

Springer Monographs in Mathematics

Marek Jarnicki
Peter Pflug

Continuous Nowhere Differentiable Functions

The Monsters of Analysis

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Preface

As its title indicates, this book aims to be a comprehensive, self-contained compendium of results on continuous nowhere differentiable functions, collecting many results hitherto accessible only in the scattered literature.

Motivation for Writing This Book

Why did the authors, both specialists in several complex variables, decide to write a book on continuous nowhere differentiable functions? Let us try to answer this question:

- (a) Whenever we would give a lecture on real analysis, we felt unsatisfied, since there was almost no time to discuss continuous nowhere differentiable functions in detail. Therefore, we could only mention the existence of such functions in most of our lectures. Moreover, whenever we wanted to deal with such functions in a proseminar, it was difficult to find a source book. Some information could be found in a master's thesis by J. Thim (see [Thi03]), which presented a more detailed description of these functions. Later, during the writing of this book, we found another survey article by A.N. Singh (see [Sin35]). With few sources available, we thought that a modern and complete description of how these functions appeared would be of great use, both for students and for colleagues creating their lectures and preparing proseminars.
- (b) Looking back to the middle of the nineteenth century, we see that that was an important time in the history of mathematics, when many arguments turned from being based more or less on heuristics into being grounded in precise definitions and proofs. We are still experiencing the consequences of this birth of mathematical precision. It is interesting to see how the methods used to discuss continuous nowhere differentiable functions has changed over time and to observe that there are still problems that have not been solved.

We hope that the reader will accept our motivation and that our book can be used for learning some very nice mathematics or for preparing proseminars or lectures on analysis.

Remarks for the Reader

To make a big part of the material accessible even to high-school graduates, we ordered the content into four main parts:

- Part I: Classical results.
In this part are collected all results from the middle of the nineteenth century up to about 1950. The proofs are based on complicated arguments, but to understand them requires only some basic facts from analysis.
- Part II: Topological methods.
This part is based on standard techniques from functional analysis that are certainly taught in any beginning course.
- Part III: Modern approach.
This part requires some more highly developed ideas from analysis, such as measure theory and Fourier transforms.
- Part IV: The Riemann function.
This part is in some sense unusual. On the one hand, it does not directly follow the theme of the book, since the Riemann function discussed here does not belong to the class of nowhere differentiable functions. On the other hand, it is more difficult and requires knowledge from several different fields of mathematics. To help the reader, we have placed such information in an appendix.

Nevertheless, we are convinced that at least 10 % of the book may be understood by high-school graduates, 40 % by students of mathematics who have completed a first analysis course, and the remainder by master's-level students.

We did not include any exercises, as they can be found in many textbooks. But the reader will find the word EXERCISE at different places in the text. It is at such points that the reader is asked to stop reading and to extend our arguments into greater detail.

Moreover, whenever some function is discussed in the book, the reader is asked to continue its study. For example, if f is claimed to be nowhere differentiable on the interval $[0, 1]$ and nothing, even later in the text, is said about infinite derivatives, then the reader should try to discuss this question on his own. In any case, any additional information in such directions that we have found in the literature has been added to the text.

Each chapter begins with a brief summary of its content. Moreover, the reader will find open problems in some chapters. They are indicated by the sign $\boxed{?} \dots \boxed{?}$. All these problems are collected at the end of the book, see List of Problems section in Appendix C. The reader is asked to work on these questions, although they do not seem to be simple to solve. For notation that may appear in the text without explanation, the reader is asked to consult Sect. B.1.

We wish to thank all our colleagues who told us about gaps in this book during its writing. In particular, we thank Dr. P. Zapałowski for all the corrections he made. It would not have been possible to reach the current level of presentation without his precise and detailed observations. Nevertheless, according to our experiences with our former books, we are sure that many errors have remained, and we are responsible for not detecting them.

We will be pleased if readers inform us about any observations they may have while studying the text. Please use the following e-mail addresses:

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Isaac Newton (1643–1727) first developed calculus having been inspired by the physical world: the orbit of a planet, the swing of a pendulum, perhaps even, as legend has it, the motion of falling fruit. His thinking led to a geometric intuition about mathematical structures. They should make sense in the same way that a physical object would. As a result, many mathematicians concentrated on “continuous” functions. Conceptually, these are the functions that can be drawn without taking pen away from paper. There will be no gaps or sudden jumps.

A first “definition” of a function was given by Leonhard Euler (1707–1783) in [Eul48], page 4: “A function of a real variable is an analytic expression that is built from the variable, numbers, and constants.”¹ Functions in that sense are automatically everywhere continuous (in the modern sense) up to possibly a discrete set of discontinuities.

Nevertheless, the notion of a function remained a vague one for a long time. It seems that in 1873, Lejeune Dirichlet (1805–1859) became the first to give a precise definition (see [DS00], §1): “Fix two values a and b . Then x may be thought as a quantity that may take all values between a and b . Assume that to every x a value $y = f(x)$ is associated such that if x runs continuously through the interval from a to b , then $y = f(x)$ changes also in a continuous way. Then y is called a *continuous function of x on the interval*. It is not necessary that y be built according to one law for each x ; even more, there is no need to think of this relation in the form of a mathematical operation.”²

Even more, Dirichlet pointed out that his definition does not require a common rule regarding how such a function should be built. It is allowed that the function may be constructed from different pieces or even more, it may be given without a common rule for its pieces.³

Note that Dirichlet defines a “continuous function,” but it is clear how the term function has to be understood out of his definition. It is important and new that a function is no

The chapter “Introduction: A Historical Journey” published in the book “Continuous Nowhere Differentiable Functions”, pages 1–6, DOI [10.1007/978-3-319-12670-8_1](https://doi.org/10.1007/978-3-319-12670-8_1), has been retracted by the request of the Editor, because portions of the text are duplicated without permission from a previously published article by Adam Kucharski.

¹ “*Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantita variabili et numeris seu quantitibus constantibus.*”

² “Man denke sich unter a und b zwei feste Werthe und unter x eine veränderliche Grösse, welche nach und nach alle zwischen a und b liegenden Werthe annehmen soll. Entspricht nun jedem x ein einziges endliches y und zwar so, dass, während x das Intervall von a bis b stetig durchläuft, $y = f(x)$ sich ebenfalls allmählich verändert, so heisst y eine stetige oder continuirliche Function von x für dieses Intervall. Es ist dabei gar nicht nöthig, dass y in diesem ganzen Intervalle nach demselben Gesetze von x abhängig sei, ja man braucht nicht einmal an eine durch mathematische Operationen ausdrückbare Abhängigkeit zu denken.”

³ “Diese Definition schreibt den einzelnen Theilen der Curve kein gemeinsames Gesetz vor; man kann sich dieselbe aus den verschiedenartigsten Theilen zusammengesetzt oder ganz gesetzlos gezeichnet denken.” See [DS00], § 153.

longer something that is given by a closed analytic expression. It is the above definition that is familiar to today's mathematicians: to any point x of a certain set X one and only one value $f(x)$ is given, and the whole association is called the function f .

Nevertheless, the experiences at that time made people believe that for every continuous curve, it was possible to find the slope at all but a finite number of points. This seemed to match intuition: a line might have a few jagged bits, but there would always be a few sections that were “smooth.” The French physicist and mathematician André-Marie Ampère (1775–1836) even published a proof of this claim (see [Amp06]). His argument was built on the “intuitively evident” fact that a continuous curve must have sections that increase, decrease, or remain flat. This meant that it must be possible to calculate the slope in those regions. Ampère did not think about what happened when the sections became infinitely small, but he claimed that he did not need to. His approach was general enough to avoid having to consider things that were “infiniment petits.” Most mathematicians were happy with his reasoning. By the middle of the nineteenth century, almost every calculus textbook quoted Ampère's proof.

But during the 1860s, rumors began circulating about a strange function that contradicted Ampère's theorem. In Germany, the great Bernhard Riemann (1826–1866) told his students that he knew of a continuous function that had no smooth sections, and for which it was impossible to calculate the derivative of the function at any point. Riemann did not publish a proof, and neither did Charles Cellérier (1818–1889), at the University of Geneva, who—despite writing that he had discovered something “very important and I think new”—stuffed the work into a folder that would become public only after his death decades later (see [Cel90]). Over the years, it was found that the function Riemann proposed does not fulfill the property of being nowhere differentiable. Although his function is, in fact, somewhere differentiable, we decided to put an extensive discussion of this function into our book, showing the current state of knowledge (see Chap. 13).

Such a monster of a function was finally publicly accessible in 1872, when Karl Weierstrass (1815–1897) announced in a lecture in front of the Königlische Akademie der Wissenschaften, Berlin, that he had found a function that was continuous everywhere and yet not smooth at any point. He had constructed it by adding together an infinitely long sequence of cosine functions. To be more precise, it is given by the following formula:

$$f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x), \quad x \in \mathbb{R},$$

where $a \in (0, 1)$, b is an odd integer, and $ab > 1 + \frac{3}{2}\pi$.

As a function, it was ugly and awkward. It was not even clear what it would look like when plotted on a graph. But that did not matter to Weierstrass. His proof consisted of equations rather than shapes, and that is what made his announcement so powerful. Not only has he created a *monster*, he has built it from concrete logic. He had taken his new, rigorous definition of a derivative and shown that it was impossible to calculate one anywhere for this new function.

The lecture by Weierstrass was not immediately published, but it seems that his example reached many mathematicians at that time. Thus Paul du Bois-Reymond (1831–1889) wrote to Weierstrass asking for details. After Weierstrass had sent him his notes, Bois-Reymond published the example (see [BR74]). Bois-Reymond added the following comment, showing the influence that this example had had on him: “There is not only no implication between continuity and differentiability at one point, but it is an exciting result that there exists a

continuous function in an interval having no differential quotient at any point of it.”⁴ This is the first example of a continuous nowhere differentiable function published in a mathematical journal.

After the Weierstrass lecture and before its publication by Bois-Reymond, Gaston Darboux (1842–1917) also observed another similar monster. He showed that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin((n+1)!x)}{n!}, \quad x \in \mathbb{R},$$

is continuous but nowhere differentiable (see [Dar75, Dar79]). His proof in the first cited paper is very sketchy, while the second paper contains more details of the proof. It is interesting to observe that in his preface to the first paper, he mentioned names like Riemann, Hankel, Schwarz, and Klein, but omitted to cite Weierstrass. This was also the case in the second paper, even though Weierstrass had protested in a letter to Bois-Reymond, claiming that the first examples were due to him (see [Wei23], page 211).

Also Ulisse Dini (1845–1918) published in 1877 a paper (see [Din77]) in which he presented another example, namely

$$F(x) = \sum_{n=1}^{\infty} \frac{a^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cos(1 \cdot 3 \cdot 5 \cdots (2n-1)x), \quad x \in \mathbb{R},$$

which is continuous but nowhere differentiable if $a > 1 + \frac{3}{2}\pi$. He referred to the example of Weierstrass, but his aim was to find other such strange functions.

This result⁵ threw the mathematics community into a state of shock. The French mathematician Émile Picard (1856–1941) pointed out that if Newton had known about such functions, he would have never created calculus. Rather than harnessing ideas about the physics of nature, he would have been stuck trying to clamber over rigid mathematical obstacles. The monster also began to trample over previous research. Results that had been “proven” began to buckle. Ampère had used the vague definitions favored by Cauchy to prove his smoothness theorem. Now his arguments began to collapse. The vague notions of the past were hopeless against the monster. Worse, it was no longer clear what constituted a mathematical proof. The intuitive geometry-based arguments of the previous two centuries seemed to be of little use. If mathematics tried to wave the monster away, it would stand firm. With one bizarre equation, Weierstrass had demonstrated that physical intuition was not a reliable foundation on which to build mathematical theories. So this new mathematics (arithmetic analysis) led to a breaking away from trusting one’s intuition, geometric or otherwise.

Established mathematicians tried to brush the result aside, arguing that it was awkward and unnecessary. They feared that pedants and troublemakers were hijacking their beloved subject. At the Sorbonne, Charles Hermite (1822–1901) wrote to Stieltjes (see [BB05], page 318): “I turn with terror and horror from this lamentable scourge of functions with no derivatives.”⁶ Henri Poincaré (1854–1912)—who was the first to call such functions monsters—

⁴ “Mit der Existenz eines Differentialquotienten hat die Bedingung der Stetigkeit nicht allein für einen einzelnen Punkt nichts zu schaffen, sondern es ist eines der ergreifendsten Ergebnisse der neueren Mathematik, dass eine Funktion in *allen* Punkten eines Intervalles stetig sein kann, ohne für einen Punkt dieses Intervalles einen bestimmten Differentialquotienten zu ergeben.”

⁵ The present paragraph and others as well are taken from the lovely article [Kuc14], sometimes word for word (see also [Vol1987, Vol1989]).

⁶ “Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions continues qui n’ont point de dérivées.”

denounced Weierstrass's work as "an outrage against common sense." He claimed that the functions were an arrogant distraction, and of little use to the subject. "They are invented on purpose to show that our ancestors' reasoning was at fault," he said, "and we shall never get anything more out of them." See [Poi99], page 159.

Many of the old guard wanted to leave Weierstrass's monster in the wilderness of mathematics. It did not help that nobody could visualize the shape of this strange function they were dealing with—only with the advent of computers did it become possible to plot it. Its hidden form made it hard for the mathematics community to grasp how such a function could exist. Weierstrass's style of proof was also unfamiliar to many mathematicians. His argument involved dozens of logical steps and ran to several pages. The trail of ideas was subtle and technically demanding, with no real-life analogies to guide the way. The general instinct was to avoid it.

But with the dawn of the twentieth century, situation changed. Even physicists began to discuss strange curves like the Ludwig Boltzmann (1844–1906) nonrectifiable H-curve, which was used to describe the movement of particles in statistical mechanics. In fact, much later, Norbert Wiener (1894–1964) was able to prove that the trajectory of a particle, in view of Brownian motion, is not rectifiable. The twentieth century has forced upon us the inadequacy of so-called ordinary curves to represent the facts of nature. Let us quote the French physicist Jean Baptiste Perrin (1870–1942), who helped to prove that atoms and molecules exist, an achievement that earned him the 1926 Nobel Prize in physics. In his 1913 book *Les atomes*, about the motion of atoms (see the English translation [Per16]), he writes in the introduction: "I wish to offer a few remarks designed to give objective justification for certain logical exigencies of the mathematicians. It is well known that before giving accurate definitions we show beginners that they already possess the idea of continuity. We draw a well-defined curve and say to them, holding a ruler against the curve, 'You see that there is a tangent at every point.' Or again, in order to impart the more abstract notion of the true velocity of a moving object at a point in its trajectory, we say, 'You see, of course, that the mean velocity between two neighbouring points on this trajectory does not vary appreciably as these points approach infinitely near to each other.' And many minds, perceiving that for certain familiar motions this appears true enough, do not see that there are considerable difficulties in this view. To mathematicians, however, the lack of rigour in these so-called geometrical considerations is quite apparent, and they are well aware of this childishness of trying to show, by drawing curves, for instance, that every continuous function has a derivative. Though derived functions are the simplest and the easiest to deal with, they are nevertheless exceptional; to use geometrical language, curves that have no tangents are the rule, and regular curves, such as the circle, are interesting though quite special cases. At first side the consideration of such cases seems merely an intellectual exercise, certainly ingenious but artificial and sterile in application, the desire for absolute accuracy carried to a ridiculous pitch. And often those who hear of curves without tangents, or undervived functions, think at first that Nature presents no such complications, nor even offers any suggestion of them. The contrary, however is true, and the logic of mathematicians has kept them nearer to reality than the practical representations employed by physicists."

Or consider Grace Chisholm Young's (1868–1944) apologia (see [You16a], §18) of continuous nowhere differentiable functions, in which she says, "We of the twentieth century are bound to recognise it in its full importance. These curves (i.e. such without tangents) afford us a means of rendering more veracious the representation of the physical universe by the realm of Mathematics." So the last resistance to this kind of new function gradually disappeared.

In addition to Cellérier, another mathematician, Bernard Bolzano (1781–1841), found a function continuous but not differentiable at many points. This function is contained in Bolzano's book *Funktionenlehre*, written around 1834, but published only in 1930.

The function itself remained unpublished until 1921, when it was discovered by the young Czech mathematician M. Jasek, who was asked by the Bohemian Academy of Sciences to go through Bolzano's manuscripts. Bolzano's function is the limit of a sequence of effectively given piecewise linear functions. Bolzano himself comments thus on his function: "The function F_x considered in I, §75, changes its increasing and decreasing behavior so many times that for no value of x does there exist a small enough w so that it is possible to believe that F_x is continuously increasing or continuously decreasing between x and $x \pm w$. This function gives us a proof that even a continuous function can have no derivative for so many values of the variable that between each two such points there is a third one for which there is also no derivative to be found."⁷

A precise proof that his function is continuous and even nowhere differentiable was given by Karel Rychlik (1855–1968) in 1922 (see his comment in [Ryc23]) and by Vojtěch Jarník (1897–1970) (see [Jar22]). Because of its late publication, this kind of function did not have as great an influence on the early discussions about continuous but nowhere differentiable functions as did the example of Weierstrass.

A number of papers dealing with new examples of continuous nowhere differentiable functions appeared. In fact, in the bibliography of Emde-Boas (see [Boa69]) there are eight articles listed before 1900 and 33 papers during the period 1901–1931; see also the bibliography in [Sin35] and the one for this book. Even more, the Weierstrass example began to appear in several textbooks, for example in U. Dini: *Grundlagen für eine Theorie der Funktionen einer veränderlichen reellen Grösse* (see [Din92]), F. Klein: *Anwendungen der Differential- und Integralrechnung auf Geometrie. Eine Revision der Prinzipien* (see [Kle02]), M. Pasch: *Veränderliche und Funktion* (see [Pas14]), E.W. Hobson: *The theory of functions of a real variable and the theory of Fourier series* (see [Hob26]). For example, let us quote U. Dini from his book, §145: "The theorems proved in the last paragraphs should be able to reject, at least from the better books, the belief up to now that a continuous function has to have a derivative."⁸ Finally, modern mathematics, such as the theory of fractals, has sufficiently proved the importance of the existence of these monster functions.

In developing the discussion of these monster functions, there are first examples that, under certain restrictions on their parameters, can be handled by simple means. The discussion of these particular functions is exactly the content of Part I. Later on, mathematicians became interested in understanding the role of the parameters that lead to a function being nowhere differentiable. More difficult reasoning became necessary to study such functions. Moreover, one-sided derivatives and also infinite derivatives became of interest. Results of this kind will be discussed in Part III.

But apart from all these examples, more is true, namely that most of the continuous functions are monster functions. This kind of investigation has its basis in the theorem of Baire. It was Stefan Banach (1892–1945) who proved that the complement of the set of continuous nowhere differentiable functions is of first category, i.e., is a rather small set. As it turned out, most continuous functions behave in a strange way and are thus themselves monsters of various types. This is the content of Part II. Note that this abstract approach

⁷ "Die in I, §75, betrachtete Function F_x , bey welcher das Steigen und Fallen so vielfach abgewechselt, dass es zu keinem Werthe von x ein w klein genug gibt, um behaupten zu können, dass F_x innerhalb x und $x \pm w$ fortwährend wachse oder fortwährend abnehme, gibt uns einen Beweis, dass eine Function sogar stetig seyn könne und doch keine abgeleitete hat für so viele Werthe ihrer Veränderlichen, dass zwischen je zwey derselben sich noch ein dritter, für welchen sie abermahls keine abgeleitete hat nachweisen."

⁸ "Den in den letzten Paragraphen bewiesenen Sätzen dürfte, wie uns scheint, die Aufgabe zufallen, künftig aus den bessern Lehrbüchern den bis in die neueste Zeit als Grundlage der Differentialrechnung figurirenden Leitsatz zu verdrängen, nach welchem die Existenz der Derivierten jeder endlichen und stetigen Function wenigstens im Allgemeinen ausser Zweifel sein sollte."

does not give any effective example of such a function. Thus it makes the study of concrete examples not superfluous at all. The notion of being of first category has certain refinements such as porosity. Looking at even stranger monsters such as continuous functions having nowhere finite or infinite one-sided derivatives ended with a negative result: those functions are rare among the continuous ones. Such functions, as was shown by Stanisław Saks (1897–1942) in 1932, are of first category among all continuous functions. So there was no immediate deduction that such functions exist. Earlier, in 1924, Abram Samoilovitch Besicovitch (1891–1970) had already constructed such an example using very difficult geometric reasoning. In Chap. 11, we will present, in addition to concrete examples, a categorial argument showing, in fact, that there are many of those monsters.

Later, at the end of the twentieth century and into the current one, there appeared authors who have constructed Weierstrass-type monsters with additional pathologies. It has been a generalized trend in mathematics toward the search for large algebraic structures of pathological objects such as the continuous nowhere differentiable functions. The lineability of this type of functions has been thoroughly studied in recent years. Recall that a subset M of a topological vector space X is called *lineable* (resp. *spaceable*) in X if there exists an infinite-dimensional linear space (resp. an infinite-dimensional closed linear space) $Y \subset M \setminus \{0\}$. These notions of lineable and spaceable were originally coined by V.I. Gurariy (1935–2005). The very first result in this direction was also due to him (see [Gur67, Gur91]). He showed that the set of continuous nowhere differentiable functions on $[0, 1]$ is lineable. Further, V.P. Fonf, V.I. Gurariy, and M.I. Kadets (see [FGK99]) proved that the set of nowhere differentiable functions on $[0, 1]$ is spaceable. To give the reader a feeling for such results, we discuss some of them in Chap. 12.

We close this discussion by emphasizing that we have given only our own historical journey. We do not claim that it is a complete survey.

Part I

Classical Results

Chapter 2

Preliminaries

Summary. This chapter contains definitions and auxiliary results related to various notions of nowhere differentiability. In particular, in § 2.3, we present a proof of the famous Denjoy–Young–Saks theorem, which may permit the reader to understand better the sense of nowhere differentiability.

2.1 Derivatives

Let $I \subset \mathbb{R}$ be an arbitrary interval containing at least two distinct points.

Definition 2.1.1. For a function $\varphi : I \longrightarrow \mathbb{C}$, set

$$\Delta\varphi(t, u) := \frac{\varphi(u) - \varphi(t)}{u - t}, \quad t, u \in I, \quad t \neq u.$$

Recall that φ has a (*finite*) *derivative* $\varphi'(t)$ at a point $t \in I$ if the limit

$$\varphi'(t) := \lim_{I \ni u \rightarrow t} \Delta\varphi(t, u)$$

exists and is finite. In the case $\varphi : I \longrightarrow \mathbb{R}$, we may also consider an *infinite derivative* $\varphi'(t)$ if the limit

$$\varphi'(t) := \lim_{I \ni u \rightarrow t} \Delta\varphi(t, u)$$

exists but is infinite, i.e., $\varphi'(t) \in \{-\infty, +\infty\}$.

Remark 2.1.2. If $\varphi : I \longrightarrow \mathbb{C}$, then

$$\Delta\varphi(u_1, u_2) = \frac{u_2 - t}{u_2 - u_1} \Delta\varphi(t, u_2) + \frac{t - u_1}{u_2 - u_1} \Delta\varphi(t, u_1),$$

$$t, u_1, u_2 \in I, \quad u_1 < t < u_2.$$

Consequently:

- (a) If a finite derivative $\varphi'(t)$ exists at an interior point $t \in \text{int } I$, then

$$\varphi'(t) = \lim_{\substack{u_1, u_2 \rightarrow t \\ u_1 < t < u_2}} \Delta\varphi(u_1, u_2);$$

note that this fact was already known to T.J. Stieltjes (cf. [Sti14]).

- (b) If $\varphi : I \rightarrow \mathbb{R}$, then

$$\min\{\Delta\varphi(t, u_2), \Delta\varphi(t, u_1)\} \leq \Delta\varphi(u_1, u_2) \leq \max\{\Delta\varphi(t, u_2), \Delta\varphi(t, u_1)\},$$

$$t, u_1, u_2 \in I, u_1 < t < u_2.$$

In particular, if an infinite derivative $\varphi'(t)$ exists at an interior point $t \in \text{int } I$, then

$$\varphi'(t) = \lim_{\substack{u_1, u_2 \rightarrow t \\ u_1 < t < u_2}} \Delta\varphi(u_1, u_2).$$

Definition 2.1.3. Let $\varphi : I \rightarrow \mathbb{C}$, $t \in I$. We say that φ has a *finite right-* (resp. *left-*) *sided derivative* $\varphi'_+(t)$ (resp. $\varphi'_-(t)$) at t if the limit

$$\varphi'_+(t) := \lim_{\substack{I \ni u \rightarrow t \\ u > t}} \Delta\varphi(t, u) = \lim_{I \ni u \rightarrow t+} \Delta\varphi(t, u)$$

$$\left(\text{resp. } \varphi'_-(t) := \lim_{\substack{I \ni u \rightarrow t \\ u < t}} \Delta\varphi(t, u) = \lim_{I \ni u \rightarrow t-} \Delta\varphi(t, u) \right)$$

exists and is finite. In the case $\varphi : I \rightarrow \mathbb{R}$, we allow *infinite one-sided derivatives* $\varphi'_\pm(t) \in \{-\infty, +\infty\}$. Notice that:

- if $t \in I$ is the right endpoint of the interval, then $\varphi'_+(t)$ is not defined and $\varphi'_-(t) = \varphi'(t)$;
- if $t \in I$ is the left endpoint of the interval, then $\varphi'_-(t)$ is not defined and $\varphi'_+(t) = \varphi'(t)$.

One-sided derivatives are also called *unilateral derivatives*.

Remark 2.1.4. Let $\varphi : I \rightarrow \mathbb{C}$.

- (a) If a finite $\varphi'_+(t)$ exists, then for every $C > 0$, we have

$$\varphi'_+(t) = \lim_{\substack{I \ni u', u'' \rightarrow t, t < u' < u'' \\ |\frac{u''-t}{u''-u'}| \leq C}} \Delta\varphi(u', u'').$$

Indeed, we have $\varphi(u) = \varphi(t) + \varphi'_+(t)(u - t) + \alpha(u)(u - t)$, $t < u \in I$, where $\lim_{u \rightarrow t+} \alpha(u) = 0$. Hence

$$\begin{aligned} \Delta\varphi(u', u'') &= \frac{\varphi(t) + \varphi'_+(t)(u'' - t) + \alpha(u'')(u'' - t)}{u'' - u'} \\ &\quad - \frac{\varphi(t) + \varphi'_+(t)(u' - t) + \alpha(u')(u' - t)}{u'' - u'} \\ &= \varphi'_+(t) + \frac{u'' - t}{u'' - u'} \alpha(u'') - \frac{u' - t}{u'' - u'} \alpha(u') \xrightarrow[I \ni u', u'' \rightarrow t]{t < u' < u''} \varphi'_+(t), \end{aligned}$$

provided $\frac{u''-t}{u''-u'}$ is bounded.

- (b) An analogous result may be easily obtained for finite left derivatives.

(c) Notice that (a) is not true for infinite unilateral derivatives.

For example, let $n_1 = 2$, $n_{k+1} = n_k^2$, $k \in \mathbb{N}$. Define $\varphi : [0, \frac{1}{4}] \rightarrow \mathbb{R}$, $\varphi(0) := 0$,

$$\varphi(u) := \frac{1}{n_k}, \quad u \in \left[\frac{1}{n_k^3}, \frac{1}{n_k^2}\right], \quad \varphi(u) := n_{k+1}u, \quad u \in \left[\frac{1}{n_{k+1}^2}, \frac{1}{n_k^3}\right], \quad k \in \mathbb{N}.$$

Observe that φ is continuous and $\varphi'_+(0) = +\infty$. In fact, for $u \in [\frac{1}{n_k^3}, \frac{1}{n_k^2}]$, we have

$$\Delta\varphi(0, u) = \frac{1}{n_k u} \geq n_k. \text{ For } u \in [\frac{1}{n_{k+1}^2}, \frac{1}{n_k^3}], \text{ we have } \Delta\varphi(0, u) = n_{k+1}.$$

Take $u'_k := \frac{1}{n_k^3}$, $u''_k := \frac{1}{n_k^2}$. Then $\Delta\varphi(u'_k, u''_k) = 0$ and $\frac{u''_k - 0}{u'_k - u'_k} \leq 2$.

(d) A finite derivative $\varphi'(t)$ exists at an interior point $t \in \text{int } I$ iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall \begin{matrix} t - \delta \leq a_i \leq t \leq b_i \leq t + \delta \\ a_i, b_i \in I, \quad a_i < b_i, \quad i = 1, 2 \end{matrix} : |\Delta\varphi(a_1, b_1) - \Delta\varphi(a_2, b_2)| < \varepsilon.$$

Indeed, if the above condition is satisfied, then taking $a_1 = a_2 = t$ (resp. $b_1 = b_2 = t$), we conclude that a finite one-sided derivative $\varphi'_+(t)$ (resp. $\varphi'_-(t)$) exists. Taking $a_1 = b_2 = t$, we get $\varphi'_+(t) = \varphi'_-(t)$. Conversely, if $\varphi'(t) \in \mathbb{R}$ exists, then we use Remark 2.1.2(a).

We will use also the following more general derivatives, introduced, e.g., by U. Dini in [Din92].

Definition 2.1.5. Let $\varphi : I \rightarrow \mathbb{R}$, $t \in I$. The *lower* (resp. *upper*) *right Dini derivative* $D_+\varphi(t)$ (resp. $D^+\varphi(t)$) of φ at t is defined as

$$D_+\varphi(t) := \liminf_{I \ni u \rightarrow t+} \Delta\varphi(t, u) \in \overline{\mathbb{R}}$$

$$\left(\text{resp. } D^+\varphi(t) := \limsup_{I \ni u \rightarrow t+} \Delta\varphi(t, u) \in \overline{\mathbb{R}} \right).$$

Analogously, the *lower* (resp. *upper*) *left Dini derivative* $D_-\varphi(t)$ (resp. $D^-\varphi(t)$) of φ at t is defined as

$$D_-\varphi(t) := \liminf_{I \ni u \rightarrow t-} \Delta\varphi(t, u) \in \overline{\mathbb{R}}$$

$$\left(\text{resp. } D^-\varphi(t) := \limsup_{I \ni u \rightarrow t-} \Delta\varphi(t, u) \in \overline{\mathbb{R}} \right).$$

Similarly to the above, $D^+\varphi(t)$ and $D_+\varphi(t)$ (resp. $D^-\varphi(t)$ and $D_-\varphi(t)$) are not defined if $t \in I$ is the right (resp. left) endpoint of the interval.

Remark 2.1.6. (a) $\varphi'_+(t)$ exists iff $D^+\varphi(t) = D_+\varphi(t)$; $\varphi'_-(t)$ exists iff $D^-\varphi(t) = D_-\varphi(t)$.

(b) $D^-\varphi = -D_-(-\varphi)$, $D_+\varphi = -D^+(-\varphi)$.

(c) $D^-\check{\varphi}(t) = -D_+\varphi(-t)$, $D_-\check{\varphi}(t) = -D^+\varphi(-t)$, where $\check{\varphi}(t) := \varphi(-t)$ (provided that $-I = I$).

Remark 2.1.7. If $\varphi : I \rightarrow \mathbb{R}$ is continuous, then the functions $D^+\varphi$, $D_+\varphi$, $D^-\varphi$, $D_-\varphi$ are Borel measurable.

We will prove that $D^+\varphi$ is Borel measurable (the remaining cases are left to the reader as an EXERCISE). We may assume that the right endpoint of I does not belong to I . It suffices to show that for every $C \in \mathbb{R}$, the set $A_C := \{t \in I : D^+\varphi(t) < C\}$ is Borel measurable. Fix a $C \in \mathbb{R}$. Let $N \in \mathbb{N}$ be such that $I_n := \{t \in I : t + \frac{1}{n} \in I\} \neq \emptyset$ for $n \geq N$. Now we need only observe that in view of the continuity of φ , we have

$$A_C = \bigcup_{n \in \mathbb{N}_N, k \in \mathbb{N}} \bigcap_{h \in \mathbb{Q} \cap (0, \frac{1}{n})} \left\{ t \in I_n : \frac{\varphi(t+h) - \varphi(t)}{h} \leq C - \frac{1}{k} \right\}.$$

Notice that the result remains true for arbitrary Borel-measurable functions $\varphi : I \rightarrow \mathbb{R}$ (cf. [Ban22]).

2.2 Families of Continuous Nowhere Differentiable Functions

Recall that our principal aim is to discuss *continuous* nowhere differentiable functions. To simplify notation related to nowhere differentiability, we define the following classes of continuous nowhere differentiable functions.

- $\mathbf{ND}(I) :=$ the set of all $\varphi \in \mathcal{C}(I, \mathbb{C})$ that are nowhere differentiable in the finite sense;
- $\mathbf{ND}^\infty(I) :=$ the set of all $\varphi \in \mathcal{C}(I)$ that are nowhere differentiable in the finite or infinite sense;
- $\mathbf{ND}_\pm(I) :=$ the set of all $\varphi \in \mathcal{C}(I, \mathbb{C})$ such that for every $t \in I$, there is neither a finite right nor a finite left derivative at t ;
- $\mathbf{ND}_\pm^\infty(I) = \mathbf{B}(I) :=$ the set of all *Besicovitch functions*, i.e., the set of all $\varphi \in \mathcal{C}(I)$ such that for every $t \in I$, there is neither a finite or infinite right nor a finite or infinite left derivative at t (cf. § 7.5);
- $\mathbf{M}(I) :=$ the set of all *Morse functions*, i.e., the set of all $\varphi \in \mathcal{C}(I)$ such that

$$\max\{|D^+\varphi(t)|, |D_+\varphi(t)|\} = \max\{|D^-\varphi(t)|, |D_-\varphi(t)|\} = +\infty, \quad t \in I;$$

- we skip the left (resp. right) $\max\{\dots\}$ if t is the right (resp. left) endpoint of the interval;
- $\mathbf{BM}(I) = \mathbf{B}(I) \cap \mathbf{M}(I) :=$ the set of all *Besicovitch–Morse functions* (cf. § 11.1).

Notice that

$$\begin{aligned} \mathbf{BM}(I) &\subset \mathbf{M}(I) \subset \mathbf{ND}_\pm(I) \subset \mathbf{ND}(I), \\ \mathbf{BM}(I) &\subset \mathbf{B}(I) = \mathbf{ND}_\pm^\infty(I) \subset \mathbf{ND}^\infty(I). \end{aligned}$$

Remark 2.2.1. Observe that if I is an open interval, then there exists a real-analytic increasing diffeomorphism $\sigma : \mathbb{R} \rightarrow I$. In particular, if a continuous function $\varphi : I \rightarrow \mathbb{C}$ belongs to one of the above classes of nowhere differentiable functions on I , then the function $\varphi \circ \sigma$ belongs to the corresponding class on $\mathbb{R}$.

The above remark permits us to transport many results from I to $\mathbb{R}$ and vice versa.

2.3 The Denjoy–Young–Saks Theorem

The following result may give some feelings for the general behavior of functions with respect to their differentiability. *On a first reading, the reader may skip the proof.*

Theorem 2.3.1 (Denjoy–Young–Saks). *Let $I \subset \mathbb{R}$ be an arbitrary nontrivial interval. Let $f : I \rightarrow \mathbb{R}$. Then there exists a set $E \subset I$ of Lebesgue measure zero such that for every $x \in I \setminus E$, either*

- *a finite $f'(x)$ exists, or*
- *$D^+f(x) = D_-f(x) \in \mathbb{R}$ and $D_+f(x) = -\infty$, $D^-f(x) = +\infty$, or*

- $D^-f(x) = D_+f(x) \in \mathbb{R}$ and $D^+f(x) = +\infty$, $D_-f(x) = -\infty$, or
- $D^-f(x) = D^+f(x) = +\infty$ and $D_-f(x) = D_+f(x) = -\infty$.

Remark 2.3.2. Symbolically, for $x \in I \setminus E$ we have the following four possibilities:

$$\begin{array}{cccc} * & \left| \begin{array}{c} +\infty \\ * \end{array} \right| & * & \left| \begin{array}{c} +\infty \\ * \end{array} \right| \\ * & \left| \begin{array}{c} * \\ -\infty \end{array} \right| & * & \left| \begin{array}{c} * \\ -\infty \end{array} \right| \\ * & \left| \begin{array}{c} * \\ -\infty \end{array} \right| & * & \left| \begin{array}{c} * \\ -\infty \end{array} \right| \end{array}$$

If f is continuous, the result was first proved by A. Denjoy in [Den15]. The case in which f is measurable was solved by G.C. Young in [You16b]. Finally, the general case was proved by S. Saks in [Sak24]. Our elementary proof is due to E.H. Hanson [Han34].

Corollary 2.3.3. *Let $f : I \rightarrow \mathbb{R}$, $f \in \mathcal{ND}(I)$. Then at almost all points of I , the function f has no one-sided (finite or infinite) derivatives.*

The following two classical results from measure theory will be important for the proof.

Theorem 2.3.4 (Vitali Covering Theorem; Cf. [KK96], Theorem 0.3.2). *Let $S \subset \mathbb{R}$ be bounded and let $\mathcal{F}$ be a family of bounded closed intervals, none consisting of a single point, such that for every $x \in S$ and $\varepsilon > 0$, there exists a $P \in \mathcal{F}$ such that $x \in P$ and $\text{diam}(P) \leq \varepsilon$. Then there exists an at most countable subfamily $\mathcal{F}^0 \subset \mathcal{F}$, consisting of pairwise disjoint intervals, such that*

$$\mathcal{L}\left(S \setminus \bigcup_{P \in \mathcal{F}^0} P\right) = 0,$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$.

Theorem 2.3.5 (Lebesgue Density Theorem; Cf. [KK96], Theorem 2.2.1). *Let $A \subset \mathbb{R}$. Then for almost all $x \in A$ and for every sequence $(P_s)_{s=1}^\infty$ of bounded intervals with $x \in P_s$ and $0 < \text{diam}(P_s) \rightarrow 0$, we have*

$$\lim_{s \rightarrow +\infty} \frac{\mathcal{L}^*(A \cap P_s)}{\mathcal{L}(P_s)} = 1,$$

where $\mathcal{L}^*$ stands for the outer Lebesgue measure on $\mathbb{R}$.

Proof of Theorem 2.3.1. Using Remark 2.2.1, we may assume that $I = \mathbb{R}$.

Step 1°. It suffices to prove that there exists a zero-measure set $E_0 = E_0(f)$ such that for every $x \in \mathbb{R} \setminus E_0$, either

- $D^+f(x) = D_-f(x) \in \mathbb{R}$, or
- $D^+f(x) = +\infty$ and $D_-f(x) = -\infty$.

Indeed, then we put $E := E_0(f) \cup E_0(-f)$.

Step 2°. The main idea of the proof is to show that:

- (a) the set $E_1 := \{x \in \mathbb{R} : D^+f(x) = +\infty, D_-f(x) \neq -\infty\}$ is of measure zero,
- (b) the set $E_2 := \{x \in \mathbb{R} : D_-f(x) = -\infty, D^+f(x) \neq +\infty\}$ is of measure zero,
- (c) the set $E_3 := \{x \in \mathbb{R} : D^+f(x) < D_-f(x) \text{ or } D^-f(x) < D_+f(x)\}$ is at most countable,
- (d) the set $E_4 := \{x \in \mathbb{R} : D^+f(x), D_-f(x) \in \mathbb{R}, D^+f(x) \neq D_-f(x)\}$ is of measure zero.

Observe that (b) follows from (a) applied to the function $-f$.

Suppose for a moment that the above properties are already proven. Put $E_0 := E_1 \cup E_2 \cup E_3 \cup E_4$ and fix an $x \in \mathbb{R} \setminus E_0$. By (d), we need to check only that if $D^+f(x)$ or $D_-f(x)$ is infinite, then $D^+f(x) = +\infty$ and $D_-f(x) = -\infty$. The configurations from (a) and (b) are excluded. Thus, there remains the case $D^+f(x) = -\infty$ (resp. $D_-f(x) = +\infty$), but then, in view of (c), $D_-f(x) = -\infty$ (resp. $D^+f(x) = +\infty$), which contradicts (b) (resp. (a)).

Step 3°. Proof of (a).

We have

$$E_1 = \bigcup_{r \in \mathbb{Q}, n \in \mathbb{N}} A_{r,n},$$

where

$$A_{r,n} := \{x \in \mathbb{R} : D^+ f(x) = +\infty, \forall_{x' \in (x - \frac{1}{n}, x)} : \Delta f(x, x') > r\}.$$

We need to prove only that each set $A_{r,n}$ is of measure zero. Fix $r, n \in \mathbb{N}$, and $b \in A_{r,n}$. Let $a \in \mathbb{R}$ be such that $0 < b - a < \frac{1}{n}$. Put $S := A_{r,n} \cap (a, b)$. Take an arbitrary $t \in \mathbb{R}$ and let

$$\mathcal{F}_t := \{[p, q] : q > p, [p, q] \subset (a, b), p \in S, \Delta f(p, q) > t\}.$$

It is clear that $(S, \mathcal{F}_t)$ satisfies the assumptions of the Vitali covering theorem. Thus there exists an at most countable subfamily $\mathcal{F}_t^0 \subset \mathcal{F}_t$, consisting of pairwise disjoint intervals, such that $\mathcal{L}(S \setminus \bigcup_{P \in \mathcal{F}_t^0} P) = 0$. Take $P_1, \dots, P_N \in \mathcal{F}_t^0$, $P_i = [p_i, q_i]$. Then $(a, b) \setminus \bigcup_{i=1}^N P_i = \bigcup_{j=1}^M (\alpha_j, \beta_j)$, where the intervals $(\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)$ are pairwise disjoint and $\beta_j \in A_{r,n}$, $j = 1, \dots, M$. In particular, $\Delta f(\alpha_j, \beta_j) > r$. Consequently,

$$\begin{aligned} f(b) - f(a) &= \sum_{j=1}^M (f(\beta_j) - f(\alpha_j)) + \sum_{i=1}^N (f(q_i) - f(p_i)) \\ &> r \sum_{j=1}^M (\beta_j - \alpha_j) + t \sum_{i=1}^N (q_i - p_i) = (t - r) \sum_{i=1}^N \mathcal{L}(P_i) + r(b - a). \end{aligned}$$

Thus

$$f(b) - f(a) \geq (t - r) \sum_{P \in \mathcal{F}_t^0} \mathcal{L}(P) + r(b - a).$$

Observe that

$$\sum_{P \in \mathcal{F}_t^0} \mathcal{L}(P) = \mathcal{L}\left(\bigcup_{P \in \mathcal{F}_t^0} P\right) \geq \mathcal{L}^*(S).$$

Consequently, for $t > r$, we get

$$f(b) - f(a) \geq (t - r) \mathcal{L}^*(S) + r(b - a).$$

Letting $t \rightarrow +\infty$, we conclude that $\mathcal{L}^*(S) = \mathcal{L}(A_{r,n} \cap (a, b)) = 0$. Hence, $\mathcal{L}(A_{r,n}) = 0$.

Step 4°. Proof of (c).

It suffices to prove that the set $A := \{x \in \mathbb{R} : D^+ f(x) < D_- f(x)\}$ is of measure zero (and then apply this result to $-f$). Observe that

$$A = \bigcup_{r \in \mathbb{Q}, n \in \mathbb{N}} A_{r,n},$$

where

$$A_{r,n} := \{x \in \mathbb{R} : \forall_{x' \in (x - \frac{1}{n}, x), x'' \in (x, x + \frac{1}{n})} : \Delta f(x, x') < r < \Delta f(x, x'')\}.$$

It is clear that if $x, y \in A_{r,n}$, then $|x - y| \geq \frac{1}{n}$. Consequently, $A_{r,n}$ is at most countable.

Step 5°. Proof of (d).

We have

$$E_4 \setminus E_3 = \bigcup_{\substack{r_1, r_2, r_3, r_4 \in \mathbb{Q} \\ r_1 > r_2 > r_3 > r_4, n \in \mathbb{N}}} A_{r_1, r_2, r_3, r_4, n},$$

where

$$A_{r_1, r_2, r_3, r_4, n} := \{x \in \mathbb{R} : r_4 < D_- f(x) < r_3 < r_2 < D^+ f(x) < r_1, \\ \forall_{x' \in (x - \frac{1}{n}, x)} : \Delta f(x, x') > r_4, \forall_{x'' \in (x, x + \frac{1}{n})} : \Delta f(x, x'') < r_1\}.$$

Fix $r_1 > r_2 > r_3 > r_4$, $n \in \mathbb{N}$, and $a, b \in A_{r_1, r_2, r_3, r_4, n}$ such that $0 < b - a < \frac{1}{n}$. Put $S := A_{r_1, r_2, r_3, r_4, n} \cap (a, b)$. In view of the proof of Step 3° with $(r, t) = (r_4, r_2)$, we get

$$f(b) - f(a) \geq (r_2 - r_4) \mathcal{L}^*(S) + r_4(b - a).$$

Let

$$\mathcal{F} := \{[p, q] : q > p, [p, q] \subset (a, b), q \in S, \Delta f(p, q) < r_3\}.$$

It is clear that $(S, \mathcal{F})$ satisfies the assumptions of the Vitali covering theorem. Thus there exists an at most countable subfamily $\mathcal{F}^0 \subset \mathcal{F}$, consisting of pairwise disjoint intervals, such that $\mathcal{L}^*(S \setminus \bigcup_{P \in \mathcal{F}^0} P) = 0$.

Take $P_1, \dots, P_N \in \mathcal{F}_t^0$, $P_i = [p_i, q_i]$. Then $(a, b) \setminus \bigcup_{i=1}^N P_i = \bigcup_{j=1}^M (\alpha_j, \beta_j)$, where the intervals $(\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)$ are pairwise disjoint and $\alpha_j \in A_{r_1, r_2, r_3, r_4, n}$, $j = 1, \dots, M$. In particular, $\Delta f(\alpha_j, \beta_j) < r_1$. Consequently,

$$f(b) - f(a) \leq (r_3 - r_1) \sum_{P \in \mathcal{F}_t^0} \mathcal{L}(P) + r_1(b - a) \leq (r_3 - r_1) \mathcal{L}^*(S) + r_1(b - a).$$

Hence

$$\frac{\mathcal{L}^*(S)}{b - a} = \frac{\mathcal{L}^*(A_{r_1, r_2, r_3, r_4, n} \cap [a, b])}{\mathcal{L}([a, b])} \leq \frac{r_1 - r_4}{r_1 - r_4 + r_2 - r_3} < 1. \quad (2.3.1)$$

Suppose that $\mathcal{L}^*(A_{r_1, r_2, r_3, r_4, n}) > 0$. Then by the Lebesgue density theorem, there exists a point $b \in A_{r_1, r_2, r_3, r_4, n}$ such that

$$\lim_{a \rightarrow b-} \frac{\mathcal{L}^*(A_{r_1, r_2, r_3, r_4, n} \cap [a, b])}{\mathcal{L}([a, b])} = 1. \quad (2.3.2)$$

In particular, in view of (2.3.1), there are no sequences $(a_s)_{s=1}^\infty \subset A_{r_1, r_2, r_3, r_4, n}$ such that $0 < b - a_s < \frac{1}{n}$ and $a_s \rightarrow b$. Thus $A_{r_1, r_2, r_3, r_4, n} \cap (b, b - \frac{1}{s}) = \emptyset$ for $s \gg 1$, which contradicts (2.3.2). $\square$

2.4 Series of Continuous Functions

Many of the functions discussed in this book will be of the form

$$\varphi(t) := \sum_{n=0}^{\infty} \varphi_n(t), \quad t \in I,$$

where $\varphi_n : I \longrightarrow \mathbb{C}$ is continuous, $n \in \mathbb{N}_0$, and the series is *normally convergent*, i.e.,

$$A := \sum_{n=0}^{\infty} (\sup_{t \in I} |\varphi_n(t)|) < +\infty.$$

In particular, such a series is *uniformly convergent*, and therefore, the function φ is continuous. Obviously, φ is bounded and $|\varphi(x)| \leq A$, $x \in I$.

Remark 2.4.1. It is well known that if, moreover, each function $\varphi_n : I \longrightarrow \mathbb{C}$ is differentiable and the series $\sum_{n=0}^{\infty} \varphi'_n$ is uniformly convergent (e.g., normally convergent) in I , then φ is differentiable and $\varphi'(t) = \sum_{n=0}^{\infty} \varphi'_n(t)$, $t \in I$.

2.5 Hölder Continuity

Definition 2.5.1. Let $\alpha \in (0, 1]$. We say that a continuous function $\varphi : I \longrightarrow \mathbb{C}$ is:

- α -Hölder continuous at a point $t \in I$ ($\varphi \in \mathfrak{H}^\alpha(I; t)$) if

$$\exists c, \delta > 0 \quad \forall h \in (-\delta, \delta) \cap (I - t) : |\varphi(t + h) - \varphi(t)| \leq c|h|^\alpha;$$

- Lipschitz at a point $t \in I$ if $\varphi \in \mathfrak{H}^1(I; t)$;
- α -Hölder continuous ($\varphi \in \mathfrak{H}^\alpha(I)$) if

$$\exists C > 0 \quad \forall t, u \in I : |\varphi(u) - \varphi(t)| \leq C|u - t|^\alpha;$$

- Lipschitz continuous if φ is 1-Hölder continuous;
- M -Lipschitz at a point $t \in I$ (where $M > 0$) if

$$\forall u \in I : |\varphi(u) - \varphi(t)| \leq M|u - t|.$$

Remark 2.5.2. (a) Observe that if $\varphi : I \longrightarrow \mathbb{C}$ is a bounded continuous function, then φ is α -Hölder continuous at t iff

$$\exists c > 0 \quad \forall u \in I : |\varphi(u) - \varphi(t)| \leq c|u - t|^\alpha \text{ (EXERCISE);}$$

in particular, φ is 1-Hölder continuous at t iff φ is M -Lipschitz at t for some $M > 0$.

- (b) If a finite derivative $\varphi'(t)$ exists, then φ is Lipschitz at t .
- (c) It is known (cf. [KK96], Theorems 1.2.8, 6.1.5, 6.1.15) that if $\varphi : I \longrightarrow \mathbb{C}$ is Lipschitz continuous, then there exists a zero-measure set $S \subset I$ such that $\varphi'(t)$ exists for all $t \in I \setminus S$.
- (d) Assume that I is a bounded closed interval and let T_M denote the set of all $\varphi \in \mathcal{C}(I, \mathbb{C})$ such that for every $t \in I$, the function φ is not M -Lipschitz at t . Consider $\mathcal{C}(I, \mathbb{C})$ as a metric space endowed with the distance $d(\varphi, \psi) := \max_I |\varphi - \psi|$. Then T_M is open in $\mathcal{C}(I, \mathbb{C})$ ¹ (EXERCISE). Consequently, the set $T := \bigcap_{M \in \mathbb{Q}_{>0}} T_M$ of all functions that are nowhere Lipschitz on I is a Borel set. Observe that $T \subset \mathfrak{ND}(I)$.

¹ Recall that a pair (X, d) is a *metric space* if $d : X \times X \longrightarrow \mathbb{R}_+$, $(d(x, y) = 0 \iff x = y)$, $d(x, y) = d(y, x)$, and $d(x, y) \leq d(x, z) + d(z, y)$. A set $A \subset X$ is called *open* if for each $a \in A$, there exists an $r > 0$ such that $\{x \in X : d(x, a) < r\} \subset A$.

Definition 2.5.3. For $\alpha > 0$, we say that a continuous function $\varphi : I \rightarrow \mathbb{C}$ is:

- *nowhere α -Hölder continuous* ($\varphi \in \mathbf{NH}^\alpha(I)$) if $\forall t \in I : \varphi \notin \mathcal{H}^\alpha(I; t)$;
- *α -anti-Hölder continuous* if

$$\exists_{\varepsilon > 0} \forall_{t \in I, \delta \in (0,1)} \exists_{\substack{h_{\pm} \in (0,\delta) \\ t \pm h_{\pm} \in I}} : |\varphi(t \pm h_{\pm}) - \varphi(t)| > \varepsilon \delta^\alpha;$$

we skip h_+ (resp. h_-) if t is the right (resp. left) endpoint of the interval;

- *weakly α -anti-Hölder continuous* if

$$\exists_{\varepsilon > 0} \forall_{t \in I, \delta \in (0,1)} \exists_{h \in (-\delta, \delta) \cap (I-t)} : |\varphi(t+h) - \varphi(t)| > \varepsilon \delta^\alpha.$$

Remark 2.5.4. Let $\alpha \in (0, 1)$.

- (a) If φ is α -anti-Hölder continuous, then $\varphi \in \mathcal{M}(I) \subset \mathbf{ND}_\pm(I)$.
- (b) If φ is weakly α -anti-Hölder continuous, then φ is nowhere 1-Hölder continuous, and hence $\varphi \in \mathbf{ND}(I)$.

Chapter 3

Weierstrass-Type Functions I

Summary. The aim of this chapter is to present various classical methods of testing the nowhere differentiability of the Weierstrass-type function $x \mapsto \sum_{n=0}^{\infty} a^n \cos^p(2\pi b^n x + \theta_n)$. More developed results will be discussed in Chap. 8.

3.1 Introduction

We will discuss the nowhere differentiability of the following *Weierstrass-type function*

$$\mathbf{W}_{p,a,b,\boldsymbol{\theta}}(x) := \sum_{n=0}^{\infty} a^n \cos^p(2\pi b^n x + \theta_n), \quad x \in \mathbb{R}, \quad (3.1.1)$$

where

$$p \in \mathbb{N}, \quad 0 < a < 1, \quad ab \geq 1, \quad \boldsymbol{\theta} := (\theta_n)_{n=0}^{\infty} \subset \mathbb{R}. \quad (3.1.2)$$

Throughout the chapter, we always assume that $p, a, b, \boldsymbol{\theta}$ satisfy (3.1.2) (cf. Figs. 3.1, 3.2, and 3.3).

Notice that the function $\mathbf{W}_{1,a,b,0}$ with $p = 1$, $b \in 2\mathbb{N} + 1$, and $ab > 1 + \frac{3}{2}\pi$, coincides with the original nowhere differentiable Weierstrass function presented by him to the Königlische Akademie der Wissenschaften on 18 July 1872; cf. [Wei86].

We will be mainly interested in a characterization of the parameters $p, a, b, \boldsymbol{\theta}$ for which the function $\mathbf{W}_{p,a,b,\boldsymbol{\theta}}$ belongs to one of the following three classes of nowhere differentiable functions: $\mathbf{ND}^{\infty}(\mathbb{R})$, $\mathbf{ND}_{\pm}(\mathbb{R})$, and $\mathbf{M}(\mathbb{R}) \cap \mathbf{ND}^{\infty}(\mathbb{R})$. Recall that $\mathbf{M}(\mathbb{R}) \subset \mathbf{ND}_{\pm}(\mathbb{R})$. We would like to point out that in general, most of the cases are not completely understood (even for $p = 1$ and $\boldsymbol{\theta} = 0$).

To simplify notation, we will use the following conventions:

- If $\theta_n = \theta$ for all $n \in \mathbb{N}_0$, then we simply write $\boldsymbol{\theta} = \theta$.
- If the parameters p, a, b are fixed, then $\mathbf{W}_{\boldsymbol{\theta}} := \mathbf{W}_{p,a,b,\boldsymbol{\theta}}$.

A special role is played by the cases in which $p = 1$ or/and ($\boldsymbol{\theta} = 0$ or $\boldsymbol{\theta} = -\frac{\pi}{2}$). In particular,

$$\mathbf{C}_{a,b}(x) := \mathbf{W}_{1,a,b,0}(x) = \sum_{n=0}^{\infty} a^n \cos(2\pi b^n x),$$

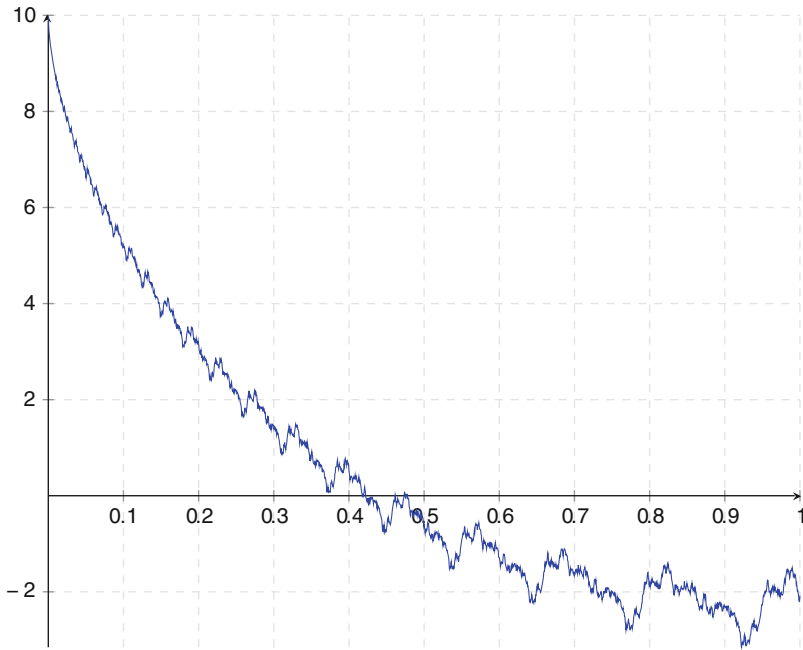


Fig. 3.1 Weierstrass-type function $\mathbb{I} \ni x \mapsto \mathbf{W}_{1,0.9,1.2,0}(x)$

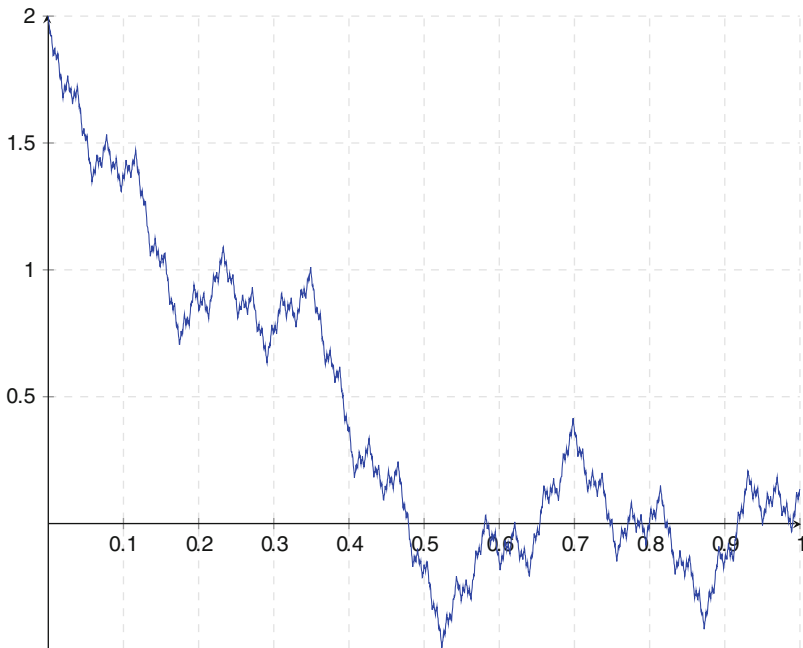


Fig. 3.2 Weierstrass-type function $\mathbb{I} \ni x \mapsto \mathbf{W}_{1,0.5,3,0}(x)$

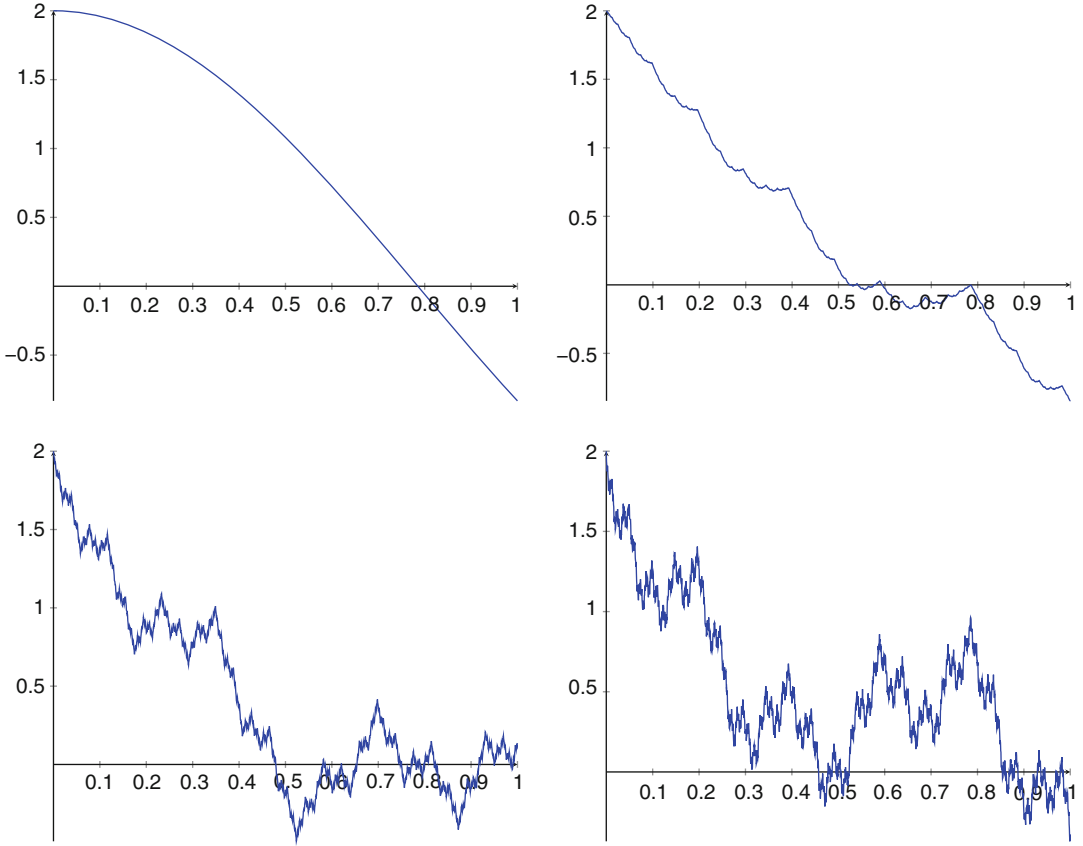


Fig. 3.3 Weierstrass-type functions $\mathbf{W}_{1,0.5,1,0}$, $\mathbf{W}_{1,0.5,2,0}$, $\mathbf{W}_{1,0.5,3,0}$, $\mathbf{W}_{1,0.5,4,0}$

$$\mathbf{S}_{a,b}(x) := \mathbf{W}_{1,a,b,-\frac{\pi}{2}}(x) = \sum_{n=0}^{\infty} a^n \sin(2\pi b^n x), \quad x \in \mathbb{R},$$

are the *classical Weierstrass functions* (cf. [BR74, Wei86]).

Remark 3.1.1. To give the reader an idea of the content of the chapter, we give below a list of results that will be presented. The list is organized in chronological order. We do not pretend that the list is complete. Most of the results will be presented in a somewhat more general form than in the original papers. Nowadays, most of these results have only historical significance. They will be essentially generalized and strengthened in Chap. 8. Nevertheless, they might give some insight into how over 120 years (1872–1992), the methods of studying nowhere differentiability have evolved.

- (1) **1872:** If $b, p \in 2\mathbb{N}_0 + 1$ and $ab > 1 + \frac{3}{2}p\pi$, then $\mathbf{W}_{p,a,b,0} \in \mathcal{M}(\mathbb{R}) \cap \mathcal{ND}^\infty(\mathbb{R}) \subset \mathcal{ND}_\pm(\mathbb{R}) \cap \mathcal{ND}^\infty(\mathbb{R})$ (Theorem 3.5.1).
- (2) **1890:** If $b \in 2\mathbb{N}$ and $b \geq 14$, then $\mathbf{W}_{1,1/b,b,\theta} \in \mathcal{ND}_\pm(\mathbb{R})$ (Theorem 3.6.1).
- (3) **1892:** If $(a < a_1(p) \text{ and } b > \Psi_1(a))$ or $(a < a_2(p) \text{ and } b > \Psi_2(a))$ (the functions a_i, Ψ_i , $i = 1, 2$, are given by effective formulas), then $\mathbf{W}_{p,a,b,\theta} \in \mathcal{ND}_\pm(\mathbb{R})$. In particular, if $(a < \frac{1}{3} \text{ and } ab > 1 + \frac{3}{2}\pi \frac{1-a}{1-3a})$ or $(a < \frac{2}{9} \text{ and } ab^2 > 1 + \frac{21}{4}\pi^2 \frac{1-a}{2-9a})$, then $\mathbf{W}_{1,a,b,\theta} \in \mathcal{ND}_\pm(\mathbb{R})$ (Theorem 3.7.1).