Fundamentals of Mechanical Vibrations

Liang-Wu Cai

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This Work is a co-publication between ASME Press and John Wiley & Sons, Ltd.

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Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

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Library of Congress Cataloging-in-Publication Data

Names: Cai, Liang-Wu author.
Title: Fundamentals of mechanical vibrations / Liang-Wu Cai.
Description: Hoboken : John Wiley & Sons, Inc., 2016. | Includes bibliographical references and index.
Identifiers: LCCN 2015044273 (print) | LCCN 2015045689 (ebook) | ISBN 9781119050124 (cloth) | ISBN 9781119050230 (pdf) | ISBN 9781119050223 (epub)
Subjects: LCSH: Machinery–Vibration. | Vibration–Mathematical models.
Classification: LCC TJ177 .C35 2016 (print) | LCC TJ177 (ebook) | DDC 621.8/11–dc23
LC record available at http://lccn.loc.gov/2015044273

A catalogue record for this book is available from the British Library.

Cover image: Getty/odmeyer

Set in 10/12pt, TimesLTStd by SPi Global, Chennai, India.

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Series Preface

The Wiley-ASME Press Series in Mechanical Engineering brings together two established leaders in mechanical engineering publishing to deliver high-quality, peer-reviewed books covering topics of current interest to engineers and researchers worldwide.

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Preface

Why a new book on mechanical vibrations?

Mechanical vibration is a core subject in mechanical engineering that has been taught for many decades, with many old classics and many more excellent contemporary textbooks available in a rather crowded market. However, after teaching this subject for several years, I feel that there are two major challenges facing the current generation of engineering students studying vibrations, and, till this day, I am still unable to identify a textbook that satisfactorily addresses these challenges. This book represents my efforts in addressing these challenges.

A vibration analysis of a system starts from the equation(s) of motion for the system. The first major challenge facing the students is reaching this starting point. Although a prerequisite, the undergraduate Newtonian dynamics course does not adequately prepare students to derive the equation(s) of motion for a system that is slightly more complex than a mass–spring–dashpot system. Without teaching them on how to get from "here" (a given system) to "there" (the equations of motion), most textbooks parade through the analyses starting from "there."

In my view, Lagrangian dynamics is the ideal approach to reach this starting point. This view can be corroborated by the fact that almost all vibration textbooks contain a brief section, often in less than 10 pages, on Lagrangian dynamics, often just before vibration analyses of multi-degree-of-freedom systems. Lagrangian dynamics is typically a graduate-level topic that is rather mathematical and abstract for undergraduate students. The abstract nature compounded with the brevity in coverage does not prepare the students to reach the starting point with confidence.

I am partial to Lagrangian dynamics in part because of where I came from. When I was a doctoral student at Massachusetts Institute of Technology, for many years, I was a teaching assistant for the dynamics course in which Lagrangian dynamics was taught to sophomores, whose previous exposure to dynamics was college physics. Professor James H. Williams, Jr., who taught the course at that time, was also writing his own textbook on Lagrangian dynamics (*Fundamentals of Applied Dynamics*, John Wiley & Sons, 1996), which was the culmination of his years of award-winning teaching of Lagrangian dynamics to undergraduate students from a classic textbook (*Dynamics of Mechanical and Electromechanical Systems*, by Crandall, Karnopp, Kurtz, and Pridmore-Brown, McGraw-Hill, 1968). I was also helping him on the preparation for the solutions manual. I observed how such a difficult topic was taught to sophomores without losing its rigor. That experience gave me the belief that Lagrangian dynamics can be taught at undergraduate level, and Professor Williams had presented a successful paradigm in his textbook.

Another major challenge facing students is the symbolic analysis. In a vibration analysis, the solution is almost always in the form of an analytical expression. The current generation of engineering students, with sophisticated scientific calculators at their handy disposal, have grown comfortable with numbers: given a problem with a set of numbers, they apply a solution process to yield another number called the *answer*. The answer, right or wrong, represents a closure, as well as a sense of accomplishment. But in a symbolic analysis, the lack of numerical values given for the parameters in the problem appears to pull them away from the physical sense of the system, and the lack of a numerical answer makes them feel uncertain whether they have reached the final solution to the problem.

In my view, a graphical representation of an analytical solution could fill in as "the answer" to a symbolic problem. Graphing an analytical expression involves mundane details and some important steps such as nondimensionalizing the expression. The mundane details can be eliminated by incorporating mathematical software such as MatLab into the learning process, essentially giving the graphing capability in their handy disposal. This could reduce their uneasiness toward symbolic analysis and eventually learn this vitally important skill.

I feel that the introductory course on vibration is a perfect place to start training students for this skill. On the one hand, by junior or senior years, students have acquired a sense for the importance of the symbolic analysis: they have the motivation. On the other hand, many traditional vibration analysis topics can be enlivened by the computational capabilities brought by MatLab. Topics such as steady-state responses due to general periodic loadings by using Fourier expansions and the convolution integral as a general-purpose numerical method for an arbitrary transient loading can benefit from numerical computations.

Empowering students with these skills and capabilities does not require the commitment of a significant amount of time and effort, but requires the right instructional materials. After teaching the vibration course at Kansas State University for a couple of years using different textbooks, I started writing my own lecture notes.

Starting from a clean slate, I focused on the following set of learning objectives that I deem the most essential for prospective engineers:

- Being able to reach the starting point of a vibration analysis with confidence.
- Being able to establish a simple mathematical model for vibration analysis of real-world structures.
- Being able to handle the mathematics of vibration analyses, and, with the aid of a computer if necessary, to interpret the physical meanings of the results.
- Being able to tackle mathematical analysis symbolically.
- Being ready to use more powerful simulation tools when the needs call for.

The materials presented in this book are accumulated and refined over the past 10-plus years. To help the students achieving the above set of learning objectives, this textbook incorporates the following carefully designed pedagogical features:

- Staying true to the fundamentals. This book emphasizes providing thorough and clear explanations to the fundamental concepts and theories. It does not stray into advanced topics that are more suitable for advanced studies in graduate school. The time (and space) saved can be better spent on honing in the skills of symbolic analysis and the capabilities afforded by MatLab.

- Treating Lagrangian dynamics with rigor while maintaining accessibility. By limiting the consideration to Newtonian particles and rigid bodies, Lagrangian dynamics is tuned down its generality and the associated mathematical complexity. Students having the prerequisite of Newtonian dynamics are offered a fresh perspective. One of the recurring comments I received from students is that they "finally learned the dynamics!"
- Provoking higher level learning and thinking by enforcing symbolic analysis. MatLab is used initially as merely a tool for visualizing analytical solutions. By limiting the use of MatLab to this rudimentary and humble goal, students are greatly empowered in their analytical skills.
- Instilling the engineers' philosophy that having a solution to a problem is not the end to the problem; in contrary, it is the beginning of an exploration. In every vibration example, a section entitled *Exploring the Solution with MatLab* is dedicated to curiosity-driven observations and explorations.
- Expanding the horizon by exploiting the capabilities of MatLab. MatLab is gradually used as a computational tool for analyzing more complicated problems such as modeling a fascinating phenomenon of levitating Slinky. In the end, a complete finite element analysis code is developed in MatLab for vibration analysis of one-dimensional beams.
- Providing a consistent approach to modeling engineering systems. This book offers the following components to prepare the students for the industry:
  - A systematic procedure for establishing lumped-parameter models for simple engineering structures and systems. This gives students the skill to produce a quick and reasonably accurate estimate of practical problems.
  - The same lump-parameter modeling procedure is used for finite element formulation. This serves two purposes: it gives students an understanding of the working principle of the finite element method; and it boosts students the confidence on the effective lump-parameter modeling.
  - A set of illustrated tutorials for vibration analyses of a real structure using a commercial finite element analysis software package is also provided.

It is my sincere hope that this new textbook will bring a new breeze to the market that is crowded with many excellent textbooks over the past few decades. I wholeheartedly welcome all criticisms and suggestions to make it a better textbook.

In writing this book, first and foremost, I am greatly indebted to my former advisor at MIT, Professor James H. Williams, Jr., for offering me the opportunity as the teaching assistant for his class and as an assistant for preparing his textbook. His teaching of dynamics forms the basis of my firm belief that Lagrangian dynamics should be the starting point of the book, which inevitability bears the signature of his paradigm. Over the years, various draft versions of this book have been used in my class, and many students have provided valuable feedback. In particular, Ms. Congrui Jin and Mr. Ryan Cater offered extensive corrections to early versions. I would also like to thank Mr. Paul Petralia, senior acquisitions editor at Wiley, for his patience and encouragements, and Clive Lawson (UK), Preethi Belkese (India), and other editors at Wiley for their assistance. Finally, I would like to thank my wife, Huimin, for her unconditional support and love during this endeavor.

Liang-Wu Cai
September 2015

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A Crash Course on Lagrangian Dynamics

1.1 Objectives

This chapter presents the fundamental concepts in Lagrangian dynamics and outlines a procedure for deriving the equation(s) of motion for holonomic mechanical systems using Lagrange's equation. Extensive examples are presented to cover a large variety of mechanical systems containing particles and rigid bodies. Finally, the procedures for finding the equilibrium position(s) and linearizing the equation(s) of motion in preparation for vibration analysis are also presented.

1.2 Concept of "Equation of Motion"

An *equation of motion* for a mechanical system is a differential equation that governs the changes in positions of components in the system with respect to time.

There are three key phrases in the above definition. Phrases *differential equation* and *with respect to time* specify a particular mathematical form for the equation and distinguish the equation of motion from its solution. That is, an equation of motion is a differential equation involving time derivatives. The phrase *positions* specifies a particular kinematic quantity, the most fundamental one from which other measures of motion, namely, displacements, velocities, and accelerations, can be obtained.

The concept of the *equation of motion* has suffered a degree of misuse in some dynamics textbooks. In these textbooks, Newton's second law, which many of us conveniently recite as $\boldsymbol{F} = m\boldsymbol{a}$, is called the equation of motion. In fact, Newton's second law is the fundamental physical law that governs the motion of any physical system. It is often a crucial tool to use in obtaining the equation(s) of motion for a system. But it is too primitive a form to be called an equation of motion. In our daily lives, we do not call a foundation as a structure. The same goes here.

Fundamentals of Mechanical Vibrations, First Edition. Liang-Wu Cai.

Companion Website: www.wiley.com/go/cai/fundamentals_mechanical_vibrations

In the mean time, we must keep our minds open to the notion that there are other ways to establishing the equation of motion for a system. *Lagrangian dynamics* is one such alternative.

Before jumping into the details of Lagrangian dynamics, let us look at the way the equation of motion for a system can be obtained by using Newton's second law.

■ Example 1.1: Simple Mass–Spring–Dashpot System

Derive the equation of motion for the mass–spring–dashpot system as shown in Fig. 1.1.

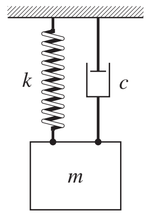

Figure 1.1 Mass–spring–dashpot system

□ Solution 1:

- Define x as the downward displacement of the mass, measured from its position when the spring is unstretched.
- *Kinematics of Mass m*: This example is simple enough to allow us to write directly:

$$\text{velocity} = \dot{x} \qquad \text{and} \qquad \text{acceleration} = \ddot{x} \tag{a}$$

  where, as in dynamics, an overhead dot represents the time derivative, and overhead double-dot represents the second derivative with respect to time t.
- *Kinetics of Mass m*: A free-body diagram can be drawn, as in Fig. 1.2, to show all the forces acting on the mass in a generic instant in time when the system is in motion. In Fig. 1.2, F_s and F_d are the forces exerted by the spring and the dashpot, respectively. Applying Newton's second law based on the free-body diagram in Fig. 1.2 gives

$$mg - F_s - F_d = m\ddot{x} \tag{b}$$

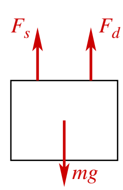

Figure 1.2 Free-body diagram for mass m at a generic time instant

- *Constitutive Relations*: Besides the mass, the system contains a spring and a dashpot, whose forces are proportional to the deflection and the velocity, respectively. That is,

$$F_s = kx \qquad \text{and} \qquad F_d = c\dot{x} \tag{c}$$

- Substituting eqns. (c) into eqn. (b) gives, with a slight rearrangement,

$$m\ddot{x} + c\dot{x} + kx = mg \tag{d}$$

which is the equation of motion for this system.

□ **Solution 2:**

- Define y as the downward displacement of the mass, measured from its static equilibrium position.
- *Kinematics of Mass m*: Kinematics of mass m is unchanged, except replacing x by y.
- *Kinetic of Mass m*: The free-body diagram remains the same as in Fig. 1.2; and hence Newton's second law gives

$$mg - F_s - F_d = m\ddot{y} \tag{e}$$

- *Constitutive Relations* for system components are

$$F_s = k(y + \Delta) \qquad \text{and} \qquad F_d = c\dot{y} \tag{f}$$

where we note that the spring is already stretched at equilibrium, denoted as Δ, and that the spring force is proportional to the total amount of deformation in the spring.
- Substituting eqn. (f) into eqn. (e) gives

$$mg - ky - k\Delta - c\dot{y} = m\ddot{y} \tag{g}$$

- To eliminate Δ from eqn. (g), we look at the static equilibrium: the dashpot does not exert any force to the mass. The free-body diagram for the mass in its static equilibrium state is shown in Fig. 1.3. The equilibrium requires

$$k\Delta = mg \tag{h}$$

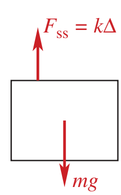

Figure 1.3 Free-body diagram for mass m at the static equilibrium

- Combining eqns. (g) and (h) gives, after a slight rearrangement,

$$m\ddot{y} + c\dot{y} + ky = 0 \tag{i}$$

which is the equation of motion for the system.

________ · ________

This example illustrates a typical process for deriving the equation(s) of motion for a mechanical system by using Newton's second law. Through this example, we can make the following observations:

- Before we can proceed to deriving the equation, some variables (such as x and y) must be defined so that the locations of the system's components can be described. In the end, the equations of motion for the system are differential equations of these variables. Such variables are called the *generalized coordinates*. We shall study this concept more thoroughly and rigorously in the next section.
- Different definitions of the generalized coordinates result in different equations of motion for the same system. That is, the equations of motion are not unique, depending on the choice of the generalized coordinates by which the system is described.
- The following three pieces of information are generally needed: (1) kinematics of the system components at a generic time instant during the motion; (2) the properties of system components, which are called the *constitutive relations*; and (3) Newton's second law that threads all these pieces of information together.

However, there are some shortcomings in this approach of deriving the equations of motion for a system. For example, how do we know how many equations should be obtained? For a complex system, which part or parts of the system should be isolated to draw the free-body diagrams? Finding the answer to these questions is *ad hoc*: we have to look into individual systems case by case. This means that we may be able to solve one problem effortlessly, but we might stumble on the next. We all have experienced the situations in which drawing a free-body diagram reveals more unknowns than desired and calls for drawing more free-body diagrams and writing out more equations. We may also recall that, in kinematics, finding the acceleration for a particle or a point in a rigid body involves substantially more work than finding the velocity.

Lagrangian dynamics avoids most of these issues and provides a structured way to analyze a system and to subsequently obtain its equation(s) of motion. It works in a way similar to the energy method: only positions and velocities are required in the formulation. The procedure is unchanged regardless of the system's complexity. Furthermore, it can be extended to handle systems in other physical domains, such as electrical, electromagnetic, and electromechanical systems.

Deriving the equation of motion is the first step toward understanding a system. Having the equation(s) of motion in hand, the first and foremost, we can find and subsequently analyze the solution to the equation(s) of motion. This plainly stated activity, in fact, encompasses almost all subjects of study in mechanical engineering, including vibration analyses. As we shall see, vibration analyses are to find, analyze, and study the solutions to a particular class of equations of motion.

Having the equation of motion, before putting all the efforts into finding the solution, a few less ambitious things having great engineering interests can be done:

- Determine the system's equilibrium configuration(s). In the equilibrium state, the system does not move at all. For the above example, setting the velocity and acceleration to zero, the equation of motion derived in *Solution 1* would become

$$kx_{\mathrm{eq}} = mg \qquad \text{or} \qquad x_{\mathrm{eq}} = \frac{mg}{k}$$

  where x_{eq} denotes the position of the mass at equilibrium. The equation of motion derived in *Solution 2* would give $y_{\mathrm{eq}} = 0$, as expected.
- Analyze the dynamic stability and other associated behaviors of the system.

- Introduce approximations to simplify the equation(s) of motion or to obtain approximate solutions.

We discuss these topics separately later in this chapter. In fact, these things must be explored before we can proceed with vibration analyses of a system.

1.3 Generalized Coordinates

Coordinates usually are associated with a particular coordinate system and usually appear in a set. For instance, in a Cartesian coordinate system, the coordinates for a point are (x, y, z). In a polar coordinate system, the coordinates are (r, θ). Adding the word *generalized* frees us from abiding to any particular coordinate system so that we can choose whatever parameter that is convenient to describe the position of a point in a system. Hence,

A *generalized coordinate* is a parameter that is used to locate a part of a system. A generalized coordinate is a scalar quantity.

A group of parameters that is used to locate a system is a *set* of generalized coordinates. To denote a set of generalized coordinates, we follow the mathematical notation for explicitly defining a set by listing all elements contained in the set. We write, for example, $\{x_1, x_2\}$.

A *complete set* of generalized coordinates is a set of generalized coordinates that completely locates all parts of a system in all geometrically admissible configurations. A *geometrically admissible configuration* is a configuration that is allowed by the geometrical constraints in the system.

An *independent set* of generalized coordinates is a set of generalized coordinates in which if all but one of them are fixed, there still exists a continuous range of values for that unfixed generalized coordinate.

A *complete and independent set* of generalized coordinates is, literally, a set of generalized coordinates that is both complete and independent.

Because generalized coordinates can be chosen or defined almost at will, generalized coordinates for a system are not unique. For this same reason, it is utterly important to define them clearly, aided by a schematic depiction if necessary. When defining a generalized coordinate, usually the following four vital aspects should be clearly indicated for each generalized coordinate: the exact physical meaning, where it is measured from, and relative to whom, and which direction is positive.

■ Example 1.2: Particle Moves Freely in Three-Dimensional Space

A particle can move freely in a three-dimensional space. Define a set of generalized coordinates and subsequently judge whether it is a complete and independent set of generalized coordinates.

□ Solution:

Before we start, as a preparatory setup, we define a Cartesian coordinate system to facilitate the discussions that follow. The Cartesian coordinate system is fixed in space.

We define $\{x, y, z\}$ as a set of generalized coordinates, where x, y, and z are the Cartesian coordinates of the particle. Note that the positive directions for the coordinates have already been defined in the Cartesian coordinate system.

- *Is It a Complete Set?* YES, because there is only one particle in the three-dimensional space, the three coordinates completely specified a point in the three-dimensional space.
- *Is It an Independent Set?* To answer this question, we need to conduct a series of tests: if all but one of the generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to change. In this problem, if x and y are fixed, z can still be freely changed along a line parallel to the z-axis. Similar conclusion can be drawn for leaving any other coordinate unfixed. So, the answer to this question is also YES.

Therefore, we conclude that $\{x, y, z\}$ as defined is a complete and independent set of generalized coordinates.

□ Discussion: Using a Coordinate System

When using coordinates in a well-established coordinate system, such as the Cartesian coordinate system in this example, as the generalized coordinates, in general, the coordinate system has already included the specifications for the directions of positive coordinates. An important task in defining a coordinate system is to specify how its origin moves, such as relative to whom, and how the coordinate axes are oriented.

■ Example 1.3: Simple Planar Pendulum

A simple pendulum is made up by a particle hung to a pivot point through a massless string. A planar pendulum is a simple pendulum whose motion is restricted to within a plane, typically the plane of paper, as shown in Fig. 1.4. Assume that the string of length l remains taut at all times. Define a set of generalized coordinates for this pendulum and subsequently judge whether it is a complete and independent set of generalized coordinates.

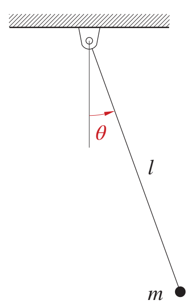

Figure 1.4 Simple planar pendulum

□ Solution 1: Angular Displacement

We define $\{\theta\}$ as a set of generalized coordinates, where θ is the angular displacement of the pendulum, measured in the counterclockwise direction from the vertical, as shown in Fig. 1.4. Note that, in the sketch, the arrow for θ goes only in one direction, indicating its positive direction.

- *Is It a Complete Set?* YES, because there is only one particle in the system and it can only move around a circle of radius l centered at the pivot point. Once the angle θ is determined, the location of the particle is uniquely determined.

- *Is It an Independent Set?* To answer this question, we need to conduct a test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary? In this case, there is only one generalized coordinate. When "all but one are fixed," actually nothing is fixed, and the unfixed one is θ. As the particle is allowed to move along the circle of radius l, θ indeed has a continuous range to vary. So, the answer to this question is also YES.

Therefore, we conclude that $\{\theta\}$ as defined is a complete and independent set of generalized coordinates.

□ Discussion: Geometrically Admissible Configurations

The definition for the completeness of a set of generalized coordinate includes a qualifying term "geometrically admissible configurations." However, this phrase is not mentioned in the preceding discussions. A geometrically admissible configuration requires the pendulum remain in the plane of paper and the string being taut. These constraints have been implicitly invoked when we say "the pendulum can only move around a circle."

□ Solution 2: Cartesian Coordinates

In the preparatory setup, we define a Cartesian coordinate system Oxy such that its origin O is fixed at the pivot point, the x-axis points horizontally to the right, and the y-axis points vertically upward.

We then define $\{x, y\}$ as a set of generalized coordinates, where x and y are Cartesian coordinates of the mass of the pendulum.

- *Is It a Complete Set?* YES, because there is only one particle that can only move within the plane of paper: once the Cartesian coordinates x and y are specified, the location of the mass is completely determined.
- *Is It an Independent Set?* To answer this question, we need to conduct the following test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary? If we fix x, since the particle can only move along the dotted circle shown in Fig. 1.5, y can be located at two possible positions as indicated: at the intersections of the vertical line and the circle. However, they are two *discrete* positions, and do not constitute a *continuous range*. This suffices to give NO as the answer to this question without conducting any further test.

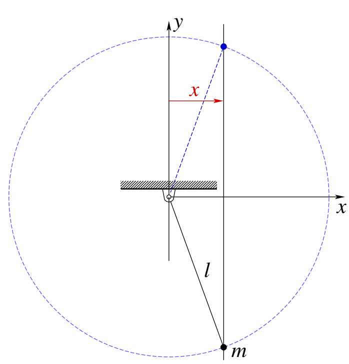

Figure 1.5 Two possible positions for the pendulum when x is fixed

Therefore, we conclude that $\{x, y\}$ as defined is not a complete and independent set of generalized coordinates. Specifically, it is a complete set but not an independent set.

□ Solution 3: Cartesian Coordinates, Remedied

Although the problem statement does not ask us to define a complete and independent set of generalized coordinates, we are still curious as how to remedy the set we just defined to make it a complete and independent set.

We define $\{x\}$ as a set of generalized coordinates, where x is the x-coordinate of the mass of the pendulum in the Cartesian coordinate system as defined in *Solution 2*.

- *Is It a Complete Set?* When x is fixed, as discussed earlier, there are two possible positions for the pendulum. In the Cartesian coordinate system, the y-coordinate can be found from the following relation:

$$x^2 + y^2 = l^2 \qquad \text{or} \qquad y = \pm\sqrt{l^2 - x^2} \tag{a}$$

  This identifies two possible y values in Fig. 1.5. The exact locations of the two possible positions are completely determined. We consider the answer to this question is YES.
- *Is It an Independent Set?* To answer this question, we need to conduct the following test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary? In this case, if we fix all but x, nothing is actually fixed, and x can vary in the range $-l \leq x \leq l$. Thus, the answer to this question is YES.

Therefore, we conclude that $\{x\}$ as defined is a complete and independent set of generalized coordinates.

□ Discussion

The condition such as the one in eqn. (a) is called a *geometric constraint* of the system. The question about which of the two positions the particle is located at a given time will be answered or become apparent by other information of the system under consideration, such as the initial conditions of the system. A mechanical system moves continuously in space from one location into an adjacent one.

■ Example 1.4: Rigid Slender Rod Moves in a Plane

A rigid slender rod of length L is allowed to move freely on the plane of paper. Define a set of generalized coordinates and subsequently judge whether it is a complete and independent set of generalized coordinates.

□ Solution:

In the preparatory setup, we define a Cartesian coordinate system Oxy that is fixed in space, as shown in Fig. 1.6.

We then define $\{x_1, x_2, y_1, y_2\}$ as a set of generalized coordinate, where x_1 and y_1 are the Cartesian coordinates of the left end of the rod and x_2 and y_2 are the Cartesian coordinates of the other end of the rod.

- *Is It a Complete Set?* YES, because once the two ends of the rod are fixed, every point on the rod can be located.

- *Is It an Independent Set?* To answer this question, we need to conduct the following test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary? We first fix x_1, y_1, and x_2 and leave y_2 unfixed. As a rigid body, the length of the rod, L, is fixed. This geometrical constraint gives y_2 as

$$y_2 = y_1 \pm \sqrt{L^2 - (x_1 - x_2)^2} \tag{a}$$

There are two possible values for y_2, as indicated by the $\pm$ sign, as shown in Fig. 1.7. They are two discrete values, but do not constitute a continuous range of values. Thus, the answer to this question is NO.

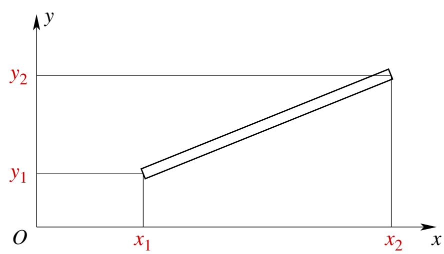

Figure 1.6 Rigid slender rod moves on the plane

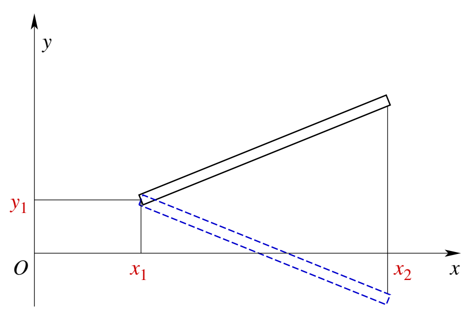

Figure 1.7 Two possible locations of the rod when x_1, y_1, and x_2 are fixed

Therefore, we conclude that $\{x_1, x_2, y_1, y_2\}$ as defined is not a complete and independent set of generalized coordinates. Specifically, it is a complete set but not an independent set.

□ **Remedied Solution:**

To define a set of complete and independent set, we only need to pick out any three of the four generalized coordinates, knowing that the fourth can be obtained from the geometrical constraint in eqn. (a).

In this solution, we pick out $\{x_1, y_1, x_2\}$ as a set of generalized coordinates. Their definitions are exactly the same as before and would not be repeated here.

- *Is It a Complete Set?* YES. As analyzed before, when needed, the remaining coordinate y_2 can be found from eqn. (a). Then, the exact location of the rod is completely determined.

- *Is It an Independent Set?* We cannot take for granted that such a choice would automatically be a complete and independent set. We still need to conduct a series of tests.
  - First, we fix x_1 and y_1 and leave x_2 unfixed. The possible scenario is that the rod can rotate about its left end, while its right end moves along a circle, as shown in Fig. 1.8. Along this circle, x_2 certainly has a continuous range to vary. Furthermore, the range is between the extreme ends of the circle, that is, $x_1 - L \leq x_2 \leq x_1 + L$.

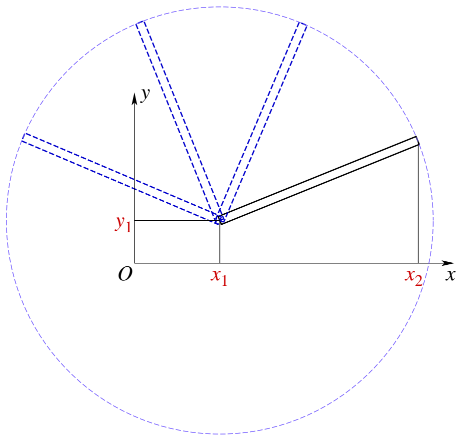

Figure 1.8 Possible locations of the rod when x_1 and y_1 are fixed

  - Next, we fix x_1 and x_2 and leave y_1 unfixed. This allows the rod to slide up and down along a vertical strip confined by x_1 and x_2, as shown in Fig. 1.9. Thus, y_2 has a continuous range to vary. Furthermore, the range is unlimited: $-\infty < y_1 < \infty$.

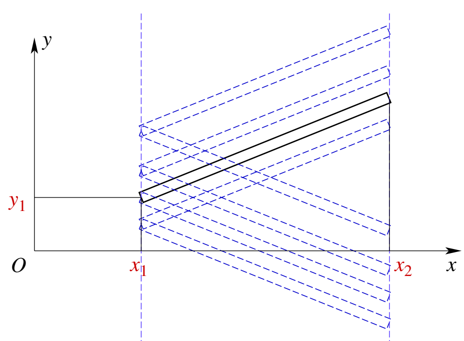

Figure 1.9 Possible locations of the rod when x_1 and x_2 are fixed

  - Lastly, we fix y_1 and x_2 and leave x_1 unfixed. The scenario is similar to a ladder sliding along a wall–floor corner while remaining in simultaneous contacts with both the wall and the floor. In this analogy, the "wall" is defined by the vertical lines $x = x_2$; while the "floor" is defined by the horizontal line $y = y_1$. The difference is that the image mirrored either by the "wall" or by the "floor" is also valid, as sketched in Fig. 1.10. Thus, x_1 has a continuous range to vary. Furthermore, the range is $x_2 - L \leq x_1 \leq x_2 + L$.

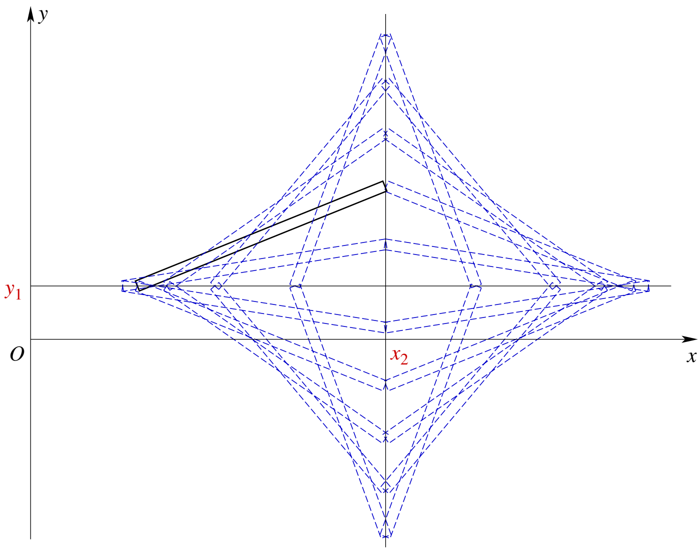

Figure 1.10 Possible locations of the rod when y_1 and x_2 are fixed

This completes the independence test. In all three tests, the unfixed coordinate always has a continuous range of values to vary. Thus, the answer to this question is YES.

We conclude that $\{x_1, y_1, x_2\}$ as defined is a complete and independent set of generalized coordinates.

■ Example 1.5: Particle Moves Along a Wire of Known Shape

A particle moves along a wire on the *xy*-plane of a known shape given by

$$y = a + bx^2$$

as shown in Fig. 1.11. Define a complete and independent set of generalized coordinates.

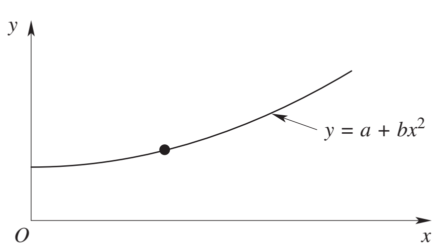

Figure 1.11 Particle moves on a wire of known shape

□ Solution:

Since a Cartesian coordinate system has already been defined, it is convenient to choose the Cartesian coordinates x and y of the particle as the generalized coordinates. However, Example 1.4 gives us sufficient reason to pause before jumping into making the declaration. A careful examination suggests that we can only choose one of the coordinates, and the function describing the wire shape provides the other coordinate.

Therefore, we define $\{x\}$ as a set of generalized coordinates, where x is the x-coordinate of the particle in the Cartesian coordinate system Oxy.

- *Is It a Complete Set?* YES, as we have just analyzed above.
- *Is It an Independent Set?* We still need to conduct the test for the independence. If all but one fixed, the only possibility is to leave x unfixed, and nothing is fixed. The particle is still free to move along the wire.

Therefore, we conclude that $\{x\}$ as defined is a complete and independent set of generalized coordinates.

■ Example 1.6: Disk Rolls Without Slip on Ground

A circular disk of radius R rolls without slip on a horizontal ground. Define a complete and independent set of generalized coordinates.

□ Solution:

In the preparatory setup, we paint a radius on the disk between its center and the contact point with ground in a reference configuration. We also define a Cartesian coordinate system Oxy such that its origin is fixed at the contact point in the reference configuration and its x-axis lies on the ground. Figure 1.12 shows both the reference and the displaced configurations of the disk. In the displaced configuration, C is the center of the disk, A is the contact point, and B is the contact at the reference configuration. Angle θ denotes the angular displacement of the painted radius.

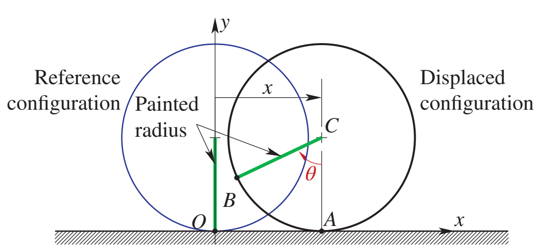

Figure 1.12 Disk rolls without slip on horizontal ground, showing reference and displaced configurations

We can now define the x-coordinate of the center C as a generalized coordinate. Once x is fixed, the location of the center C is completely determined. As a rigid body, locating the center is not sufficient. We need to fix its orientation. The angle θ seems to be a perfect candidate for another generalized coordinate. However, since the disk rolls without slip, it requires that the length OA equals to the arc length $\widehat{AB}$. Since length OA equals x, given the radius R of the disk, θ can be uniquely determined.

Therefore, we define $\{x\}$ as a set of generalized coordinate, where x is the x-coordinate of the center of the disk in the Cartesian coordinate system Oxy.

- *Is It a Complete Set?* YES. As we have just analyzed, this is a complete set.
- *Is It an Independent Set?* We need to conduct the following test: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary. If we fix all but x, actually nothing is fixed, the disk can roll on the ground, and hence x has a continuous range to vary. Thus, the answer to this question is YES.

Finally, we conclude that $\{x\}$ as defined is a complete and independent set of generalized coordinates.

■ Example 1.7: Disk Rolls on Ground, Slipping Allowed

A circular disk of radius R rolls on a horizontal ground. The disk is allowed to slip while rolling. Define a complete and independent set of generalized coordinates.

□ Solution:

The preparatory setup is the same as in Example 1.6.

We define $\{x, \theta\}$ as a set of generalized coordinate, where x is the x-coordinate of the center of the disk and θ is the clockwise angular displacement of the painted radius.

- *Is It a Complete Set?* Based on the analysis in Example 1.6, we can readily conclude that, YES, it is a complete set.
- *Is It an Independent Set?* We need to conduct the following series of tests: if all but one generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to vary. We first fix x and allow θ to vary. Since slipping is allowed, when the center is fixed, the disk can still rotate just like a spinning wheel of a car on a jack stand. If we fix θ and allow x to vary, because slipping is allowed, the disk can slide without rotation, just like a locked wheel skidding on ice. So, YES, it is an independent set.

Therefore, we conclude that $\{x, \theta\}$ is a complete and independent set of generalized coordinates.

1.4 Admissible Variations

A *variation* is a hypothetical small change in a generalized coordinate.

A variation is a close cousin of a *virtual displacement*. Let us discuss the virtual displacement first. A virtual displacement differs from a real displacement in two aspects: (1) It occurs instantaneously without advancing the time. Or, put it differently: it is the difference between the real position and an alternative position, as if we say to ourselves: "what if at this particular moment this point is located there instead of here." (2) It is *small* in the sense of a differential change.

Now let us look at the relation between a virtual displacement and a variation. A real change in position is a *displacement* (vector). Since the system is described by the chosen complete and independent set of generalized coordinates, this displacement is expressible in terms of this set of generalized coordinates (scalars). A virtual displacement is a *hypothetical* and *small* displacement. It is expressible in *hypothetical* and *small* changes in generalized coordinates, which are defined as *variations*. Normally, displacements are vectors, so are the virtual displacements. Generalized coordinates are scalars, so are the variations.

An *admissible variation* is a hypothetical small change in a generalized coordinate that is allowed by the geometrical constraints of the system.

Since the admissible variations are associated with the generalized coordinates, once a set of generalized coordinates has been defined, a set of admissible variations can be naturally derived. In other words, it does not need to be defined. For a set of generalized coordinates, we usually write, for example, $\{\delta x_1, \delta x_2\}$ as the associated set of admissible variations. Mathematically, the variation operator δ follows the same rules as the differential operator d. But, what is important is to conduct similar tests to see whether the set of admissible variations is *complete and independent*. Passing both completeness and independence tests makes the set a *complete and independent set* of admissible variations.

■ Example 1.8: Rigid Slender Rod Moves on Plane, Variations

Determine whether the set of variations associated with the set of generalized coordinates defined in Example 1.4 is a complete and independent set of admissible variations.

□ Solution:

In Example 1.4, a complete and independent set of generalized coordinate has been defined as $\{x_1, y_1, x_2\}$, where x_1, x_2, and y_1 are all Cartesian coordinates. The corresponding set of admissible variations is $\{\delta x_1, \delta y_1, \delta x_2\}$. Being "admissible" means that the varied configuration remains entirely on the Oxy plane.

Assume a varied configuration of the system, shown as the dashed configuration in Fig. 1.13. The left end is shown in an enlarged view, in which $\delta \boldsymbol{r}_A$ is its virtual displacement; δx_1 and δy_1 are the admissible variations associated with x_1 and y_1, respectively. They, along with x_1 and y_1, locate the left end of rod in the varied configuration. Similarly, δx_2 and x_2 locate the x-coordinate of the right end of the rod in the varied configuration. With the geometrical constraint of a fixed length L, the right end of the rod in the varied configuration is thus located. Consequently, every point in the rod in the varied configuration can be located. Thus, this set of admissible variations is complete.

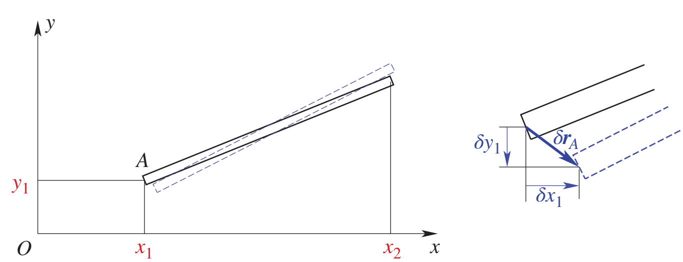

Figure 1.13 Varied configuration of slender rod on plane. The enlarged view of left end A on the right shows virtual displacement $\delta \boldsymbol{r}_A$ being represented by variations δx_1 and δy_1