Lecture Notes of the Unione Matematica Italiana

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Nonlocal Diffusion and Applications

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Preface

The purpose of these pages is to collect a set of notes that are a result of several talks and minicourses delivered here and there in the world (Milan, Cortona, Pisa, Roma, Santiago del Chile, Madrid, Bologna, Porquerolles, and Catania to name a few). We will present here some mathematical models related to nonlocal equations, providing some introductory material and examples.

Of course, these notes and the results presented do not aim to be comprehensive and cannot take into account all the material that would deserve to be included. Even a thorough introduction to nonlocal (or even just fractional) equations goes way beyond the purpose of this book.

Using a metaphor with fine arts, we could say that the picture that we painted here is not even impressionistic, it is just naïf. Nevertheless, we hope that these pages may be of some help to the young researchers of all ages who are willing to have a look at the exciting nonlocal scenario (and who are willing to tolerate the partial and incomplete point of view offered by this modest observation point).

Milano, Italy — Claudia Bucur
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Contents

Introduction

In the recent years the fractional Laplace operator has received much attention both in pure and in applied mathematics. Starting from the basics of the nonlocal equations, in this set of notes we will discuss in detail some recent developments in four topics of research on which we focused our attention, namely:

- A problem arising in crystal dislocation (which is related to a classical model introduced by Peierls and Nabarro)
- A problem arising in phase transitions (which is related to a nonlocal version of the classical Allen–Cahn equation)
- The limit interfaces arising in the above nonlocal phase transitions (which turn out to be nonlocal minimal surfaces, as introduced by Caffarelli, Roquejoffre, and Savin)
- A nonlocal version of the Schrödinger equation for standing waves (as introduced by Laskin)

Many fundamental topics slipped completely out of these notes: just to name a few, the topological methods and the fine regularity theory in the fractional cases are not presented here; the fully nonlinear or singular/degenerate equations are not taken into account; only very few applications are discussed briefly; important models such as the quasi-geostrophic equation and the fractional porous media equation are not covered in these notes; we will not consider models arising in game theory such as the nonlocal tug-of-war; the parabolic equations are not taken into account in detail; unique continuation and overdetermined problems will not be studied here, and the link to probability theory that we consider here is not rigorous and only superficial (the reader interested in these important topics may look, for instance, at [8, 10, 14, 15, 17–20, 31, 36–38, 41, 53, 65, 66, 70, 71, 88, 95, 97–100, 109, 113, 127, 133]). Also, a complete discussion of the nonlocal equations in bounded domains is not available here (for this, we refer to the recent survey [119]). In terms of surveys, collections of results, and open problems, we also mention the very nice website [2], which gets[1] constantly updated.

This set of notes is organized as follows. To start with, in Chap. 1, we will give a motivation for the fractional Laplacian (which is the typical nonlocal operator for our framework) that originates from probabilistic considerations. As a matter of fact, no advanced knowledge of probability theory is assumed from the reader, and the topic is dealt with at an elementary level.

In Chap. 2, we will recall some basic properties of the fractional Laplacian, discuss some explicit examples in detail, and point out some structural inequalities that are due to a fractional comparison principle. This part continues with a quite surprising result, which states that every function can be locally approximated by functions with vanishing fractional Laplacian (in sharp contrast with the rigidity of the classical harmonic functions). We also give an example of a function with constant fractional Laplacian on the ball.

In Chap. 3 we deal with extended problems. It is indeed a quite remarkable fact that in many occasions nonlocal operators can be equivalently represented as local (though possibly degenerate or singular) operators in one dimension more. Moreover, as a counterpart, several models arising in a local framework give rise to nonlocal equations, due to boundary effects. So, to introduce the extension problem and give a concrete intuition of it, we will present some models in physics that are naturally set on an extended space to start with and will show their relation with the fractional Laplacian on a trace space. We will also give a detailed justification of this extension procedure by means of the Fourier transform.

As a special example of problems arising in physics that produce a nonlocal equation, we consider a problem related to crystal dislocation, present some mathematical results that have been recently obtained on this topic, and discuss the relation between these results and the observable phenomena.

Chapters 4, 5, and 6 present topics of contemporary research. We will discuss in particular: some phase transition equations of nonlocal type; their limit interfaces, which (below a critical threshold of the fractional parameter) are surfaces that minimize a nonlocal perimeter functional; and some nonlocal equations arising in quantum mechanics.

We remark that the introductory part of these notes is not intended to be separated from the one which is more research oriented, namely, even the chapters whose main goal is to develop the basics of the theory contain some parts related to contemporary research trends.

[1] It seems to be known that Plato did not like books because they cannot respond to questions. He might have liked websites.

Chapter 1
A Probabilistic Motivation

The fractional Laplacian will be the main operator studied in this book. We consider a function $u: \mathbb{R}^n \to \mathbb{R}$ (which is supposed[1] to be regular enough) and a fractional parameter $s \in (0, 1)$. Then, the fractional Laplacian of u is given by

$$(-\Delta)^s u(x) = \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \, dy, \tag{1.1}$$

where $C(n, s)$ is a dimensional[2] constant.

One sees from (1.1) that $(-\Delta)^s$ is an operator of order $2s$, namely, it arises from a differential quotient of order $2s$ weighted in the whole space. Different fractional operators have been considered in literature (see e.g. [39, 111, 128]), and all of them come from interesting problems in pure or/and applied mathematics. We will focus here on the operator in (1.1) and we will motivate it by probabilistic considerations (as a matter of fact, many other motivations are possible).

The probabilistic model under consideration is a random process that allows long jumps (in further generality, it is known that the fractional Laplacian is an infinitesimal generator of Lèvy processes, see e.g. [7, 13] for further details). A more detailed mathematical introduction to the fractional Laplacian is then presented in the subsequent Sect. 2.1.

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[1] To write (1.1) it is sufficient, for simplicity, to take here u in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of smooth and rapidly decaying functions, or in $C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We refer to [131] for a refinement of the space of definition.

[2] The explicit value of $C(n, s)$ is usually unimportant. Nevertheless, we will compute its value explicitly in formulas (2.10) and (2.15). The reason for which it is convenient to divide $C(n, s)$ by a factor 2 in (1.1) will be clear later on, in formula (2.5).

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1.1 The Random Walk with Arbitrarily Long Jumps

We will show here that the fractional heat equation (i.e. the "typical" equation that drives the fractional diffusion and that can be written, up to dimensional constants, as $\partial_t u + (-\Delta)^s u = 0$) naturally arises from a probabilistic process in which a particle moves randomly in the space subject to a probability that allows long jumps with a polynomial tail.

For this scope, we introduce a probability distribution on the natural numbers $\mathbb{N}^* := \{1, 2, 3, \cdots\}$ as follows. If $I \subseteq \mathbb{N}^*$, then the probability of I is defined to be

$$P(I) := c_s \sum_{k \in I} \frac{1}{|k|^{1+2s}}.$$

The constant c_s is taken in order to normalize P to be a probability measure. Namely, we take

$$c_s := \left(\sum_{k \in \mathbb{N}^*} \frac{1}{|k|^{1+2s}} \right)^{-1},$$

so that we have $P(\mathbb{N}^*) = 1$.

Now we consider a particle that moves in $\mathbb{R}^n$ according to a probabilistic process. The process will be discrete both in time and space (in the end, we will formally take the limit when these time and space steps are small). We denote by τ the discrete time step, and by h the discrete space step. We will take the scaling $\tau = h^{2s}$ and we denote by $u(x, t)$ the probability of finding the particle at the point x at time t.

The particle in $\mathbb{R}^n$ is supposed to move according to the following probabilistic law: at each time step τ, the particle selects randomly both a direction $v \in \partial B_1$, according to the uniform distribution on ∂B_1, and a natural number $k \in \mathbb{N}^*$, according to the probability law P, and it moves by a discrete space step khv. Notice that long jumps are allowed with small probability. Then, if the particle is at time t at the point x_0 and, following the probability law, it picks up a direction $v \in \partial B_1$ and a natural number $k \in \mathbb{N}^*$, then the particle at time $t + \tau$ will lie at $x_0 + khv$.

Now, the probability $u(x, t + \tau)$ of finding the particle at x at time $t + \tau$ is the sum of the probabilities of finding the particle somewhere else, say at $x + khv$, for some direction $v \in \partial B_1$ and some natural number $k \in \mathbb{N}^*$, times the probability of having selected such a direction and such a natural number. This translates into

$$u(x, t + \tau) = \frac{c_s}{|\partial B_1|} \sum_{k \in \mathbb{N}^*} \int_{\partial B_1} \frac{u(x + khv, t)}{|k|^{1+2s}} \, d\mathscr{H}^{n-1}(v).$$

Notice that the factor $c_s/|\partial B_1|$ is a normalizing probability constant, hence we subtract $u(x,t)$ and we obtain

$$\begin{aligned} u(x,t+\tau)-u(x,t) &= \frac{c_s}{|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x+khv,t)}{|k|^{1+2s}}\,d\mathcal{H}^{n-1}(v)-u(x,t)\\ &= \frac{c_s}{|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x+khv,t)-u(x,t)}{|k|^{1+2s}}\,d\mathcal{H}^{n-1}(v). \end{aligned}$$

As a matter of fact, by symmetry, we can change v to $-v$ in the integral above, so we find that

$$u(x,t+\tau)-u(x,t) = \frac{c_s}{|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x-khv,t)-u(x,t)}{|k|^{1+2s}}\,d\mathcal{H}^{n-1}(v).$$

Then we can sum up these two expressions (and divide by 2) and obtain that

$$\begin{aligned} &u(x,t+\tau)-u(x,t)\\ &\quad= \frac{c_s}{2\,|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{|k|^{1+2s}}\,d\mathcal{H}^{n-1}(v). \end{aligned}$$

Now we divide by $\tau = h^{2s}$, we recognize a Riemann sum, we take a formal limit and we use polar coordinates, thus obtaining:

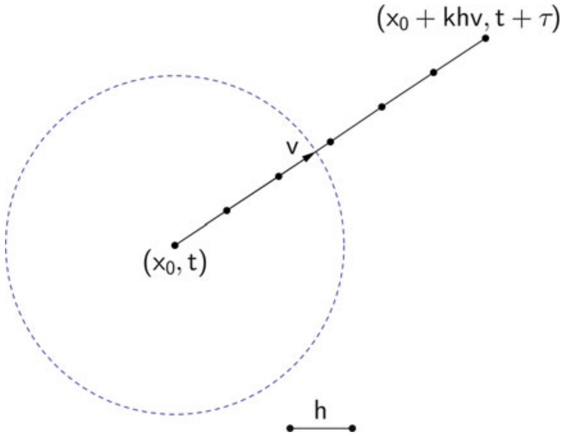
$$\begin{aligned} \partial_t u(x,t) &\simeq \frac{u(x,t+\tau)-u(x,t)}{\tau}\\ &= \frac{c_s\,h}{2\,|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{|hk|^{1+2s}}\,d\mathcal{H}^{n-1}(v)\\ &\simeq \frac{c_s}{2\,|\partial B_1|}\int_0^{+\infty}\int_{\partial B_1}\frac{u(x+rv,t)+u(x-rv,t)-2u(x,t)}{|r|^{1+2s}}\,d\mathcal{H}^{n-1}(v)\,dr\\ &= \frac{c_s}{2\,|\partial B_1|}\int_{\mathbb{R}^n}\frac{u(x+y,t)+u(x-y,t)-2u(x,t)}{|y|^{n+2s}}\,dy\\ &= -\,c_{n,s}\,(-\Delta)^s u(x,t) \end{aligned}$$

for a suitable $c_{n,s} > 0$.

This shows that, at least formally, for small time and space steps, the above probabilistic process approaches a fractional heat equation.

We observe that processes of this type occur in nature quite often, see in particular the biological observations in [90, 140], other interesting observations in [118, 126, 142] and the mathematical discussions in [84, 93, 104, 107, 110].

Fig. 1.1 The random walk with jumps



Roughly speaking, let us say that it is not unreasonable that a predator may decide to use a nonlocal dispersive strategy to hunt its preys more efficiently (or, equivalently, that the natural selection may favor some kind of nonlocal diffusion): small fishes will not wait to be eaten by a big fish once they have seen it, so it may be more convenient for the big fish just to pick up a random direction, move rapidly in that direction, stop quickly and eat the small fishes there (if any) and then go on with the hunt. And this "hit-and-run" hunting procedure seems quite related to that described in Fig. 1.1.

1.2 A Payoff Model

Another probabilistic motivation for the fractional Laplacian arises from a payoff approach. Suppose to move in a domain Ω according to a random walk with jumps as discussed in Sect. 1.1. Suppose also that exiting the domain Ω for the first time by jumping to an outside point $y \in \mathbb{R}^n \setminus \Omega$, means earning $u_0(y)$ sestertii. A relevant question is, of course, how rich we expect to become in this way. That is, if we start at a given point $x \in \Omega$ and we denote by $u(x)$ the amount of sestertii that we expect to gain, is there a way to obtain information on u?

The answer is that (in the right scale limit of the random walk with jumps presented in Sect. 1.1) the expected payoff u is determined by the equation

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ u = u_0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.2}$$

To better explain this, let us fix a point $x \in \Omega$. The expected value of the payoff at x is the average of all the payoffs at the points $\tilde{x}$ from which one can reach x, weighted by the probability of the jumps. That is, by writing $\tilde{x} = x + khv$, with $v \in \partial B_1$,