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M. di Bernardo C.J. Budd A.R. Champneys
P. Kowalczyk

Piecewise-smooth Dynamical Systems

Theory and Applications

Springer

Mario di Bernardo, Laurea; PhD
Department of Engineering Mathematics
University of Bristol, Bristol BS8 1TR, UK
and
Department of Systems and Computer Science
University of Naples Federico II, Naples, Italy

Christopher J. Budd, MA; DPhil
Department of Mathematical Sciences
University of Bath, Bath BA2 7AY, UK

Alan R. Champneys, BSc; DPhil
Department of Engineering Mathematics
University of Bristol, Bristol BS8 1TR, UK

Piotr Kowalczyk, MSc; PhD
School of Engineering
Computer Science and Mathematics,
University of Exeter, Exeter EX4 4QF, UK
and
Department of Engineering Mathematics
University of Bristol, Bristol BS8 1TR, UK

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Preface

Many dynamical systems that occur naturally in the description of physical processes are piecewise-smooth. That is, their motion is characterized by periods of smooth evolutions interrupted by instantaneous events. Traditional analysis of dynamical systems has restricted its attention to smooth problems, thus preventing the investigation of non-smooth processes such as impact, switching, sliding and other discrete state transitions. These phenomena arise, for example, in any application involving friction, collision, intermittently constrained systems or processes with switching components.

Literature that draws attention to piecewise-smooth systems includes the comprehensive work of Brogliato [38, 39], the detailed analysis of Kunze [165], the books on bifurcations in discontinuous systems [193, 177] and various related edited volumes [268, 35]. These books contain many examples largely drawn from mechanics and control. Also there is a significant literature in the control and electronics communities; see for example the book [193], which has many beautiful examples of chaotic dynamics induced by non-smooth phenomena. Earlier studies of non-smooth dynamics appeared in the Eastern European literature; for instance the pioneering work of Andronov *et al.* on non-smooth equilibrium bifurcations [5], Feigin [98, 80] on C-bifurcations, Peterka [216] and Babitskii [19] on impact oscillators, and Filippov [100] on sliding motion. Delving into this and other literature, one finds that piecewise-smooth systems can feature rich and complex dynamics.

In one sense, jumps and switches in a system's state represent the grossest form of nonlinearity. On the other hand, many examples appear benign at first glance since they are composed of pieces of purely linear systems, which are solvable closed form. However, this solvability is in general an illusion since one does not know *a priori* the times at which the switches occur. Nevertheless, the analysis of such dynamics is not intractable, and indeed, many tools of traditional bifurcation theory may be applied. However, it has become increasingly clear that there are distinctive phenomena unique to discontinuous systems, which can be analyzed mathematically but fall outside the usual methodology for smooth dynamical systems.

Indeed, for smooth systems, governed by ordinary differential equations, there is now a well established qualitative, topological theory of dynamical systems that was pioneered by Poincaré, Andronov and Kolmogorov among others. This theory has led to a mature understanding of bifurcations and routes to chaos—see, for example the books by Kuznetsov [168], Wiggins [273], Arrowsmith & Place [9], Guckenheimer & Holmes [124] and Seydel [232]. The key step in the analysis is to use topological equivalence, Poincaré maps, center manifolds and normal forms to reduce all possible transitions under parameter variation to a number of previously analyzed cases. These ideas have also informed modern techniques for the numerical analysis of dynamical systems and, via time-series analysis, techniques for the analysis of experimental data from nonlinear systems. The bifurcation theory methodology has shown remarkable success in describing dynamics observed in many areas of application including, via center-manifold and other reduction techniques, spatially extended systems. However, most of these successes are predicated on the dynamical system being smooth.

The purpose of this book then is to introduce a similar qualitative theory for non-smooth systems. In particular we shall propose general techniques for analyzing the bifurcations that are unique to non-smooth dynamical systems, so-called *discontinuity-induced* bifurcations (DIBs for short). This we propose as a general term for all transitions in dynamics specifically brought about through interaction of invariant sets of the system ('attractors') with a boundary in phase space across which the system has some kind of discontinuity. First and foremost, we shall give a consistent classification of all known DIBs for piecewise-smooth continuous-time dynamical systems (flows), including such diverse phenomena as sliding, chattering, grazing and corner collision. We will then describe a unified analytical framework for reducing the analysis of each such bifurcation involving a periodic orbit to that of an appropriately defined Poincaré map. This process is based on the construction of so-called *discontinuity mappings* [198, 64], which are analytical corrections made to account for crossing or tangency with discontinuity boundaries. We introduce the notion of the *degree of smoothness* depending on whether the state, the vector field or one of its derivatives has a jump across a discontinuity boundary. We show how standard examples such as impact oscillators, friction systems and relay controllers can be put into this framework, and show how to construct discontinuity mappings for tangency of each kind of system with a discontinuity boundary.

The analysis is completed by a classification of the dynamics of the Poincaré maps so-obtained. Thus we provide a link between the theory of bifurcations in piecewise-smooth flows and that associated with discontinuity crossings of fixed points of piecewise-smooth maps—so-called *border-collision* bifurcations [207, 21], which are just particular examples of a DIB. The presentation is structured in such a manner to make it possible for a reader to follow a series of steps to take a non-smooth dynamical system arising in an application from an outline description to a consistent mathematical characterization.

Throughout, the account will be motivated and illustrated by copious examples drawn from several areas of applied science, medicine and engineering; from mechanical impact and friction oscillators, through power electronic and control systems with switches, to neuronal and cardiac and models. In each case, the theory is compared with the results of a numerical analysis or, in some cases, with data from laboratory experiments. More general issues concerning the numerical and experimental investigation of piecewise-smooth systems are also discussed.

The manner of discourse will rely heavily on geometric intuition through the use of sketch figures. Nevertheless, care will be taken to single out as theorems those results that do have a rigorous proof, and where the proof is not presented, a reference will be given to the appropriate literature.

The level of mathematics assumed will be kept to a minimum: nothing more advanced than multivariable calculus, differential equations and linear algebra traditionally taught at undergraduate level on mathematics, engineering or applied science degree programs. A familiarity with the basic concepts of nonlinear dynamics would also be useful. Thus, although the book is aimed primarily at postgraduates and researchers in any discipline that impinges on nonlinear science, it should also be accessible to many final-year undergraduates.

We now give a brief outline each chapters.

Chapter 1. Introduction. This serves as a non-technical motivation for the rest of the book. It can in fact be read in isolation and is intended as a primer for the non-specialist. After a brief motivation of why piecewise-smooth systems are worthy of study, the main thrust of the chapter is to immerse the reader in the kind of dynamics that are unique to piecewise-smooth systems via a series of case studies. The first case study is the single-degree-of-freedom impact oscillator. The notion of grazing bifurcation is introduced along with the dynamical complexity that can result from this seemingly innocuous event. Agreement is shown among theory, numerics and physical experiment. After brief consideration of bi-linear oscillators, we then consider two mathematically related systems that can exhibit recurrent sliding motion: a relay controller and a stick-slip friction system. The next case study concerns a well-used electronic circuit with a switch, the so-called DC–DC converter. Finally, we consider one-dimensional maps that arise through the study of these flows, including a simple model of heart attack prediction. Here we introduce the ubiquitous *period-adding cascade* that is unique to non-smooth systems.

Chapter 2. Qualitative theory of non-smooth dynamical systems. The aim here is to set out concisely the mathematical and notational framework of the book. We present a brief introduction to the qualitative theory of dynamical systems for smooth systems, including a brief review of standard bifurcations, stressing which of these also makes sense for piecewise-smooth systems. The formalism of piecewise-smooth systems is introduced, although no specific attempt is made to develop an existence and uniqueness theory. However, a brief introduction is given to the extensive literature on other more rigorous mathematical formulations for non-smooth dynamics, such as differential inclusions, complementarity systems and hybrid dynamical systems. A working definition of discontinuity-induced bifurcation is given from a topological point of view, which motivates a brief list of the kinds of discontinuity-induced bifurcations that are likely to occur as a single parameter is varied. The notion of discontinuity mapping is introduced, and such a map is carefully derived in the case of transverse crossing of a discontinuity boundary. The chapter ends with a discussion on numerical techniques, both direct and indirect, that will be used throughout the rest of the book for investigating the dynamics of example systems and calculating the appropriate bifurcation diagrams.

Chapter 3. Border collision in piecewise-smooth continuous maps. This chapter contains results on the dynamics of discrete-time continuous maps that are locally composed of two linear pieces. First border-collision bifurcations are analyzed whereby a simple fixed point passes through the boundary between the two map pieces. General criteria are established for the existence and stability of simple period-one and -two fixed points created or destroyed in such transitions, by using information only on the characteristic polynomial of the matrix representation of the two sections of the map. Analogs of simple fold and period-doubling bifurcations are shown to occur, albeit where the bifurcating branch has a non-smooth rather than quadratic character. The cases of one and two dimensions are considered in detail. Here, more precise information can be established such as conditions for the existence of period-adding, and cascades of such as another parameter (representing the slope of one of the linear pieces) is varied. Finally, we consider maps that are noninvertible in one part of their domain. For such maps, conditions can be found for the creation of robust chaos, which has no embedded periodic windows.

Chapter 4. Bifurcations in general piecewise-smooth maps. Here the analysis of the previous chapter is generalized to deal with maps that crop up as normal forms of the grazing and other non-smooth bifurcations analyzed in subsequent chapters, and which change their form across a discontinuity boundary. First, we treat maps that are piecewise-linear but discontinuous. We then proceed to study continuous maps that are a combination of a linear and a square-root map, and finally maps that combine a linear map with an $\mathcal{O}(3/2)$ or a quadratic map. In each case we study the existence of both periodic and chaotic behavior and look at the transitions between these states. Of particular interest will be the identification of *period-adding* behavior in which, under the variation of a parameter, the period of a periodic state increases in arithmetic progression, accumulating onto a chaotic solution.

Chapter 5. Boundary equilibrium bifurcations in flows. This chapter collects and reviews various results on the global consequences of an equilibrium point encountering the boundary between two smooth regions of phase space in a piecewise-smooth flow. Cases are treated where the vector field is continuous across the boundary and where it is not (and indeed where the boundary may itself be attracting—the Filippov case). In two dimensions, a more or less complete theory is possible since the most complex attractor is a limit cycle, which may be born in a non-smooth analog of a Hopf bifurcation. In the Filippov case, so-called pseudo-equilibria that lie inside the sliding region can be created or destroyed on the boundary, as they can for impacting systems.

Chapter 6. Limit cycle bifurcations in impacting systems. We return to the one-degree-of-freedom impact oscillator from the Introduction, stressing a more geometrical approach to understanding the broad features of its dynamics. Within this approach, grazing events are thought of as leading to singularities in the phase space of certain Poincaré maps. These singularities are shown to organize the shape of strange attractors and also the basins of attraction of competing attractors. An attempt is made to generalize such geometrical considerations to general n-dimensional hybrid systems of a certain class. The narrative then switches to dealing with grazing bifurcations of limit cycles within this general class. The discontinuity mapping idea is used to derive normal form maps that have a square-root singularity. The technique is shown to work on several example systems. The chapter also includes a treatment of chattering (a countably infinite sequence of impacts in a finite time) and multiple impacts, including a simple example of a triple collision.

Chapter 7. Limit cycle bifurcations in piecewise-smooth flows. This chapter treats the general case of non-Filippov flows and two specific kinds of bifurcation event where a periodic orbit grazes with a discontinuity surface. In the first kind the periodic orbit becomes tangent to a smooth surface. In the second kind the periodic orbit passes through a non-smooth junction between two surfaces. For both kinds, discontinuity mappings are calculated and normal form mappings derived that can be analyzed using the techniques of the earlier chapters. Examples of the theory are given including general bilinear oscillators, a certain stick-slip system and the DC–DC convertor introduced in Chapter 1.

Chapter 8. Sliding bifurcations in Filippov systems. The technique of discontinuity mappings is now applied to the situations where flows can slide along the attracting portion of a discontinuity set in the case where the vector fields are discontinuous. Four non-generic ways that periodic orbits can undergo sliding are identified that lead to four bifurcation events. Each event involves the fundamental orbit involved in the bifurcation gaining or losing a sliding portion. The mappings derived at these events typically have the property of being non-invertible due to the loss of initial condition information inherent in sliding. So a new version of the theory of Chapters 3 and 4 has to be derived, dealing with this added complication. Examples of relay controllers and friction oscillators introduced in Chapter 1 are given further treatment in the light of this analysis.

Chapter 9. Further applications and extensions. This chapter contains a series of additional case study applications that serve to illustrate further bifurcations and dynamical features, a detailed analysis of which would be beyond the scope of this book. Each application arises from trying to understand or model some experimental or in service engineered or naturally occurring system. The further issues covered include the notion of parameter fitting to experimental data, grazing bifurcations of invariant tori and examples of codimension-two bifurcations.

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The study of piecewise-smooth dynamical systems as a subject in its own right is relatively new. We have tried to use a consistent terminology and notation, which we accept may not be to everyone's taste. We have been helped in this task by the unofficial Non-smooth Standardization Agency that has provided a sounding board for agreeing on certain basic nomenclature. Its members include Harry Dankowicz, John Hogan, Yuri Kuznetsov, Arne Nordmark, Petri Piiroinen and Gerard Olivar.

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Contents

Glossary

:=	Is equal to by definition.
α	The free parameter in Filipov's method, $\alpha \in [0, 1]$; see, (2.30).
β	Utkin's equivalent control for a Filippov system, $\beta \in [-1, 1] = 2\alpha - 1$; see (2.29).
$\beta(x, y)$	Vector multiplied by $y = \sqrt{-H_{\min}}$ in the leading-order expression for the discontinuity mapping at grazing.
δ, δ_i	Unknown small time(s) in construction of discontinuity mapping.
ε	Small perturbation in state variable x.
λ, λ_i	Eigenvalue of a matrix, eigenvalue of an equilibrium point of a flow, multiplier of a fixed point of a map. Unknown parameter in Complementarity system. Unknown parameter in sticking flow, see (2.43).
μ	Parameter; $\mu \in \mathbb{R}^p$ or $\mu \in \mathbb{R}$, so that we write $F(x, \mu)$ etc. where the dependence on a parameter is imporant.
ϕ	Abstract dynamical system $\phi^t : X \to X$, $x_t = \phi^t(x_0)$.
Ψ	Notation for the flow operator of a hybrid system.
Φ, Φ_i	Flow operator corresponding to ODE $\dot{x} = F_i(x)$.
Σ_{ij}, Σ	Discontinuity manifold, sometimes called impacting set, switching set, or discontinity boundary.
$\widehat{\Sigma}_{ij}$	Sliding region of a switching set Σ_{ij} in a Filippov system.
$\partial\widehat{\Sigma}_{ij}^{+}$, $\partial\widehat{\Sigma}^{+}$	Boundary of sliding region corresponding to $\beta = +1$.
$\partial\widehat{\Sigma}_{ij}^{-}$, $\partial\widehat{\Sigma}^{-}$	Boundary of sliding region corresponding to $\beta = -1$.
a	Symbol to represent iterate of two-zone nonsmooth map in region S_1. Acceleration in a single zone impacting system $a(x) = (H_x F)_x F = \mathcal{L}_F^2 H(x)$.
A	Symbol representing iterate in S_1 of part of a stable orbit of a two-zone nonsmooth map.

b	Symbol to represent iterate of two-zone nonsmooth map in region S_2.
B	Symbol representing iterate in S_2 of part of a stable orbit of a two-zone nonsmooth map.
BEB	Boundary equilibrium bifurcation; see Chapter 5.
C^r	The space of continuously r times differentiable functions.
C^T	Row vector multiplying linearisation with respect to x of smooth function $H(x) = C^T x + D\mu$ representing discontinuity surface Σ.
D	Linearisation of discontinuity surface H with respect to parameter μ.
$\mathcal{D}$	Domain of definition of a piecewise-linear system, $x \in \mathcal{D} \subset \mathbb{R}^n$.
DAE	Differential algebraic equation.
DIB	Discontinuity-induced bifurcation.
DM	Discontinuity mapping (for transversal or non-transversal crossing).
DoF	Degree of freedom.
E	Rank-one matrix representing difference between linearisations $N_1 - N_2$ in piecewise-linear map $x \to N_1 x$, if $C^T x > 0$, $x \to N_2 x$ if $C^T x < 0$. Vector E multiplying the scalar $y = \sqrt{-H(x)}$ for $x < 0$ in square root map written in simpest form.
f	General expression for a (smooth or nonsmooth) vector field $\dot{x} = f(x)$, or map $x \mapsto f(x)$.
F_i	Smooth vector field or map applying in region S_i; $\dot{x} = F(x_i)$.
F_{ij}, F_s	Sliding vector field for $x \in \widehat{\Sigma}_{ij}$.
G	Grazing region of a discontinuity boundary Σ. If $\Sigma = \{x \in \mathcal{D} \vert H(x) = 0\}$, then $G := \{x \in \Sigma \vert H_x F(x) = 0\}$.
$\mathcal{G}$	Grazing manifold. Image of G under the flow.
G_Π	Intersection of grazing manifold with Poincaré section Π.
H_{ij}, H	Smooth function defining discontiny manifold $\Sigma_{ij} := \{x \in \mathcal{D} \subset \mathbb{R}^n \vert H(x) = 0\}$.
$\mathcal{L}_f h(x)$	Lie derivative $\mathcal{L}_f h(x) = \frac{\partial h}{\partial x} f(x)$.
n	The dimension of phase space, $x \in \mathbb{R}^p$.
N, N_i	Linearisation with respect to state of (piecewise) linear map. $x \mapsto N_i x + M_i \mu$ if $x \in S_i$.
M, M_i	Linearisation with respect to parameter of (piecewise) linear map.
O	"Big O". $O(\varepsilon^n)$ means of the same size as ε^n as $\varepsilon \to 0$. That is, $O(\varepsilon^n)/\varepsilon^n$ tends to a finite limit as $\varepsilon \to 0$. Also $O(n)$ means $O(\varepsilon^n)$ for an implied small quantity ε.
o	"Little o". $o(\varepsilon^n)$ means asymptotically smaller than ε^n. That is, $o(\varepsilon^n)/\varepsilon^n \to 0$ as $\varepsilon \to 0$.

ODE	Ordinary differential equations.
P	General notation for a Poincaré map, $x \to P(x)$.
p	The dimension of parameter space, $\mu \in \mathbb{R}^p$.
PDM	Poincaré-section discontinuity mapping; see Definition 2.34.
PWS	Piecewise smooth.
Q	Discontinuity mapping $x \to Q(x)$.
R or R_{ij}	Reset (or restitution) map in a hybrid system, sometimes called the impact map $x \to R(x)$ or $x^+ = R(x^-)$.
S_i	Open region of phase space in which dynamics is governed by $\dot{x} = F_i(x)$ or $x \to F_i(x)$.
$\overline{S}_i$	The closed region S_i plus its boundary.
S^+	Physical region of phase space for which $H(x) > 0$ in an impacting hybrid system with a single impact boundary $\Sigma = \{x \vert H(x) > 0\}$.
S^-	Unphysical region for which $H(x) < 0$ in an impacting hybrid system with a single impact boundary.
s	Phase variable $s = \omega t/2\pi$ where ω is the angular frequency of a forcing function $w(t)$.
t	Time.
u	Co-ordinate of a single-degree-freedom system.
v	Velocity. In an impacting hybrid system we have $v(x) = H_x F(x) = \mathcal{L}_F H(x)$.
W	Smooth vector representing simplest form of impact law $x + W(x)H_x F$.
w	Forcing function $w(t)$ in a single-degree-of-freedom system.
x	State variable $x \in \mathbb{R}^n$.
x_*	Grazing point of a trajectory. Sometimes also used for equilibrium point of flow or fixed point of map.
x^*	Equilibrium point of flow, or fixed point of map.
y	Scalar variable $\sqrt{-H(x)}$ for $H(x) < 0$.
$\mathcal{Z}$	Sticking region of impacting system.
ZDM	Zero-time discontinuity mapping; see Definition 2.35.

1

Introduction

1.1 Why piecewise smooth?

Dynamical systems theory has proved a powerful tool to analyze and understand the behavior of a diverse range of problems. There is now a well-developed qualitative, or geometric, approach to dynamical systems that typically relies on the system evolution being defined by a smooth function of its arguments. This approach has proved extremely effective in helping to understand the behavior of many important physical phenomena such as fluid flows, elastic deformation, nonlinear optical and biological systems. However, this theory excludes many significant systems that arise in practice. These are dynamical system containing terms that are *non-smooth* functions of their arguments. Problems of this nature arise everywhere! Important examples are electrical circuits that have switches, mechanical devices in which components impact with each other (such as gear assemblies) or have freeplay, problems with friction, sliding or squealing, many control systems (including their implementation via adaptive numerical methods) and models in the social and financial sciences where continuous change can trigger discrete actions. Such problems are all characterized by functions that are piecewise-smooth but are *event driven* in the sense that smoothness is lost at instantaneous events, for example, upon application of a switch. They have fascinating dynamics with significant practical application and a rich underlying mathematical structure. It a serious omission that their behavior is not easily described in terms of the modern qualitative theory of dynamical systems.

A commonly expressed reason for this omission is that there is strictly speaking no such thing as a piecewise-smooth dynamical system and that in reality all physical systems are smooth (at least at all length scales greater than the molecular). However, this statement is misleading. The timescales over which transitions such as an impact or a control-law switch occur in an engineering system can be remarkably small compared with that of the overall dynamics, and thus, the correct global model is certainly discontinuous on a macroscopic timescale. Furthermore, relatively simple phenomena when considered from the point of view of piecewise-smooth systems often turn out to be natural limits of far more complex scenarios observed in smoother systems. For example, it is quite natural for a piecewise-smooth system to undergo a sudden jump from strongly stable periodic motion to full scale chaotic motion under variation of a parameter. In a smooth system, such a scenario would typically require an infinite sequence of bifurcations to occur, such as the famous Feigenbaum cascade of period-doubling bifurcations, leading to chaos.

A second reason for the exclusion of piecewise-smooth systems from the established literature is that they challenge many of our assumptions about dynamics. For example, how can we define concepts such as structural stability, bifurcation and qualitative measures of chaos in such systems? By making careful assumptions about the problems we investigate, which are not inconsistent with the physical problems leading to them, it will become apparent that many of the concepts once thought to be the domain of smooth systems only, naturally extend to piecewise-smooth ones as well. But, and this is the main thrust of this book, there are also dynamical phenomena that are unique to piecewise-smooth systems that are, nevertheless, straightforward to analyze.

The purpose of this introductory chapter is to be a self-contained and non-technical guide to piecewise-smooth dynamical systems, which will outline the more detailed treatment given in the later chapters; but can be read independently from them. We will establish the basic foundations for discussion of non-smooth dynamics in an informal, non-technical and applications-oriented setting, through the description of case study examples arising from physical models. We will also show how bifurcations in piecewise-smooth *flows* (systems of ordinary differential equations) naturally generate piecewise-smooth *mappings*, or *maps* (discrete-time iteration processes), which is a connection that lies at the heart of this book. The chapter is essentially designed to be read like an extended essay. *Italicized terms* are used to introduce mathematical concepts that will be defined more accurately later on in the book. Also, the application-oriented nature of the essay is aimed at answering the question of why piecewise-smooth systems are worth studying.

As a first motivating example of a piecewise-smooth system, consider the operation of a domestic central heating system that is trying to achieve a desired temperature θ. If this temperature is exceeded, a thermostat causes a switch to turn off the power supply to a boiler. The system then evolves smoothly with the heating *off*, until the temperature falls below θ. At this point the system dynamics changes, as the boiler is turned *on* and a different set of evolution rules apply. Thus, if we view the switching process as taking an infinitesimally short time compared with the heating and cooling phases, we can view the dynamics of the temperature $T(t)$ as being that of a continuous piecewise-smooth flow. Two different smooth flow regimes describe the off and on states, with switching occurring when the dynamics crosses the boundary $T(t) = \theta$ between them.

Let us suspend belief for a moment and imagine an instantly responsive heating system. The natural dynamics would then be a state known as *sliding* in which $T(t)$ is permanently set to the threshold value θ, with the thermostat poised between the on and off positions. As the temperature rises above threshold, the boiler is switched off, which instantaneously causes the temperature to fall below threshold. Thus, the boiler is reignited, causing the temperature to rise above threshold, and so on. We shall see shortly that sliding corresponds to a natural state of so-called *relay* controllers and also to the stick phase of systems with dry friction that can exhibit stick-slip motion.

Returning to the more realistic situation where changes in temperature lag behind the turning on or off of the boiler, we can consider the dynamics of this example as being driven by *events*. The events are the times t at which $T(t) = \theta$ and switching occurs. The system evolves smoothly between events such that we can easily define a discrete-time *event map* that expresses the system state at one switching as a function of the state at the previous switch. This map, which may be smooth or non-smooth, effectively has a lower-dimensional state space since we know that the temperature is at threshold. Suppose now that the heating system has a timer device that switches on and off the heating at fixed times each day. In this case we could consider sampling the temperature at 24-hour intervals, producing a *stroboscopic* map that expresses the system state at a fixed time each day as a function of the state at the same time the previous day. This map is unlikely to be smooth, because the dynamics of a system that starts with a temperature above θ is likely to be different from one that starts below.

This simple example demonstrates that any discussion of piecewise-smooth systems should naturally include both flows and maps. A third naturally arising kind of piecewise-smooth system is a combination of a flow and a map, and we shall call these *hybrid* systems. Such systems arise when the effect of the flow reaching the switching threshold is to cause an instantaneous jump in the flow (which in effect becomes discontinuous). In the heating system, this might occur if the result of the temperature dropping to θ is to instantaneously turn on an electric fire that heats the house very much more rapidly than the boiler, so that (on a 24-hour timescale) we see an effectively instantaneous temperature rise. We begin our more detailed discussion of case studies with a class of hybrid systems that have played a key role in the historical development of the theory of piecewise-smooth systems.

1.2 Impact oscillators

Consider the motion of an elastic ball bouncing vertically on a rigid surface such as a table. In unconstrained motion the ball falls smoothly under gravity between impacts and has an 'instantaneous' reversal of its velocity at each impact. Suppose that a simple Newtonian restitution law applies such that reversed velocity is a coefficient $0 \leq r \leq 1$ times the incoming velocity. Typical

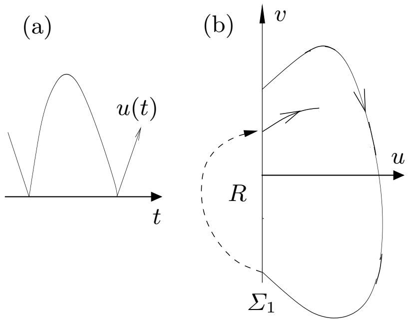

Fig. 1.1. Sketch figure of both (a) the position $u(t)$ of a bouncing ball against time and in (b), (u, v)-*phase space*, where $v(t)$ is the velocity of the ball. Here R is the map that takes v to $-rv$.

motion of the ball is represented in Fig. 1.1. Note that if $r < 1$ then a state where the ball is at rest (*stuck*) on the table can be reached by simply releasing the ball. After an infinite number of impacts (an accumulation of a *chattering sequence*), but a *finite* time, the ball comes to rest. If we were to allow the possibility of an oscillating rigid table (like a tennis player bouncing the ball on his racket between rallies), then the dynamics can be incredibly rich [124, Ch. 2], as we are about to see in a related model.

A bouncing ball is just a simple example of what are termed *impact oscillators*, which are low-degree-of-freedom mechanical systems with hard constraints that feature *impacts* (like the bounce of the ball on the table). impact oscillators have played an important role in the historical development of piecewise-smooth systems. Their dynamics has been studied in the Czech and Russian literature since the 1950s (see, e.g., [98, 19] and references therein, especially [216, 217]), much or which was essentially rediscovered in the Western literature in the 1980s and 1990s [237, 236, 251, 264, 197, 43, 102, 18, 67].

Impacting behavior is found in a large number of mechanical systems ranging from gear assemblies [146, 149, 229, 249], impact print hammers [128, 256], walking robots [138], boiler tube dynamics [212, 122], metal cutters [267], car suspensions [29], vibration absorbers [234], [20], percussive drilling and moling [269, 163] and many-body particle dynamics [228]; see also Fig. 1.1. The effect of the rigid collisions is to make these systems highly nonlinear, and chaotic behavior becomes the rule rather than the exception. Collisions also lead to associated wear on the components of the system. If these components are, for example, the tubes in a boiler [122] or gear teeth [146], then it is crucial to estimate the average wear that might occur in certain operating conditions.

We will not consider the detailed physics of the impacting process in this book. Such processes can be highly subtle, especially when involving the impact of rough bodies, which may also involve friction. It is well covered in the many texts on impact mechanics and tribology; see for example [243, 279]. Instead, like in the bouncing ball example, we shall take simple coefficient of restitution impact laws, which, despite their simplicity, we will see can give a close match to experimental observations.

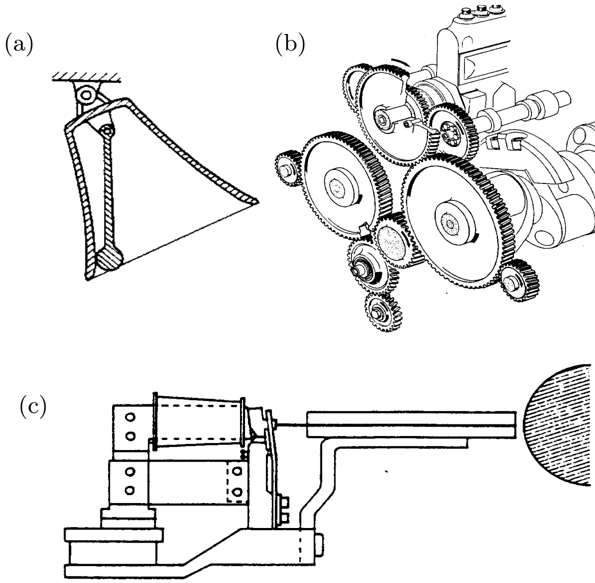

Fig. 1.2. Some examples of vibro-impact systems taken from [208], (a) a bell, (b) a gear assembly and (c) an impact print hammer. (Reprinted from [208] with permission from ASME).

We shall look at two case study examples of one-degree-of-freedom impacting systems. First, we consider a simple model that contains an instantaneous impact, where we find analytical, numerical and experimental evidence for complex dynamics and the period-adding route to chaos. Second, in Sec. 1.2.4 we consider how this dynamics might arise via taking the limit of a sequence of, possibly more realistic, continuous models that feature compliant impact. Chapters 4, 6 and 7 will complete these studies by first presenting a general theory of non-smooth maps, then of hybrid systems of arbitrary dimensions (which includes impact oscillators as a special case), and finally of continuous flows. The presentation of these case studies will draw heavily on work by Peterka [216, 217], Nordmark [197], Whiston [264, 263, 265], Chillingworth [53], Shaw & Holmes [237, 236], Thompson & Bishop [251, 103, 30, 31, 260], Budd [171, 42, 43, 44, 45] and their co-workers.

1.2.1 Case study I: A one-degree-of-freedom impact oscillator

Consider the motion of a body in one spatial dimension, which is completely described by the position $u(t)$ and velocity $v(t) = \frac{du}{dt}$ of its center of mass. Thus we think of this body as a single particle in space. When in free motion, we suppose that there is a linear spring and dashpot that attach this particle to a datum point so that its position satisfies the dimensionless differential equation

$$\frac{d^2u}{dt^2} + 2\zeta\frac{du}{dt} + u = w(t), \quad \text{if} \quad u > \sigma. \tag{1.1}$$

Here, the mass and stiffness have been scaled to unity, 2ζ measures the viscous damping coefficient, and $w(t)$ is an applied external force. We assume that motion is free to move in the region $u > \sigma$, until some time t_0 at which $u = \sigma$ where there is an impact with a rigid obstacle. Then, at $t = t_0$, we assume that $(u(t_0), v(t_0)) := (u_-, v_-)$ is mapped in zero time to (u^+, v^+) via an *impact law*

$$u^+ = u^- \quad \text{and} \quad v^+ = -rv^-, \tag{1.2}$$

where $0 < r < 1$ is Newton's coefficient of restitution. An idealized mechanical model of this system is given in Fig. 1.3.

The simplest form of forcing function $w(t)$ can arise from an excitation of the lower part of the oscillator. An equivalent problem is to set $w(t) = 0$ in (1.1) but to introduce an excitation on the whole system by moving the obstacle (so that σ becomes a function of time) and using a collision law that takes into account the relative velocity between the particle and the moving obstacle so that

$$v^+ - d\sigma/dt = -r\left(v^- - d\sigma/dt\right).$$

A simple translation in space, setting $\widehat{u}(t) = u(t) - \sigma(t)$, and $\widehat{v}(t) = v(t) - d\sigma/dt$, and dropping the hats recovers (1.1).

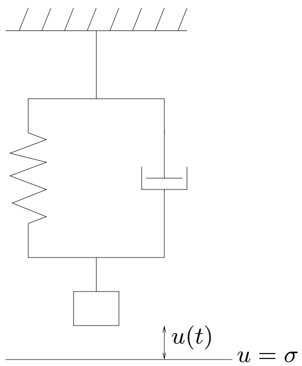

Fig. 1.3. A simple impact oscillator.

Note that r^2 measures the percentage of the kinetic energy that is absorbed in the impact. The case $r = 1$ gives an *elastic* collision [170] (often assumed in simulations of granular media, for example, [228]) and $r = 0$ is a completely dissipative collision [238] (modeling, for example, the behavior of a clapper inside a church bell [33]). In experiments, e.g. [209], a value of $r = 0.95$ is found to be reasonable to model the case of a steel bar impacting with a rigid point, whereas in [260], a different value of r was found to provide the best fit for an impacting cantilever beam; see also [91]. This indicates that the value r depends not only on the material properties of the impacting components, but also on their geometry. This is because the restitution law represents the overall effect of a much more rapid process of energy dissipation through the propagation of shock waves (those of which in the audible range we hear as the crack or bang associated with impact).

There have been many analytical and experimental investigations of the forced impact oscillator with different types of forcing function $w(t)$; see [30] for a survey. In this case study we concentrate on periodic sinusoidal forcing:

$$w(t) = \cos(\omega t), \qquad \text{with period} \quad T = 2\pi/\omega. \tag{1.3}$$

However, the literature also includes discussions of forcing caused by an external flow such as vortices shed from a boiler tube [57] or from an ocean wave [172], quasi-periodic forcing [215, 214], stochastic forcing [276, 45] and problems where $w(t)$ is the solution of another problem, for example, a further impact oscillator. The latter case arises quite commonly when energy is transmitted via impacts in a loosely fitting mechanical structure, of which the executive toy 'Newton's cradle' is a simple example.

It is difficult, in practice, to realize such a system exactly in an experiment. There is no such thing as a perfect, instantaneous impact, as the action of the impact excites higher oscillatory modes in almost any vibrating system. This difficulty can be reduced (although not entirely eliminated) by using a highly massive moving object. Such an experimental impact oscillator used by Popp and co-authors [209, 132] is depicted in Fig. 1.4. Here, a massive beam is mounted on an almost frictionless air bearing and is allowed to move freely under the restoring force of two springs that are carefully engineered to behave elastically. The beam is excited by an electromagnetic field and repeatedly comes into contact with a rigid stop. The velocity of the beam is measured at discrete time intervals by using a laser-Doppler device, and this measurement converted to a position measurement by integration. Results from this experiment will be referred to several times in what follows and will be compared with the results of numerical simulation of (1.1)–(1.3).

In the absence of impacts, the system (1.1) is *linear* and is therefore easy to analyze. Its solutions comprise exponentially decaying free oscillations converging to driven periodic motions at frequency ω. The form of these periodic solutions is unique, up to phase, independent of initial conditions, and does not change a great deal under parameter variation, provided that we avoid

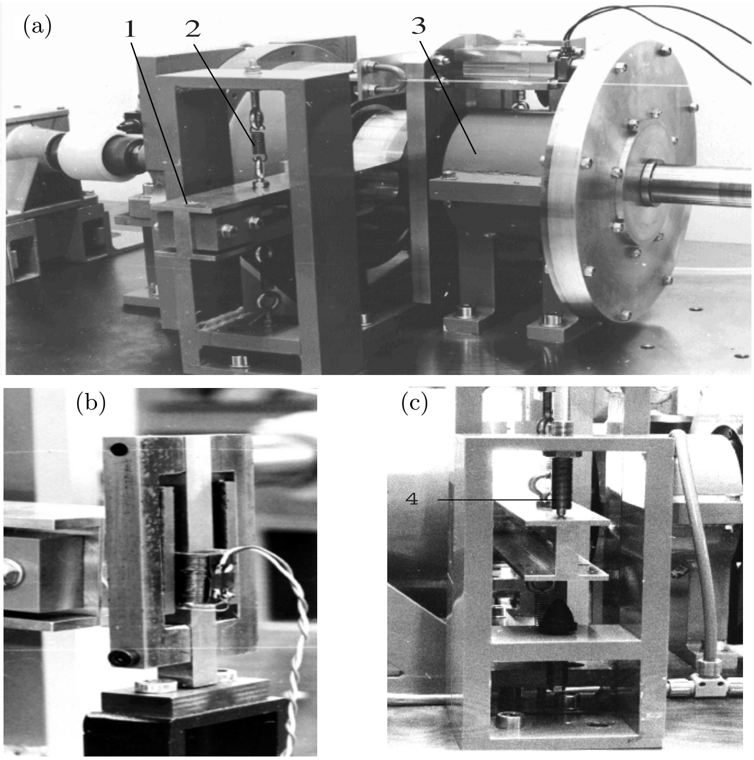

Fig. 1.4. An experimental impact oscillator, after [208]. In this figure (a) shows (1) the beam (2) the restoring springs and (3) the frictionless air bearing. Panel (b) shows the electromagnetic excitation and (c) the impact with the rigid obstacle. (Reprinted from [208] with permission from ASME).

natural resonances $\omega = n$ for any integer n. This state of affairs changes completely when impacts occur, introducing a strong *nonlinearity* into the system. Then we observe a multitude of different possible recurrent behaviors, which include *periodic motions* of both higher and lower frequency than ω, and much more irregular *chaotic* motions in which the *orbit* $u(t)$ is highly irregular and is acutely sensitive to its initial conditions The number and nature of these different types of behavior now depend sensitively on the different parameters in the system.

We can easily look at the dynamics of different types of such orbit by plotting the solution *trajectories* of the solution in the *phase plane* (u, v). Note that the phase space of this system is actually three-dimensional because for a complete description of the dynamics we must include the phase variable

$$s = t \bmod 2\pi/\omega.$$

Examples of three qualitatively different solutions of the idealized simple impact oscillator are given in Fig. 1.5 for three differing, nearby input frequencies. Here we see (a) periodic motion with two impacts per period, (b) more complicated periodic motion and (c) chaotic motion.

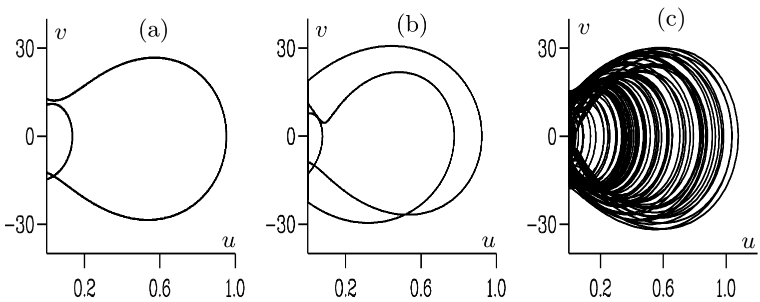

Fig. 1.5. Solutions of the idealized impact oscillator (1.1)–(1.3) in which $\sigma = 0$, $r = 0.95$, $\zeta = 0$ and (a) $\omega = 3$, (b) $\omega = 2.76$, (c) $\omega = 2.9$. (Reprinted from [208] with permission from ASME).

It is valuable to compare the solutions of this simple model with those seen in an experiment. For the experimental set up illustrated above, at the same parameter values as in the simulation, we have the phase plane plots seen in Fig. 1.6. The quantitative and qualitative agreement with the simulations is striking. The main difference between model and experiment is the excitation of a rapidly decaying higher mode of oscillation immediately after the impact. However, this does not seem to have significant effect upon the global dynamics. Note that the chaotic motion is entirely the result of the impacting behavior and is quite different from solutions to a linear differential equation, even though the motion between impacts is completely described by a linear model.

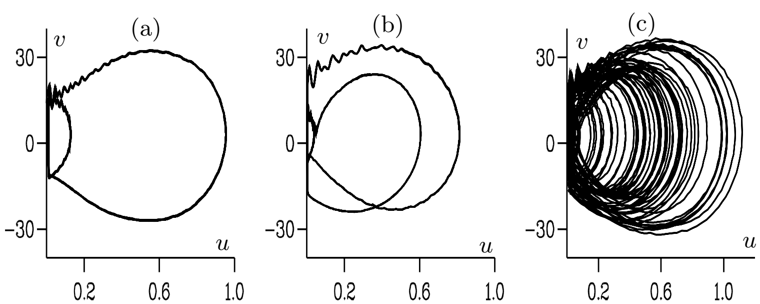

Fig. 1.6. The dynamics of the experimental impact oscillator at the corresponding parameter values to those in Fig. 1.5. (Reprinted from [208] with permission from ASME).

Consider now the general motion of an impact oscillator. It is simplest to start the analysis by assuming that the particle described by the oscillator starts at the obstacle with an initial velocity of $v_0 > 0$, at a time t_0 and a corresponding phase s_0. The motion of the particle is then described by the linear system (1.1) with initial data $u(t_0) = \sigma$ and $v(t_0) = v_0$. Provided that $v_0 > 0$, this motion will initially lie in the region $u > \sigma$, and in general (certainly if $\zeta = 0$), the trajectory will strike the obstacle at a later time t_1 with velocity $-v_1/r < 0$. After the impact, the velocity is v_1. Setting $v = v_1$ and $t = t_1$ the motion then continues as above. The overall dynamics is thus a series of smooth flows, interrupted by discontinuous changes in velocity.

Things are rather different if $v_0 = 0$ at the point of release at $t = t_0$. If $d^2u/dt^2 = f(t_0) - \sigma < 0$, then the particle cannot move and remains *stuck* to the obstacle until it has a positive acceleration. (A simple example of this being the motion of any particle under gravity, which, when placed on an obstacle with zero velocity will simply stay stuck to that obstacle.) The region over which sticking occurs is called the *sticking region* $\mathcal{Z} = \{(u, v, t) = (\sigma, 0, t) | w(t) - \sigma < 0\}$. If $0 < r < 1$, then the particle generically enters a sticking phase via an *infinite* sequence of impacts, a *chattering sequence.* (If $r = 0$, a particle impacting with $f(t_0) - \sigma < 0$ will stick immediately.)

Returning to the case with $v_0 > 0$, let us try to construct solutions analytically. It is easiest to look at the case of no viscous damping $\zeta = 0$ (which we shall henceforth assume unless otherwise stated), which is with little loss of generality if $r < 1$, because the restitution law provides the largest source of damping on the system. The differential equation (1.1) is linear and so can be solved using elementary methods. Taking the initial condition $u(0) = s_0$, $\frac{du}{dt}(0) = v(0) = v_0$, we get

$$u(t; v_0, s_0) = (\sigma - \gamma C_0)\cos(t - s_0) + (v_0 + \omega\gamma S_0)\sin(t - s_0) + \gamma C(t), \quad (1.4)$$

where

$$\gamma = \frac{1}{1 - \omega^2}, \quad C(t) = \cos(\omega t), \quad S(t) = \sin(\omega t), \quad C_0 = C(s_0), \quad S_0 = S(s_0). \quad (1.5)$$

Now suppose that the orbit described by the flow (1.4) impacts with the obstacle at a later time t_1 so that

$$u(t_1; v_0, s_0) = \sigma, \quad (1.6)$$

with a velocity $-v_1/r$ before impact, and velocity v_1 after impact [Fig. 1.7(a)]. Such trajectories implicitly define an *impact map* P_I relating the time (phase) and velocity of one impact to that of the next,

$$P_I(s_0, v_0) = (s_1, v_1). \quad (1.7)$$

We can continue this analysis further to look at subsequent impacts at times t_i with velocities $v_i > 0$ immediately after impact, so that $(t_{i+1}, v_{i+1}) =$

$P_I(t_i, v_i)$. As we are considering a system that is periodically forced with period T, we can also define an alternative *stroboscopic* Poincaré map:

$$P_S(u(t), v(t)) = (u(t+T), v(t+T)), \tag{1.8}$$

which we use a lot in the later analysis of the impacting system. Note that in computing P_S we must determine all impacts in the interval $(t, t+T)$. Even for the simple linear system described in (1.1) the computation of the impact time t_1 from (1.6) using (1.4) involves solving a (nonlinear) transcendental equation. Hence, even though the system is piecewise-linear, we should regard the system as fully nonlinear, since its evolution requires knowledge of t_1. Indeed, the general impossibility of solving such equations as (1.6) in closed form renders the distinction between piecewise-linear and piecewise-smooth systems essentially meaningless. For both, the grossest nonlinearity is usually that introduced by interaction with a discontinuity surface. Fortunately, efficient numerical methods exist to compute the smooth flows, to determine the impact times and to follow these as the solution parameters vary.

A key feature of all the analysis in this book is a study of how solutions close to certain distinguished trajectories of piecewise-linear systems behave. Let us consider such analysis in the context of the impact oscillator, for the case of a trajectory that impacts. To begin with consider the case in Fig. 1.7(a) where the velocity v_1 is not small and the trajectory τ impacts with the obstacle at times t_1, t_2, etc. In this case if we look at a trajectory that starts close to τ (so that it leaves the obstacle at a time close to t_0 with an initial velocity close to v_0), then it will remain close to τ at least up to the time t_2.

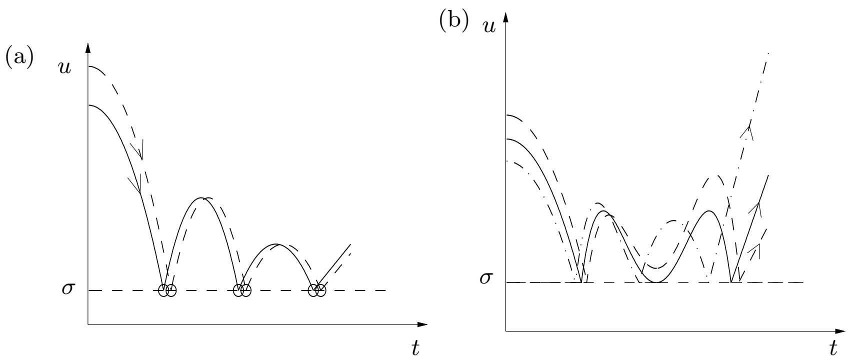

Fig. 1.7. (a) An impacting trajectory (solid) with a high-velocity impact and a nearby trajectory (dashed) projected onto the (t, u)-plane. (b) An impacting trajectory (solid) with a zero velocity impact at t_1 and two nearby trajectories, one (dashed) with no impact close to t_1 and one (dot-dashed) with a low velocity impact close to t_1.