WILEY HANDBOOKS IN
FINANCIAL ENGINEERING
AND ECONOMETRICS

HANDBOOK OF HIGH-FREQUENCY TRADING AND MODELING IN FINANCE

EDITED BY

Ionut Florescu
Maria C. Mariani
H. Eugene Stanley
Frederi G. Viens

WILEY

Handbook of High-Frequency Trading and Modeling in Finance

Wiley Handbooks in

FINANCIAL ENGINEERING AND ECONOMETRICS

Advisory Editor

Ruey S. Tsay

The University of Chicago Booth School of Business, USA

A complete list of the titles in this series appears at the end of this volume.

Handbook of High-Frequency Trading and Modeling in Finance

Edited by

IONUT FLORESCU
MARIA C. MARIANI
H. EUGENE STANLEY
FREDERI G. VIENS

WILEY

Published by John Wiley & Sons, Inc., Hoboken, New Jersey
Published simultaneously in Canada

Library of Congress Cataloging-in-Publication Data:

Names: Florescu, Ionut, 1973- editor.
Title: Handbook of high-frequency trading and modeling in finance / edited by Ionut Florescu, Maria C. Mariani, H. Eugene Stanley, Frederi G. Viens.
Description: Hoboken, NJ : John Wiley & Sons, Inc., [2016] | Includes index.
Identifiers: LCCN 2015043237 (print) | LCCN 2016000501 (ebook) | ISBN 9781118443989 (cloth) | ISBN 9781118593400 (pdf) | ISBN 9781118593325 (epub)
Subjects: LCSH: Investment analysis–Mathematical models. | Investments–Mathematical models. | Finance–Mathematical models.
Classification: LCC HG4529 .H35863 2016 (print) | LCC HG4529 (ebook) | DDC 332.64/20285–dc23
LC record available at http://lccn.loc.gov/2015043237

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Contents

Notes on Contributors

Editors

Ionut Florescu, Financial Engineering Division, Stevens Institute of Technology, Hoboken, NJ, USA

Maria C. Mariani, Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX, USA

H. Eugene Stanley, Boston University, Boston, MA, USA

Frederi G. Viens, Purdue University, West Lafayette, IN, USA

List of Contributors

K. Basu, Department of Mathematics, Occidental College, Los Angeles, CA, USA

M. P. Beccar-Varela, Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX, USA

Claas Becker, Studiengang Angewandte Mathematik, Hochschule Rhein-Main, 65197 Wiesbaden, Germany

P. Bezdek, Department of Mathematics, The University of Utah, Salt Lake City, UT, USA

F. Biney, Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX, USA

Michael Carlisle, Department of Mathematics, Baruch College, City University of New York, New York, NY, USA

Bernardo Creamer, Universidad de las Américas, Quito, Ecuador

Germán G. Creamer, School of Business, Stevens Institute of Technology, Hoboken, NJ, USA

Alexis Fauth, SAMM, Université de Paris 1 Panthéon-Sorbonne, 90, rue de Tolbiac, 75634, Paris, France and Invivoo, 13 rue de l'Abreuvoir, 92400 Courbevoie, France

Olympia Hadjiliadis, Department of Mathematics and Statistics, Hunter College, City University of New York, and Departments of Computer Science and Mathematics, Graduate Center, City University of New York, New York, NY, USA

Baron Law, Purdue University, West Lafayette, IN, USA

Forrest Levin, Financial Engineering Division, Stevens Institute of Technology, Hoboken, NJ, USA

Michael Marzec, Stevens Institute of Technology, Hoboken, NJ, USA

Ambar N. Sengupta, Department of Mathematics, Louisiana State University, Baton Rouge, LA, USA

I. SenGupta, Department of Mathematics, North Dakota State University, Fargo, ND, USA

Mark B. Shackleton, Department of Accounting and Finance, Lancaster University Management School, Lancaster, England, United Kingdom

M. Shpak, NeuroTexas Institute, St. David's Medical Center, Austin, TX, USA

Ioannis Stamos, Department of Computer Science, Hunter College, City University of New York, and Department of Computer Science, Graduate Center, City University of New York, New York, NY, USA

Ping-Chen Tsai, Department of Finance, Southern Taiwan University of Science and Technology, Yongkang, Tainan City, Taiwan

Ciprian A. Tudor, Laboratoire Paul Painlevé, Université de Lille 1, F-59655 Villeneuve d'Ascq, France, and Department of Mathematics, Academy of Economical Studies, Bucharest, Romania

Preface

This Handbook is a collection of chapters that describe a range of current empirical and analytical work on financial industry data sampled at high frequency (HF).

Our contemporary Age of Information is a world dominated by ever-increasing quantitative elements that decision makers are expected to take into account. Many fields are confronted with large amounts of data. The phenomenon is particularly challenging in the finance industry, in that evidently relevant data can be sampled with increasingly HF, a trend that started in earnest more than a decade ago and does not seem to be letting down. Some of the special challenges posed by these now staggering amounts of data stem from the uncomfortable evidence that traditional models and information technology tools can be poorly suited to grapple with their size and complexity.

Probabilistic modeling and statistical data analysis attempt to uncover order from apparent disorder. By illustrating this methodological framework in the context of HF finance, the current volume may serve as a guide to various new systematic approaches concerning how to implement these quantitative activities with HF financial data. The chapters herein cover a wide range of topics related to the analysis and modeling of data sampled with HF, principally in finance, as well as in other fields where new ideas may prove helpful to HF finance applications. The first chapters cover the dynamics and complexity of futures and derivatives markets as well as a novel take on the portfolio optimization problem using quantum computers. The following chapters are dedicated to estimating complex model parameters using HF data. The final chapters create links between models used in financial markets and models used in other research areas such as geophysics, fossil records, and earthquake studies.

The editors express their deepest gratitude to all the contributors for their talent and labor in bringing together this Handbook, to the many anonymous referees who helped the contributors perfect their work, and to Wiley for making the publication a reality.

Ionut Florescu
Hoboken, NJ

Maria C. Mariani
El Paso, TX

H. Eugene Stanley
Boston, MA

Frederi G. Viens
Washington, DC

August 9, 2015

Chapter One

Trends and Trades

Michael Carlisle[1], Olympia Hadjiliadis[2], and Ioannis Stamos[3]

[1]Department of Mathematics, Baruch College, City University of New York, New York, NY, USA
[2]Department of Mathematics and Statistics, Hunter College, City University of New York, and Departments of Computer Science and Mathematics, Graduate Center, City University of New York, New York, NY, USA
[3]Department of Computer Science, Hunter College, City University of New York, and Department of Computer Science, Graduate Center, City University of New York, New York, NY, USA

1.1 Introduction

High-frequency data in finance is often characterized by fast fluctuations and noise (see, e.g., [7]), a trait that is known to make the volatility of the data very hard to estimate (see, e.g., [13]). Although this characteristic creates many challenges in modeling, it offers itself to the study of distinguishing "signal" from "noise," a topic of interest in the area of *quickest detection* (see [25], [5]). One of the most popular algorithms used in quickest detection is known as the *cumulative sum* (CUSUM) stopping rule first introduced by Page [24]. In this work, we employ a sequence of CUSUM stopping rules to construct an online trading strategy. This strategy takes advantage of the relatively frequent number of alarms CUSUM stopping times may provide when applied to high-frequency data as a result of the fast fluctuations present therein. The trading strategy implemented settles frequently and thus eliminates the risk of large positions. This makes the

Handbook of High-Frequency Trading and Modeling in Finance, First Edition.
Edited by Ionut Florescu, Maria C. Mariani, H. Eugene Stanley and Frederi G. Viens.

strategy implementable in practice. Prior work has been done by Lam and Yam [20] on drawing connections between CUSUM techniques and the filter trading strategy, yet both the filter trading strategy (see [2, 3]), or its equivalent, the buy and hold strategy (see [12]), run high risks of great losses mainly due to the randomness associated with settling. The well-known trailing stops strategy whose properties have been thoroughly studied in the literature (see, e.g., [15] or [1]) is also related to the filter strategy and thus suffers similar risks.

Although our proposed rule presents clear merits in terms of minimizing the risk of large positions by taking advantage of the high volatility frequently present in high-frequency data, the main purpose of this chapter is to present and illustrate the use of detection techniques (in this case the CUSUM) in high-frequency finance. In particular, the strategy proposed is based on running in parallel two CUSUM stopping rules: one detects an upward (+) change and the other a downward (−) change in the mean of the observations. Once an upward/downward CUSUM alarm (called a "signal") goes off, there is a buy/short sale of one unit of the underlying asset. At that moment, we repeat a CUSUM stopping rule, and for every alarm of the same sign, we continue buying or short selling one unit of the underlying asset until a CUSUM alarm of the opposite sign is set off, at which time we sell off all of what we bought or buy up all of what we short sold. The high frequency of CUSUM alarms in high-frequency tick data permits the implementation of this rule in practice since large exposures on one side, whether on the buy or on the sell side, are settled relatively quickly.

The algorithmic strategy proposed is applied on real tick data of a 30-year asset and a 5-year note sold at auction on various individual days. It is seen that the algorithm is most profitable in the presence of upward or downward trends (which we call "subperiods"), even in the presence of noise, and is less profitable on periods of price stability. The proposed strategy is, in fact, a trend-following algorithm.

To quantify the performance of the proposed algorithmic strategy, we calculate its expected reward in a simple random walk model. Our diagnostic plots indicate that the more biased the random walk is, the more profitable the proposed strategy becomes, which is consistent with the actual findings when the strategy is applied to real data. This is because in the presence of a bias, trends are more likely to form than in the absence of a bias.

We take the analytical approach of discrete data and a linear random walk model, rather than taking the continuous approach via, for example, the geometric Brownian motion model, because we are analyzing the movement of individual ticks of a price, quantized in a linear fashion (e.g., at the level of 1 cent, $\frac{1}{32}$ cent, or $\frac{1}{64}$ cent). Our models focus on tracking the motion of an asset price via these ticks, and so a linear approach is a more realistic setting, when short interest rate effects would be minimal.

We begin our analysis in Section 1.2 by describing a general trading strategy based on following upward or downward trends in a data stream, without specifying the timing mechanism behind such a strategy. We then develop the notion of gain over the time period of an individual trend. In Section 1.3, we build a timing scheme stemming from quickest detection considerations and give a preliminary performance evaluation of the overall strategy on real tick data. Next, in Section 1.4, we analyze the specific case of random walk-based data and calculate the expected value of the gain over a trend in this case. We give an explicit formula for this gain in the special case of simple asymmetric random walk on asset tick changes. Then, in Section 1.5, we give results of Monte Carlo simulations for the asymmetric lazy simple random walk and symmetric lazy random walk on tick changes. In Section 1.6, we discuss the effect of the CUSUM threshold parameter on the trading strategy. We conclude in Section 1.7 by a discussion of ways in which the proposed strategy may be improved with suggestions for further work.

1.2 A trend-based trading strategy

Let $\{S_n\}_{n=0,1,2...}$ be a sequence of data points; for our purposes, they will be samples of the price of an asset. We assume that $S_0 = s$ is a constant, and $S_k = 0$ for some k implies that $S_n = 0$ for all $n > k$. Let $T_0 = 0$, and define T_k, $k = 1, 2, ...$ as an increasing sequence of (stopping) times, called *signals*, noting some trend in the sequence. We call T_k the k-th *signal*.

1.2.1 SIGNALING AND TRENDS

In this subsection, we construct a trading strategy in the case that there are two types of signals: "+ signals" (declaring the detection of an upward trend in the data) and "− signals" (declaring the detection of a downward trend in the data). Let "Property $+(k)$" be the property that causes a + signal to occur as the kth signal, and denote this event by $\{T_k = T_k^+\}$. Likewise, let "Property $-(k)$" be the property that causes a − signal to occur as the k-th signal, and denote this by $\{T_k = T_k^-\}$. Only one type of trend can be detected at a time, so we formally define T_k^+ and T_k^- by

$$T_k^+ := \begin{cases} T_k & \text{if Property } +(k) \text{ occurs} \\ \infty & \text{if Property } -(k) \text{ occurs} \end{cases} \tag{1.1}$$

$$T_k^- := \begin{cases} T_k & \text{if Property } -(k) \text{ occurs} \\ \infty & \text{if Property } +(k) \text{ occurs} \end{cases} \tag{1.2}$$

Thus, $T_k = T_k^+ \wedge T_k^-$ for every $k = 1, 2, ...$

Next, we state what it means for the data to stay in a trend. We define the sequence of signal indices $\alpha(l)$ as follows: let $\alpha(0) = 0$, so $T_{\alpha(0)} = 0$, and for $l \geq 1$, with $k \geq 2$, define the properties

$$\begin{aligned} &\text{"Property } +(l,k)\text{" : } \; T_j = T_j^- \text{ for every } \alpha(l-1) < j < k \text{ and } T_k = T_k^+ \\ &\text{"Property } -(l,k)\text{" : } \; T_j = T_j^+ \text{ for every } \alpha(l-1) < j < k \text{ and } T_k = T_k^-. \end{aligned}$$

Then, we define the *lth shift point* as, for $l = 1, 2, ...,$

$$\alpha(l) := \inf\{k \geq \alpha(l-1) + 2\text{: Property} + (l,k) \text{ or Property} - (l,k) \text{ holds}\}. \tag{1.3}$$

Note that $T_{\alpha(l)}$ is at least two signals after $T_{\alpha(l-1)}$. Definition (1.3) is equivalent to

$$\alpha(l) := \inf\{k \geq \alpha(l-1) + 2\text{:} T_k \text{ has different sign than } T_j, \alpha(l-1) < j < k\}. \tag{1.4}$$

A sequence of the same type of signal will be called a *subperiod* of the sample points. A shift point denotes the *end of a subperiod* of the same type of signal.

Let Δ_n be the number of shares of the asset S held at time n. Set $\Delta_0 = 0$. Note that, for every $n \in (T_{\alpha(l)}, T_{\alpha(l+1)})$, the sign of Δ_n is invariant, that is, either $\Delta_n > 0$ holds for every $n \in (T_{\alpha(l)}, T_{\alpha(l+1)})$ or $\Delta_n < 0$ holds for every $n \in (T_{\alpha(l)}, T_{\alpha(l+1)})$.

Our trading strategy is as follows:

$$\Delta_{n+1} = \begin{cases} \Delta_n & \text{if no signal at time } n, \text{ i.e. } n \neq T_j \ \forall j \text{ (no change)} \\ \Delta_n + 1 & \text{if } n = T_j = T_j^+ \text{ for some } j,\ \alpha(l) < j < \alpha(l+1) \\ & \quad \text{for some } l \text{ (buy one during a } + \text{ subperiod)} \\ \Delta_n - 1 & \text{if } n = T_j = T_j^- \text{ for some } j,\ \alpha(l) < j < \alpha(l+1) \\ & \quad \text{for some } l \text{ (sell one during a } - \text{ subperiod)} \\ 0 & \text{if } n = T_{\alpha(l)} \text{ for some } l \geq 1 \\ & \quad \text{(buy-up if } T_{\alpha(l)}^+;\ \text{sell-off if } T_{\alpha(l)}^-). \end{cases} \tag{1.5}$$

We assume a market in which all market orders are instantly fulfilled. The intent of this strategy is to profit from following subperiods of + or − signals by the old adage "buy low, sell high." The success of this strategy relies mainly on the length of such subperiods.

1.2.2 GAIN OVER A SUBPERIOD

We wish to analyze the gain G_l, $l = 1, 2, ...$, for this trading strategy over the time period $(T_{\alpha(l-1)}, T_{\alpha(l)}]$, called *subperiod* l; this is the amount of cash earned or lost by liquidating the transactions made from signals $T_{\alpha(l-1)+1}, ..., T_{\alpha(l)-1}$ at $T_{\alpha(l)}$.

Note that a subperiod is determined by the first signal on that run: if $T_1 = T_1^+$, then the run from signal 1 to signal $\alpha(1) - 1$ is a "bull run" subperiod of individual buy orders followed by a sell-off at time $T_{\alpha(1)} = T_{\alpha(1)}^-$; if $T_1 = T_1^-$, then this run is a "bear run" subperiod of individual short sales followed by a buy-up at $T_{\alpha(1)} = T_{\alpha(1)}^+$. Define G_l to be the gain on subperiod l; thus, G_1 is the gain on the first subperiod, starting at signal $T_{\alpha(0)+1} = T_1$ and ending at signal $T_{\alpha(1)}$. We require, as a condition, the sign of the first signal of the subperiod. Let $c \geq 0$ be the percentage cost per transaction, and define

$$A_l := 1_{\{T_{\alpha(l-1)+1} = T^-_{\alpha(l-1)+1}\}}, \quad Y_l := \alpha(l) - \alpha(l-1) - 1. \tag{1.6}$$

The gain on a subperiod is calculated as follows:

$$G_l := \begin{cases} (1-c)\sum_{j=\alpha(l-1)+1}^{\alpha(l)-1} S_{T_j} - (1+c)(\alpha(l)-\alpha(l-1)-1)S_{T_{\alpha(l)}} \\ \quad \text{if } T_{\alpha(l-1)+1} = T^-_{\alpha(l-1)+1}, \\ (1-c)(\alpha(l)-\alpha(l-1)-1)S_{T_{\alpha(l)}} - (1+c)\sum_{j=\alpha(l-1)+1}^{\alpha(l)-1} S_{T_j} \\ \quad \text{if } T_{\alpha(l-1)+1} = T^+_{\alpha(l-1)+1} \end{cases}$$
$$:= \begin{cases} (1-c)\sum_{j=1}^{Y_l} S_{T_{j+\alpha(l-1)}} - (1+c)(Y_l)S_{T_{\alpha(l)}} & \text{if } T_{\alpha(l-1)+1} = T^-_{\alpha(l-1)+1}, \\ (1-c)(Y_l)S_{T_{\alpha(l)}} - (1+c)\sum_{j=1}^{Y_l} S_{T_{j+\alpha(l-1)}} & \text{if } T_{\alpha(l-1)+1} = T^+_{\alpha(l-1)+1}. \end{cases} \tag{1.7}$$

For example, if $c = 0.01$, $T_1 = T_1^+$, and $\alpha(1) = 4$, then $T_{\alpha(1)} = T_4 = T_4^-$. Say the prices at the buy-signal times are $S_{T_1} = 5$, $S_{T_2} = 7$, $S_{T_3} = 9$, and we sell everything off at $S_{T_4} = 8$. Then $\Delta_{T_0} = 0$, $\Delta_{T_1} = 1$, $\Delta_{T_2} = 2$, $\Delta_{T_3} = 3$, and we liquidate at time T_4 to $\Delta_{T_4} = 0$. The gain on the first subperiod would then be $G_1 = (0.99)(3)(8) - (1.01)(5 + 7 + 9) = 2.55$.

Combining the $1 - c$ terms and adding on the random variable $2cY_l S_{\alpha(T_l)}$, we have after some algebra a sum of price increments:

$$\begin{aligned} G_l + 2cY_l S_{\alpha(T_l)} &= \left(c + (-1)^{A_l}\right)\left[Y_l S_{T_{\alpha(l)}} - \sum_{j=1}^{Y_l} S_{T_{j+\alpha(l-1)}}\right] \\ &= \left(c + (-1)^{A_l}\right)\sum_{j=1}^{Y_l}(S_{T_{\alpha(l)}} - S_{T_{j+\alpha(l-1)}}). \end{aligned} \tag{1.8}$$

We can rewrite each difference in the sum as a telescoping sum: setting

$$Z_k := S_{T_{k+1}} - S_{T_k}, \quad k = 1, 2, ..., \tag{1.9}$$

as the incremental price change between signals k and $k + 1$, we have

$$S_{T_{\alpha(l)}} - S_{T_{j+\alpha(l-1)}} = \sum_{k=j+\alpha(l-1)}^{\alpha(l)-1}(S_{T_{k+1}} - S_{T_k}) = \sum_{k=j+\alpha(l-1)}^{\alpha(l)-1} Z_k = \sum_{k=j}^{Y_l} Z_{k+\alpha(l-1)}.$$

Substituting this back into (1.8) yields

$$G_l + 2cY_l S_{\alpha(T_l)} = \left(c + (-1)^{A_l}\right)\sum_{j=1}^{Y_l}\left[\sum_{k=j+\alpha(l-1)}^{\alpha(l)-1} Z_k\right] = \left(c + (-1)^{A_l}\right)\sum_{j=1}^{Y_l} jZ_{j+\alpha(l-1)}. \tag{1.10}$$

Therefore, by (1.11), the gain over subperiod l is

$$G_l = \left(c + (-1)^{A_l}\right) \sum_{j=1}^{Y_l} jZ_{j+\alpha(l-1)} - 2cY_l S_{\alpha(T_l)}. \tag{1.11}$$

Note that, in the absence of transaction costs (i.e., $c = 0$), the expected gain G_l is entirely determined by price increments and the sign of the first signal of the subperiod.

1.3 CUSUM timing

Next, we describe a version of the CUSUM statistic process and its associated CUSUM stopping rule, which we will use to devise a timing scheme based on the quickest detection of trends, and incorporate this scheme to our trading strategy.

1.3.1 CUSUM PROCESS AND STOPPING TIME

In this section, we begin by introducing the measurable space $(\Omega, \mathcal{F})$, where $\Omega = \mathbb{R}^\infty$, $\mathcal{F} = \cup_n \mathcal{F}_n$, and $\mathcal{F}_n = \sigma\{Y_i, i \in \{0, 1, \ldots, n\}\}$. The law of the sequence Y_i, $i = 1, \ldots$, is described by the family of probability measures $\{P_\nu\}$, $\nu \in \mathbb{N}^*$. In other words, the probability measure P_ν for a given $\nu > 0$, playing the role of the *change point*, is the measure generated on Ω by the sequence Y_i, $i = 1, \ldots$, when the distribution of the Y_i's changes at time ν. The probability measures P_0 and P_∞ are the measures generated on Ω by the random variables Y_i when they have an identical distribution. In other words, the system defined by the sequence Y_i undergoes a "regime change" from the distribution P_0 to the distribution P_∞ at the change point time ν.

The *CUSUM statistic* is defined as the maximum of the log-likelihood ratio of the measure P_ν to the measure P_∞ on the σ-algebra $\mathcal{F}_n$. That is,

$$C_n := \max_{0 \le \nu \le n} \log \left. \frac{dP_\nu}{dP_\infty} \right|_{\mathcal{F}_n} \tag{1.12}$$

is the CUSUM statistic on the σ-algebra $\mathcal{F}_n$. The *CUSUM statistic process* is then the collection of the CUSUM statistics $\{C_n\}$ of (1.12) for $n = 1, \ldots$.

The *CUSUM stopping rule* is then

$$T(h) := \inf \left\{ n \geq 0 : \max_{0 \leq \nu \leq n} \log \frac{dP_\nu}{dP_\infty}\bigg|_{\mathcal{F}_n} \geq h \right\}, \tag{1.13}$$

for some threshold $h > 0$. In the CUSUM stopping rule (1.13), the CUSUM statistic process of (1.12) is initialized at

$$C_0 = 0. \tag{1.14}$$

The CUSUM statistic process was first introduced by Page [24] in the form that it takes when the sequence of random variables Y_i is independent and Gaussian; that is, $Y_i \sim N(\mu, 1)$, $i = 1, 2, \ldots$, with $\mu = \mu_0$ for $i < \nu$ and $\mu = \mu_1$ for $i \geq \nu$. Since its introduction by Page [24], the CUSUM statistic process of (1.12) and its associated CUSUM stopping time of (1.13) have been used in a plethora of applications where it is of interest to perform detection of abrupt changes in the statistical behavior of observations in real time. Examples of such applications are signal processing (see [10]), monitoring the outbreak of an epidemic (see [29]), financial surveillance (see [14] and [9]), and more recently computer vision (see [19] or [30]). The popularity of the CUSUM stopping time (1.13) is mainly due to its low complexity and optimality properties (see, for instance, [21], [22, 23], [6] and [27] or [26]), in both discrete and continuous time models.

As a specific example, we now derive the form in which Page [24] introduced the CUSUM. To this effect, let $Y_i \sim N(\mu_0, \sigma^2)$ that change to $Y_i \sim N(\mu_1, \sigma^2)$ at the change point time ν. We now proceed to derive the form of the CUSUM statistic process (1.12) and its associated CUSUM stopping time (1.13) in the example set forth in this section. To this effect, let us now denote by $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ the Gaussian kernel. For the sequence of random variables Y_i given earlier, we can now compute (see also [28] or [25]):

$$\begin{aligned} C_n &= \max_{0 \leq \nu \leq n} \log \frac{dP_\nu}{dP_\infty}\bigg|_{\mathcal{F}_n} = \max_{0 \leq \nu \leq n} \log \frac{\prod_{i=1}^{\nu-1} \phi\left(\frac{Y_i - \mu_0}{\sigma}\right) \prod_{i=\nu}^{n} \phi\left(\frac{Y_i - \mu_1}{\sigma}\right)}{\prod_{i=1}^{n} \phi\left(\frac{Y_i - \mu_0}{\sigma}\right)} \\ &= \frac{1}{\sigma^2} \max_{0 \leq \nu \leq n} (\mu_1 - \mu_0) \sum_{i=\nu}^{n} \left[Y_i - \frac{\mu_1 + \mu_0}{2} \right]. \end{aligned} \tag{1.15}$$

In view of (1.14), we initialize the sequence (1.15) at $Y_0 = \frac{\mu_1+\mu_0}{2}$ and proceed to distinguish the following two cases:

- Case 1: $\mu_1 > \mu_0$: divide out $\mu_1 - \mu_0$, multiply by the constant σ^2 in (1.15), and use (1.13) to obtain the CUSUM stopping rule T^+ :

$$T^+(h^+) = \inf\left\{ n \geq 0 : \max_{0\leq \nu \leq n} \sum_{i=\nu}^{n} \left[Y_i - \frac{\mu_1 + \mu_0}{2} \right] \geq h^+ \right\} \tag{1.16}$$

for an appropriately scaled threshold $h^+ > 0$.

- Case 2: $\mu_1 < \mu_0$: divide out $\mu_1 - \mu_0$, multiply by the constant σ^2 in (1.15), and use (1.13) to obtain the CUSUM stopping rule T^-:

$$T^-(h^-) = \inf\left\{ n \geq 0 : \max_{0\leq \nu \leq n} \sum_{i=\nu}^{n} \left[\frac{\mu_1 + \mu_0}{2} - Y_i \right] \geq h^- \right\} \tag{1.17}$$

for an appropriately scaled threshold $h^- > 0$.

As shown in the study [24] or [11], we can reexpress the stopping times (1.16) and (1.17) in terms of the recurrence relations

$$u_0 = 0;\ u_n := \max\left\{ 0, u_{n-1} + \left(Y_n - \frac{\mu_1 + \mu_0}{2} \right) \right\} \tag{1.18}$$

$$d_0 = 0;\ d_n := \max\left\{ 0, d_{n-1} - \left(Y_n - \frac{\mu_1 + \mu_0}{2} \right) \right\}, \tag{1.19}$$

which lead to

$$T^+(h^+) = \inf\{ n > 0 : u_n \geq h^+ \}, \tag{1.20}$$

$$T^-(h^-) = \inf\{ n > 0 : d_n \geq h^- \}. \tag{1.21}$$

The sequences u_n and d_n of (1.18) and (1.19), respectively, form a CUSUM according to the deviation of the monitored sequential observations Y_n from the average of their pre- and postchange means. The first time that one of these sequences reaches its threshold (in (1.20) or (1.21)), the respective alarm T^+ or T^- fires.

Although the stopping times (1.16) and (1.17) and their respective equivalents (1.20) and (1.21) can be derived by formal CUSUM regime change considerations using the example set forth in this section, they may also be used as general nonparametric stopping rules directly applied to sequential observations as seen in the study by Brodsky and Darkhovsky

[8] or Devore [11]. The former can be used as a general stopping rule to detect an upward change in the mean while the latter a downward one. In many applications, it is of interest to monitor an upward or downward change in the mean of sequential observations simultaneously. This gives rise to the two-sided CUSUM (2-CUSUM), which was first introduced by Barnard [4], and whose optimality properties have been established in Hadjiliadis [17], Hadjiliadis and Moustakides [16], and Hadjiliadis et al. [18]. In the context presented in this section, the 2-CUSUM stopping time takes the form

$$T^{+}(h^{+}) \wedge T^{-}(h^{-}), \tag{1.22}$$

where $T^{+}(h^{+})$ appears in (1.20) and $T^{-}(h^{-})$ in (1.21). The symmetric version of the 2-CUSUM stopping time is that of (1.22) when $h^{+} = h^{-} = h$.

1.3.2 A CUSUM TIMING SCHEME

We now apply the aforementioned CUSUM stopping rule of (1.22) to a stream of data representing the value of the underlying asset without any model assumptions. In other words, the underlying asset is not necessarily assumed to be independent or normally distributed. That is, we apply the forms (1.16) and (1.17) in a nonparametric fashion. Let $M > 0$ denote the "tick size" of the asset being monitored (presuming that S changes in increments of M; we do not know the probability distribution of these changes), and $h > 0$ be a given threshold. Given that $S_0 = s$, recall that $T_0 = 0$. We monitor the progress of upward or downward adjustments in the price S_n of the underlying, by individual ticks.

In view of the previous subsection at time T_k, μ_0 is set to the value of the underlying at time T_k, namely $\mu_0 = S_{T_k}$, and $\mu_1^u = S_{T_k} + M$ and $\mu_1^d = S_{T_k} - M$ are the two "new" mean levels to be monitored against. Thus, as in equations (1.18) and (1.19), which cumulate the deviations of the monitored sequence from the average of their pre- and postchange means, we now monitor the deviations of the underlying sequence S_n, $n = 1, 2 \ldots$, from the quantities

$$\begin{aligned} m_k^u &:= \frac{(S_{T_k} + M) + S_{T_k}}{2} = S_{T_k} + \frac{M}{2}, \\ m_k^d &:= \frac{(S_{T_k} - M) + S_{T_k}}{2} = S_{T_k} - \frac{M}{2}, \end{aligned} \tag{1.23}$$

where $k \geq 0$. To this effect, set $u_0^k = d_0^k = 0$, and for $n \geq 1$, define the CUSUM statistics

$$u_n^k := \max\{0, u_{n-1}^k + (S_{n+T_k} - m_k^u)\},$$
$$d_n^k := \max\{0, d_{n-1}^k - (S_{n+T_k} - m_k^d)\}. \quad (1.24)$$

Thus, for $k \geq 0$, the CUSUM timing scheme for our trend-following trading strategy is defined by using (1.20) and (1.21) (and coming from (1.1) and (1.2)),

$$\text{Property } +(k+1): \ u_n^k \geq h; \quad \text{Property } -(k+1): \ d_n^k \geq h$$
$$j_k^* := \min\{n > 0: \ \text{Property } +(k+1) \text{ or } -(k+1) \text{ occurs}\} \quad (1.25)$$
$$T_{k+1} := T_k + j_k^*.$$

In other words, each T_k is the symmetric 2-CUSUM stopping time of (1.22) for cycle k. Finally, at the "end of day," that is, on the final tick, we close out our position, inducing a final shift point to end trading, for algorithmic purposes.

1.3.3 US TREASURY NOTES, CUSUM TIMING

The following figures and chart describe the CUSUM timing scheme (1.25) applied to the trading strategy (1.5) for US Treasury notes sold at auction in 2011. Gains quoted are in increments of \$1000. In Figure 1.1, we show the

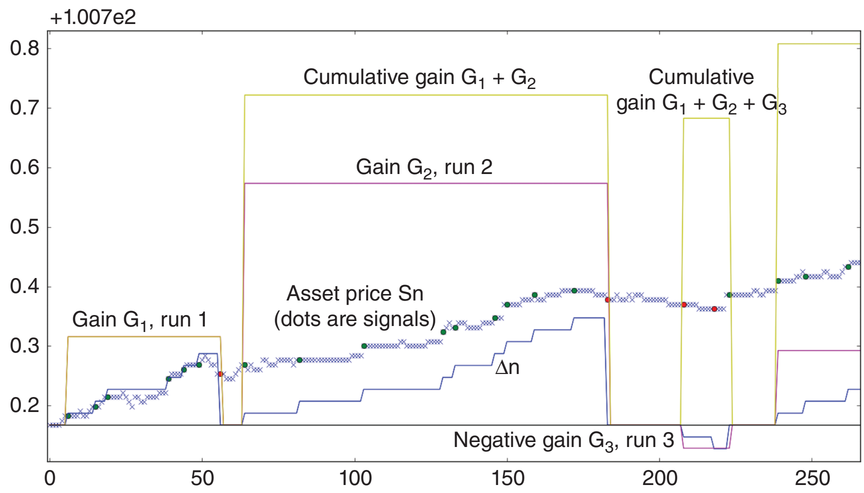

FIGURE 1.1 Plot of the first subperiods, and cumulative gain, for the CUSUM strategy, August 2, 2011, US 5-year treasury note.

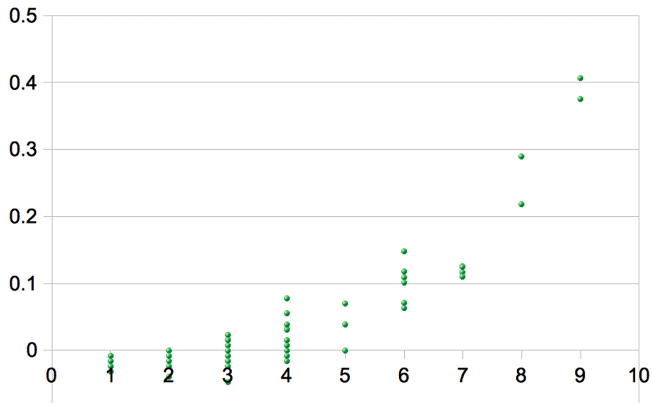

FIGURE 1.2 Lengths of subperiods versus gains, August 2, 2011, US 5-year treasury note.

asset price, along with the number of shares held, per-subperiod gain, and running total gain. Figures 1.2, 1.3, 1.4, 1.5 and 1.6 show the individual subperiod gains, plotted by the number of signals during a subperiod, of the gain for 5-year and 30-year treasury notes, and Figure 1.7 aggregates the data from Figures 1.3, 1.4, 1.5 and 1.6 for 30-year notes.

1.4 Example: Random walk on ticks

We now describe a simple example to model the asset price motions. Assume that $\exists N > 0$ such that the sequence $\{X_j\}_{j\in\mathbb{N}}$ are the steps of a

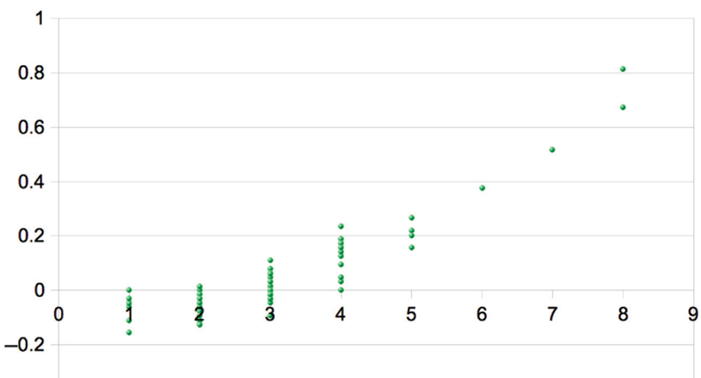

FIGURE 1.3 Subperiod length versus gain, July 29, 2011, US 30-year treasury note.