but also because the current research was motivated to a great extent by Chandrasekhar's treatise that explored similar questions in the context of inviscid fluids.

The constitutive model that we employ depends on the situation. For example, when considering equilibrium, or its stability, the granular aggregate is modeled as a rigid-perfectly-plastic material obeying a pressure-dependent yield criterion, e.g., the Drucker–Prager yield criterion, and deforming post-yield as per an appropriate flow rule. However, when studying the disruptive effects of a tidal flyby, the aggregate is taken to be an ensemble of dissipative spheres whose macroscopic behavior is determined through an application of kinetic theory. The current framework has the advantage that it allows us to improve the kinematic approximation in a structured manner as well as to explore a wide variety of constitutive laws.

The book is divided into four parts. The first part introduces the necessary mathematics and continuum mechanics, as well as, describes affine dynamics that forms the basis of all subsequent development. Part II investigates the equilibrium of rubble asteroids, satellites, and binaries, and applies it to known or suspected cases. Equilibrium, here, refers to possible ellipsoidal shapes that a rubble asteroid can take, and to both shape and orbital separation for granular satellites and binaries. In Part III, we develop a linear stability criterion specifically for rotating granular aggregates, which is then applied to the equilibria obtained in Part I. Finally, in Part IV we provide a pair of examples of dynamical evolution. These relate to the disruption and possible re-agglomeration of rubble piles during tidal flybys.

Finally, I confess to some nervousness. It is dangerous to write a book that may be viewed by some as a would-be successor to Chandrasekhar's classic treatise. Followers in the footsteps of giants risk sinking or getting lost. But then, there are worse ways to go.

Kanpur, India — Ishan Sharma
September 2015

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This book would not have been possible without the help and guidance of Profs. Joseph A. Burns and James T. Jenkins of Theoretical & Applied Mechanics at Cornell University. It was they who introduced me to this research area during my Ph.D., and later inspired me to write this book. They also read through the book and provided critical feedback that has helped improve this work significantly. The parts of this book that may appeal to the reader are directly a result of their academic influence; the errors, of course, are entirely my own.

I am also grateful to Prof. Kolumban Hutter who, as editor, gave me the opportunity to write this book as part of the Springer series on Advances in Geophysical and Environmental Mechanics and Mathematics. Not only did his regular prompting help me complete this book, but his thorough review of the final manuscript was extremely helpful.

It is also a pleasure to thank many colleagues from the Mechanics & Applied Mathematics Group (www.iitk.ac.in/mam) at the Indian Institute of Technology Kanpur: S. Mahesh, Shakti Singh Gupta, Sovan Lal Das, Anindya Chatterjee, Chandrashekhar Upadhyay, Prakash M. Dixit, P. Venkitinarayanan, Sumit Basu, Pankaj Wahi, Anurag Gupta, Mahendra K. Verma, V. Shankar, and Basant Lal Sharma. Several of them have, possibly unknowingly, helped this work since our time together at the Department of Theoretical & Applied Mechanics, Cornell. Other member of the larger IIT Kanpur fraternity whose support has been critical through the last decade while I was writing this book include, in no particular order, Profs. Asok Mallik, K. Muralidhar, (Late) Himanshu Hatwal, V. Eswaran, B.N. Banerjee, Amitabha Ghosh, C. Venkatesan, P.K. Panigrahi, N.N. Kishore, Muthukumar T., Jayant K. Singh, Koumudi P. Patil, Sunil Simon, Preena Samuel, Akash Anand, Shikha Prasad, Anandh Subrahmaniam, Brijesh Eshpuniyani, Dr. Bhuvana T., Shirolly Anand, and Dr. Ashish Bhateja. I would also like to mention the help and support of Satya Prakash Mishra, Bharat Singh, Ram Lakhan, and Lakshmi.

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Finally, no goal may possibly be reached without the best wishes and prayers of one's immediate and extended family, and close friends, and I thank each and every one of them. I mention my late grandparents – Prof. Govind Chandra Pande, Major (retd.) Gokul Chandra Sharma and Smt. Sarah Sharma – who would have been the happiest to see this book in print, but desist from mentioning anyone else by name in order to save the publisher ten or so pages.

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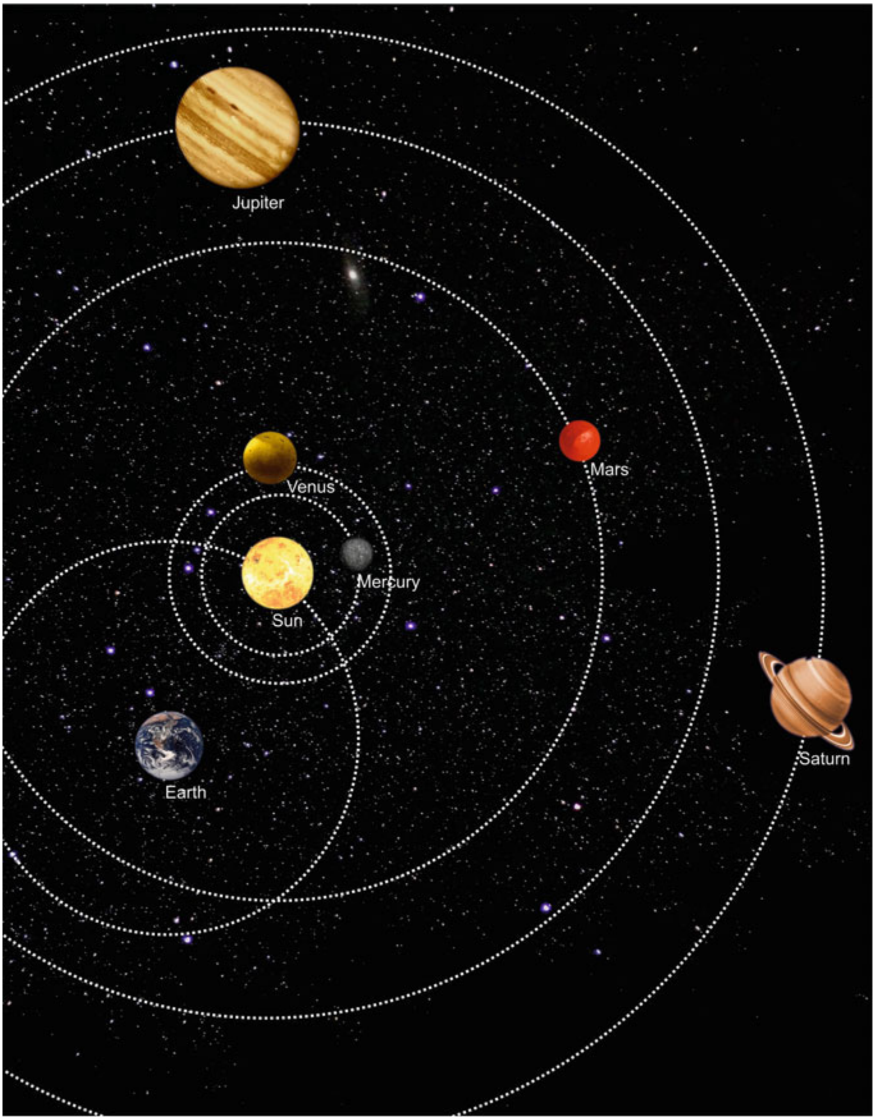

Nilkaṇṭha's 15th century cosmological model showing the five planets in eccentric motion about a mean Sun. This s the best that can be arrived at by naked-eye observation. This model predates the better known Tychonic system.

From ***Tantrasaṅgraha*** of *Nilkaṇṭha Somayāji*
by K. Ramasubramaniam and M. S. Sriram

[Image: Koumudi Patil, IIT Kanpur]

Chapter 1
Mathematical Preliminaries

In this chapter, we quickly summarize necessary tensor algebra and calculus, and introduce the notation employed in this text. We assume familiarity with matrix algebra and indical notation. More information may be obtained from standard texts such as Strang (2005) or Knowles (1998).

1.1 Coordinate Systems

We will exclusively employ right-handed cartesian coordinate systems. The coordinate system of choice may be stationary, translating, or rotating, or both. We will employ calligraphic capital letters to identify coordinate systems. In this text we will typically employ three coordinate systems $\mathscr{O}$, $\mathscr{P}$ and $\mathscr{S}$ with associated unit vectors $\hat{\mathbf{e}}_i$, $\hat{\mathbf{e}}_i'$ and $\hat{\mathbf{e}}_i''$, respectively. This will be indicated by, e.g., $\left\{\mathscr{O}, \hat{\mathbf{e}}_i\right\}$.

1.2 Vectors

A vector $\mathbf{a}$ is represented as thus. The components of $\mathbf{a}$ in $\mathscr{O}$ will be denoted by a_i and in $\mathscr{P}$ by a_i', so that we have the identities

$$\mathbf{a} = (\mathbf{a} \cdot \hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i = a_i\hat{\mathbf{e}}_i = (\mathbf{a} \cdot \hat{\mathbf{e}}_i')\hat{\mathbf{e}}_i' = a_i'\hat{\mathbf{e}}_i', \tag{1.1}$$

where ‘ · ’ is the usual vector dot product. The magnitude or *norm* of $\mathbf{a}$ is

$$|\mathbf{a}| = (\mathbf{a} \cdot \mathbf{a})^{1/2} = (a_i a_i)^{1/2} = \left(a_i' a_i'\right)^{1/2}. \tag{1.2}$$

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We now collect several useful formulae:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_i b_i, \tag{1.3a}$$

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = |\mathbf{a}||\mathbf{b}|\ \sin\theta\,\hat{\mathbf{e}}_c = \varepsilon_{ijk}\hat{\mathbf{e}}_i a_j b_k, \tag{1.3b}$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \varepsilon_{ijk}a_i b_j c_k \tag{1.3c}$$

and

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \varepsilon_{ijk}\varepsilon_{klm}\hat{\mathbf{e}}_i a_j b_l c_m. \tag{1.3d}$$

where θ is the angle between vectors **a** and **b**, $\hat{\mathbf{e}}_c$ is a unit vector normal to the plane containing **a** and **b**, and ε_{ijk} is the *alternating tensor* defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i,j \text{ and } k \text{ are an even permutation} \\ -1 & \text{if } i,j \text{ and } k \text{ are an odd permutation} \\ 0 & \text{otherwise;} \end{cases} \tag{1.4}$$

cf. Sect. 1.3.2.

We will typically limit ourselves to three-dimensional vectors.

1.3 Tensors

A first-order tensor is simply a vector. A second-order tensor is a linear transformation that maps a vector to another vector. Third- and fourth-order tensors relating lower-order tensors to other lower-order tensors may be similarly defined.

1.3.1 Second-Order Tensors

A second-order tensor A is probed by its action '$\cdot$' on vectors. We employ the same symbol as for the dot-product of vectors because of similarities between the two operations. We define the resultant **b** of A's operation, specifically a *right*-operation, on **a** by

$$\mathbf{b} = \mathsf{A}\cdot\mathbf{a}.$$

Similarly a *left*-operation may be defined. As with vectors, we will typically limit ourselves to second-order tensors that operate on and result in three-dimensional vectors.

The addition $\mathsf{A}+\mathsf{B}$ and multiplication $\mathsf{A}\cdot\mathsf{B}$ of two tensors A and B result in tensors C and D, respectively, that are defined in terms of how they operate on some vector **a**, i.e.,

$$(\mathsf{A}+\mathsf{B})\cdot\mathbf{a} = \mathsf{C}\cdot\mathbf{a} := \mathsf{A}\cdot\mathbf{a} + \mathsf{B}\cdot\mathbf{a} \tag{1.5a}$$

and

$$(\mathsf{A}\cdot\mathsf{B})\cdot\mathbf{a} = \mathsf{D}\cdot\mathbf{a} := \mathsf{A}\cdot(\mathsf{B}\cdot\mathbf{a}). \tag{1.5b}$$

It is understood that the two tensors A and B relate vectors belonging to the same set.

To better understand tensors, it is useful to generalize the concept of a unit vector to a tensorial basis. Such a generalization is furnished by the *tensor product* $\mathbf{a} \otimes \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$. The entity $\mathbf{a} \otimes \mathbf{b}$ is a second-order tensor that can act on another vector $\mathbf{c}$ in two different ways – the left- and right- operations – to yield another vector:

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} \quad \text{and} \quad \mathbf{c} \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}\,, \tag{1.6}$$

where the '$\cdot$' on the left-hand sides denotes a tensor operation, and the usual vector dot product on the right-hand sides. Contrasting the computation

$$\mathbf{a} \otimes \mathbf{b} = a_i\hat{\mathbf{e}}_i \otimes b_j\hat{\mathbf{e}}_j = (a_i b_j)\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{1.7}$$

with (1.1), suggests that a tensorial basis may be constructed by taking appropriate order tensor products of the unit vectors. We note that the above represents the linear combination of $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$. Thus, a second-order tensorial basis in the coordinate system $\mathscr{O}$ is given by the nine *unit tensors* $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$. A second-order tensor A may then be written as

$$\mathsf{A} = A_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j, \tag{1.8}$$

in terms of A's components A_{ij} in $\mathscr{O}$. These components, obtained by appealing to (1.6), are given by the equations

$$A_{ij} = \hat{\mathbf{e}}_i \cdot \mathsf{A} \cdot \hat{\mathbf{e}}_j, \tag{1.9}$$

that are reminiscent of analogous ones for vector components; see (1.1). We will refer to the nine A_{ij}'s as the "*matrix* of A in $\{\mathscr{O}, \hat{\mathbf{e}}_i\}$" denoted by $[\mathsf{A}]$. In another coordinate system, say $\mathscr{P}$, the tensorial basis is given by $\hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j$, while $A'_{ij} = \hat{\mathbf{e}}'_i \cdot \mathsf{A} \cdot \hat{\mathbf{e}}'_j$ constitute the "matrix of A in $\{\mathscr{P}, \hat{\mathbf{e}}'_i\}$" denoted by $[\mathsf{A}]'$. A second-order tensor's interactions with vectors and other second-order tensors may be obtained by repeated (if required) application of (1.6). These operations are summarized below:

$$\mathsf{A} \cdot \mathbf{a} = A_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \cdot a_m\hat{\mathbf{e}}_m = A_{ij}a_j\hat{\mathbf{e}}_i, \tag{1.10a}$$

$$\mathbf{a} \cdot \mathsf{A} = a_m\hat{\mathbf{e}}_m \cdot A_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = a_i A_{ij}\hat{\mathbf{e}}_j, \tag{1.10b}$$

$$\mathsf{A} \cdot \mathsf{B} = A_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \cdot B_{mn}\hat{\mathbf{e}}_m \otimes \hat{\mathbf{e}}_n = A_{ij}B_{jn}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_n \tag{1.10c}$$

and

$$\mathsf{A} : \mathsf{B} = A_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j : B_{mn}\hat{\mathbf{e}}_m \otimes \hat{\mathbf{e}}_n = A_{ij}\hat{\mathbf{e}}_i \cdot B_{jn}\hat{\mathbf{e}}_n = A_{ij}B_{ji}, \tag{1.10d}$$

where the first two operations produce vectors, the next another second-order tensor, and the third a scalar. The double-dot product ':', as its form suggests, denotes a sequential application of dot products, as illustrated above. The tensor A's actions on higher-order tensors may be analogously defined. When there is no confusion, second-order tensors are referred to simply as tensors.

The *transpose* A^T of a tensor A is defined by the following formula

$$(\mathsf{A} \cdot \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathsf{A}^T \cdot \mathbf{b}), \tag{1.11}$$

for any two vectors **a** and **b**. The above results in $[\mathsf{A}^T]_{ij} = A_{ji} = [\mathsf{A}]_{ji}$. From the above definition of a tensor's transpose the following identities are easily proved:

$$(\mathsf{A} + \mathsf{B})^T = \mathsf{A}^T + \mathsf{B}^T \quad \text{and} \quad (\mathsf{A} \cdot \mathsf{B})^T = \mathsf{B}^T \cdot \mathsf{A}^T. \tag{1.12}$$

The *trace* of a tensor A with components A_{ij} is obtained by *contracting* the indices i and j:

$$\mathrm{tr}\, \mathsf{A} = A_{ii} = A_{11} + A_{22} + A_{33}. \tag{1.13}$$

We see below that the trace of a tensor is independent of the coordinate system in which it is computed. The following identities regarding transposes are easily proved:

$$\mathrm{tr}\, \mathsf{A} = \mathrm{tr}\, \mathsf{A}^T \Rightarrow \mathrm{tr} \prod_{i=1}^{n} \mathsf{A}_i = \mathrm{tr} \left(\prod_{i=n}^{1} \mathsf{A}_i^T \right), \tag{1.14a}$$

$$\mathrm{tr} \prod_{i=1}^{n} \mathsf{A}_i = \mathrm{tr} \left(\prod_{i=n-k}^{n} \mathsf{A}_i \cdot \prod_{i=1}^{n-k-1} \mathsf{A}_i \right), \; 0 \leqslant k \leqslant n-1, \; \text{and} \; \prod_{i=1}^{0} \mathsf{A}_i = 1, \tag{1.14b}$$

where $\prod_{i=1}^{n} \mathsf{A}_i = \mathsf{A}_1 \cdot \mathsf{A}_2 \cdot \cdots \cdot \mathsf{A}_n$.

A tensor is said to be *symmetric* if $\mathsf{A} = \mathsf{A}^T$, and *anti-/skew-symmetric* if $\mathsf{A} = -\mathsf{A}^T$. Given an arbitrary tensor A we define its symmetric part

$$\mathrm{sym}\ \mathsf{A} = \frac{1}{2}\left(\mathsf{A} + \mathsf{A}^T\right), \tag{1.15}$$

and anti-symmetric part

$$\mathrm{asym}\ \mathsf{A} = \frac{1}{2}\left(\mathsf{A} - \mathsf{A}^T\right), \tag{1.16}$$

so that any tensor A may be written as a sum of a symmetric and anti-symmetric tensor

$$\mathsf{A} = \mathrm{sym}\ \mathsf{A} + \mathrm{asym}\ \mathsf{A}. \tag{1.17}$$

An anti-symmetric tensor has at most three independent components in any coordinate system. Thus, for any anti-symmetric tensor W, it is possible to associate an *axial* vector denoted by **w** with the property that for all vectors **b**,

$$\mathsf{W} \cdot \mathbf{b} = \mathbf{w} \times \mathbf{b}. \tag{1.18}$$

The operations of constructing anti-symmetric tensors from vectors and extracting axial vectors from anti-symmetric tensors are denoted by sk $\mathbf{w}$ (= W) and ax W (= $\mathbf{w}$), respectively. The relationship between $\mathbf{w}$ and W may be expressed in indical notation employing the alternating tensor of (1.4):

$$\operatorname{ax}\mathsf{W} = \mathbf{w} = -\frac{1}{2}\varepsilon_{ijk} W_{jk}\hat{\mathbf{e}}_i \quad \text{and} \quad \operatorname{sk}\mathbf{w} = \mathsf{W} = -\varepsilon_{ijk} w_i \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k. \tag{1.19}$$

Employing (1.3b), it is straightforward to check that the above prescription for ax W and sk w will satisfy (1.18).

For most tensors, and almost all tensors occurring in this book, it is possible to find three unit vectors that are simply scaled under that tensor's operation, i.e., given A, there (almost always) exist three unit vectors $\hat{\mathbf{v}}_i$ and correspondingly three scalars λ_i, such that

$$\mathsf{A} \cdot \hat{\mathbf{v}}_i = \lambda_i \hat{\mathbf{v}}_i \quad \text{(no sum)}. \tag{1.20}$$

These special vectors $\hat{\mathbf{v}}_i$ are the *eigenvectors* of A, and the corresponding scalings λ_i are A's *eigenvalues*. In the coordinate system described by the three eigenvectors, the tensor's matrix is *diagonalized* with the tensor's eigenvalues as the diagonal entries. This simple diagonal nature makes employing the eigen-coordinate system very tempting for computation. Unfortunately, there is no guarantee that the eigenvector triad are mutually orthogonal, so that the coordinate system they describe may not be cartesian. However, if the tensor is symmetric, it is *always* possible to diagonalize it, and, moreover, the eigenvectors are orthogonal, so that the coordinate system they describe is frequently a convenient operational choice. Thus, given a symmetric tensor S, it is possible to find three eigenvectors $\hat{\mathbf{v}}_i$ and corresponding eigenvalues λ_i, so that S is simply

$$\mathsf{S} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{v}}_i \otimes \hat{\mathbf{v}}_i. \tag{1.21}$$

The operation of a symmetric S therefore corresponds to a linear scaling along three mutually orthogonal eigen-directions.

If in case all the eigenvalues of a symmetric tensor are non-zero and positive, the symmetric tensor is said to be *positive definite*. Finally, it is important to mention that for any tensor the number of eigenvalues equals the dimension of the underlying space, whether or not it is diagonalizable. For example, in three dimensions, every tensor has three eigenvalues even if it doesn't have three eigenvectors. These eigenvalues are either all real, or a mixture of real and complex conjugate pairs.

While the components of a tensor depend on the coordinate system, its eigenvalues do not. Therefore, functions of these eigenvalues, called *principal invariants*, also remain unaffected by the choice of the coordinate system; the number of these invariants equaling the dimension of the underlying space. In three dimensions, a second-order tensor A with eigenvalues λ_i has the three invariants

$$\mathrm{I}_{\mathsf{A}} = \sum_{i=1}^{3} \lambda_i = A_{ii} = \mathrm{tr}\ \mathsf{A}, \tag{1.22a}$$

$$\mathrm{II}_{\mathsf{A}} = \sum_{i \neq j} \lambda_i \lambda_j = \frac{1}{2}\left(\mathrm{I}_{\mathsf{A}}^2 - \mathrm{I}_{\mathsf{A}^2}\right) \tag{1.22b}$$

and

$$\mathrm{III}_{\mathsf{A}} = = \prod_{i=1}^{3} \lambda_i = \det \mathsf{A}, \tag{1.22c}$$

where the last invariant represents the *determinant* of A that may also be computed via standard formulae after finding A's matrix in *any* coordinate system. Finally, as for vectors, it is possible to measure the magnitude of a tensor, by employing the double-dot product ' : ' introduced in (1.10). The *norm* of a tensor A is defined by

$$|\mathsf{A}| := \sqrt{\mathsf{A} : \mathsf{A}^T} = \sqrt{\mathrm{I}_{\mathsf{A}^2}} = \sqrt{A_{ij} A_{ij}}. \tag{1.23}$$

Frequently, and again, for all tensors considered in this book, it is possible to define associated *inverse* tensors, i.e., given A taking **a** to **b**, the *inverse* tensor A^{-1} brings **b** back to **a**. It is easy enough to see that a tensor and its inverse share the same eigenvectors, but inverse eigenvalues. Thus, if A has a zero eigenvalue, its inverse does not exist. The following identities regarding inverses are easily verified:

$$\mathsf{A} \cdot \mathsf{A}^{-1} = \mathsf{A}^{-1} \cdot \mathsf{A} = 1, \tag{1.24a}$$

$$(\mathsf{A} \cdot \mathsf{B})^{-1} = \mathsf{B}^{-1} \cdot \mathsf{A}^{-1}, \tag{1.24b}$$

$$\det \mathsf{A}^{-1} = (\det \mathsf{A})^{-1} \tag{1.24c}$$

and

$$(\mathsf{A}^T)^{-1} = (\mathsf{A}^{-1})^T = \mathsf{A}^{-T}. \tag{1.24d}$$

An important class of tensors that will occur frequently in the text is the *orthogonal* tensor Q that has the property that given any vector **a**,

$$|\mathsf{Q} \cdot \mathbf{a}| = |\mathbf{a}|, \tag{1.25}$$

i.e., Q preserves a vector's length. From this the following properties follow:

$$\mathsf{Q}^{-1} = \mathsf{Q}^T \quad \text{and} \quad \det \mathsf{Q} = \pm 1. \tag{1.26}$$

In applications to follow, all orthogonal tensors will have determinant one. Such *proper orthogonal* tensors are called *rotation tensors*. Physically, as its name suggests, a rotation tensor represents rotation about the origin. It may be shown that of a rotation tensor's three eigenvalues, two are complex conjugates of norm one and the third is unity; see e.g., Knowles (1998, p. 51). The eigenvector corresponding to the unitary eigenvalue provides the axis of rotation. The amount of rotation is provided by the argument of the complex eigenvalue.

Symmetric and rotation tensors come together in the *polar decomposition theorem* (Knowles 1998, p. 57), which states that for any tensor A with $\det \mathsf{A} > 0$, it is possible to find a rotation tensor R and positive definite tensors U and V, so that

$$\mathsf{A} = \mathsf{R} \cdot \mathsf{U} = \mathsf{V} \cdot \mathsf{R} \tag{1.27}$$

uniquely. Thus, $\mathsf{U} = \mathsf{R}^T \cdot \mathsf{V} \cdot \mathsf{R}$, and U and V share the same eigenvalues, while their eigenvectors are related through R. We recall that transformation via a symmetric tensor's operation corresponds to linearly and independently scaling three mutually perpendicular directions. Any linear transformation may thus be viewed as a rotation followed (preceded) by three scalings along the orthogonal eigen-coordinate system of V (U).

We have already mentioned the tensor product of two vectors in (1.7). Amongst other things, the tensor product helps in "tensorizing" the vector operations of taking dot- and cross- products, viz.,

$$\mathbf{a} \cdot \mathbf{b} = \operatorname{tr} \mathbf{a} \otimes \mathbf{b} = \mathbf{a} \otimes \mathbf{b} : 1, \tag{1.28a}$$

and

$$\mathbf{a} \times \mathbf{b} = -2 \operatorname{ax} \operatorname{sk} \mathbf{a} \otimes \mathbf{b}. \tag{1.28b}$$

Some additional identities that are easily proved, and will often be used are

$$\mathbf{a} \otimes \mathsf{A} \cdot \mathbf{b} = \mathbf{a} \otimes \mathbf{b} \cdot \mathsf{A}^T, \tag{1.29a}$$

$$\mathbf{a} \cdot \mathsf{A} \cdot \mathbf{b} = \mathbf{a} \otimes \mathbf{b} : \mathsf{A}^T, \tag{1.29b}$$

$$\operatorname{sk} \mathsf{A} : \mathsf{B} = \operatorname{sk} \mathsf{A} : \operatorname{sym} \mathsf{B} + \operatorname{sk} \mathsf{A} : \operatorname{sk} \mathsf{B} = \operatorname{ax} \operatorname{sk} \mathsf{A} \cdot \operatorname{ax} \operatorname{sk} \mathsf{B}, \tag{1.29c}$$

$$\mathsf{A} \cdot \mathsf{B} : \mathsf{C} = \mathsf{C} \cdot \mathsf{A} : \mathsf{B} = \mathsf{B} \cdot \mathsf{C} : \mathsf{A} \tag{1.29d}$$

and

$$\mathsf{S} : \mathsf{W} = \operatorname{tr} (\mathsf{S} \cdot \mathsf{W}) = 0, \tag{1.29e}$$

where S and W are, respectively, symmetric and anti-symmetric tensors.

1.3.2 Third- and Fourth-Order Tensors

First, consider third-order tensors. In terms of the third-order tensorial bases, $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k$ in $\{\mathscr{O}, \hat{\mathbf{e}}_i\}$, a third-order tensor is defined as

$$\mathscr{A} = A_{ijk} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k, \tag{1.30}$$

so that A_{ijk} are $\mathscr{A}$'s components in this coordinate system. The actions of $\mathscr{A}$ on vectors and other tensors of various orders are defined in a manner similar to that of a second-order tensor (1.10), e.g.,

$$\mathscr{A} \cdot \mathbf{a} = A_{ijk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \cdot a_m\hat{\mathbf{e}}_m = A_{ijk}a_k\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{1.31a}$$

and

$$\mathbf{a} \cdot \mathscr{A} = a_m\hat{\mathbf{e}}_m \cdot A_{ijk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k = a_iA_{ijk}\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k. \tag{1.31b}$$

An important example of a third-order tensor is the *alternating tensor* that has already been defined by (1.4).

Fourth-order tensors are formed in a manner analogous to third-order tensors,

$$\mathbf{A} = A_{ijkl}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l, \tag{1.32}$$

and their operations on vectors and tensors of various orders may be developed by following (1.10) and (1.31), for example,

$$\mathbf{A} : \mathsf{B} = A_{ijkl}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l : B_{mn}\hat{\mathbf{e}}_m \otimes \hat{\mathbf{e}}_n = A_{ijkl}B_{lk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j. \tag{1.33}$$

1.4 Coordinate Transformation

We will need to find the components of vectors and second-order tensors in one coordinate system, say $\{\mathscr{O}, \hat{\mathbf{e}}_i\}$, given its matrix in the other, say $\{\mathscr{P}, \hat{\mathbf{e}}'_i\}$. This may be done by expressing the unit vectors of $\mathscr{P}$ in terms of those of $\mathscr{O}$ as,

$$\hat{\mathbf{e}}'_j = (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}'_j)\,\hat{\mathbf{e}}_i,$$

and substituting this relationship into (1.1) and (1.9). It may be proved that the tensor

$$\mathsf{R} = (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}'_j)\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{1.34}$$

is, in fact, a rotation tensor with the property that

$$\hat{\mathbf{e}}'_i = \mathsf{R} \cdot \hat{\mathbf{e}}_i. \tag{1.35}$$

This represents the geometrically intuitive fact that, because both $\mathscr{O}$ and $\mathscr{P}$ are right-handed Cartesian coordinate systems, it is possible to obtain one from the other by a rotation. The components of R are

$$R_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}'_j. \tag{1.36}$$

Substituting the previous two equations into (1.1) and (1.9), we obtain the coordinate transformation rules

$$[\mathbf{a}] = [\mathsf{R}][\mathbf{a}]', \quad \text{so that} \quad a_i = R_{ij}a'_j \tag{1.37a}$$

and

$$[\mathsf{A}] = [\mathsf{R}][\mathsf{A}]'[\mathsf{R}]^T, \quad \text{so that} \quad A_{ij} = R_{ik}A'_{kl}R_{jl}, \tag{1.37b}$$

for vector and second-order tensor components, respectively. As a special case, when $\mathsf{A} = \mathsf{R}$, we find that

$$[\mathsf{R}] = [\mathsf{R}]', \quad \text{so that} \quad R_{ij} = R'_{ij}, \tag{1.38}$$

i.e., the rotation tensor R has the same components in the two frames that it relates.

Higher-order tensor transformation formulae may be developed similarly when required.

1.5 Calculus

1.5.1 Gradient and Divergence. Taylor's Theorem

Let $\Phi(\mathbf{x})$ be a nth-order *tensor field* defined over three-dimensional space with $\mathbf{x}$ being a position vector. The *gradient* of Φ is defined by

$$\nabla \Phi = \frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} = \Phi,_i \hat{\mathbf{e}}_i, \tag{1.39}$$

where the comma denotes differentiation and

$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial \mathbf{x}} = \frac{\partial(\cdot)}{\partial x_i}\hat{\mathbf{e}}_i = \frac{\partial(\cdot)}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial(\cdot)}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial(\cdot)}{\partial x_3}\hat{\mathbf{e}}_3$$

is the *gradient operator*. In particular, we have the formulae

$$\nabla a = \frac{\partial a}{\partial x_i}\hat{\mathbf{e}}_i, \quad \nabla \mathbf{b} = \frac{\partial b_i}{\partial x_j}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \quad \text{and} \quad \nabla \mathsf{C} = \frac{\partial C_{ij}}{\partial x_k}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k, \tag{1.40}$$

where a, $\mathbf{b}$ and C are, respectively, scalar, vector and tensor fields. Contracting any two free indices of the gradient of Φ, we obtain its *divergence* denoted by $\nabla \cdot \Phi$; this, therefore, is applicable only when Φ is not a scalar field. We obtain the formulae

$$\nabla \cdot \mathbf{b} = \frac{\partial b_i}{\partial x_i} = b_i,_i \quad \text{and} \quad \nabla \cdot \mathsf{C} = \frac{\partial C_{ij}}{\partial x_j}\hat{\mathbf{e}}_i = C_{ij},_j \hat{\mathbf{e}}_i. \tag{1.41}$$

Note that in the last equation we could have alternatively defined $\nabla \cdot \mathsf{C}$ as $C_{ij},_i \hat{\mathbf{e}}_j$. The gradient of a field identifies the direction of steepest change through its eigenvectors, while its divergence is an estimate of the field's local flux.

A field $\Phi(\mathbf{x})$ may be expanded in a Taylor's series about a location $\mathbf{x}_0$ provided some smoothness conditions are satisfied; see, e.g., Sokolnikoff (1980, p. 311). The expansion may be expressed as