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Monotone Complete C*-algebras and Generic Dynamics

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Abstract

This book is about monotone complete C^*-algebras, their properties, and the new classification theory (using spectroid invariants and a classification semigroup). A basic account of generic dynamics is included because of its important connections to these algebras. Each bounded, upward-directed net of real numbers has a limit. Monotone complete algebras of operators have a similar property. In particular, every von Neumann algebra is monotone complete, but the converse is false. The small von Neumann factors can be labelled by the set of real numbers. But there are many more ($2^{\mathbb{R}}$) small monotone complete C^*-algebras which are factors. The aim of this book is to give an account of monotone complete C^*-algebras which includes recent advances but also indicates the many mysteries and open problems which remain.

Preface

First, we wish to thank Professor Takesaki who encouraged us to write this book. Without him, we would never have started; without the unfailing support of Yuko and Harvinder, we would never have finished.

Monotone complete C^*-algebras are now entering a new and exciting stage of their development. The new classification semigroup and spectroid invariants are useful, but we are a very long way from a full understanding; many mysteries remain. The second author particularly wishes to thank Dennis Sullivan for stimulating discussions on generic dynamics and related topics.

We would both like to thank the many distinguished mathematicians who have shared their insights with us. There are too many to list them all, but among them are J.Bonet, J.K. Brooks, L.J. Bunce, Cho-Ho Chu, C.M. Edwards, P. de Lucia, Masanao Ozawa, A.Peralta, P. Ptak, B. Weiss, and K. Ylinen.

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Chapter 1
Introduction

This book is about monotone complete C^*-algebras, their properties and their classification. We also give a basic account of generic dynamics because of its useful connections to these algebras.

1.1 Monotone Complete Algebras of Operators

Fundamental to analysis is the completeness of the real numbers. Each bounded, monotone increasing sequence of real numbers has a least upper bound. Monotone complete algebras of operators have a similar property.

Let A be a C^*-algebra. Its self-adjoint part, A_{sa}, is a partially ordered, real Banach space whose positive cone is $\{zz^* : z \in A\}$. If each upward directed, norm-bounded subset of A_{sa}, has a least upper bound then A is said to be *monotone complete*. Every von Neumann algebra is monotone complete but the converse is false.

Recently there have been major advances in the theory of monotone complete C^*-algebras; for example the construction of classification semigroups [144]. This followed an important breakthrough in [66], which introduced huge numbers of new examples. But much remains to be discovered. The purpose of this book is to expound the new theory. We want to take readers from the basics to the frontiers of the subject. We hope they will be stimulated to work on the many fascinating open problems. Our intention is to strive for clarity rather than maximal generality. Our intended reader has a grounding in elementary functional analysis and point set topology and some exposure to the fundamentals of C^*-algebras, say, the first chapters of [161]. But prior knowledge of von Neumann algebras or operator systems is not essential. However, in this introduction, we may use terminology with which some readers are unfamiliar. If so, we apologise and reassure them that all necessary technicalities will be discussed later in the text.

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Algebras of operators on Hilbert space, including C^*-algebras, von Neumann algebras and their generalisations, are the focus of intense research activity worldwide. They are fundamental to non-commutative geometry and intimately related to work on operator systems and operator spaces and have connections to many other fields of mathematics and quantum physics. But the first to be investigated (with a different name and a more "spatial" viewpoint) were the von Neumann algebras.

Monotone complete C^*-algebras arise in several different areas. There are close connections with operator systems, with operator spaces and with generic dynamics. In the category of operator systems, with completely positive maps as morphisms, each injective object can be given the structure of a monotone complete C^*-algebra in a canonical way. Injective operator spaces can be embedded as "corners" of monotone complete C^*-algebras, see Theorem 6.1.3 and Theorem 6.1.6 [38] and [25, 59, 60]. When a monotone complete C^*-algebra is commutative, its lattice of projections is a complete Boolean algebra. Up to isomorphism, every complete Boolean algebra arises in this way.

Let A be a monotone complete C^*-algebra then A is a von Neumann algebra precisely when it has a separating family of normal states. If a monotone complete C^*-algebra does not possess any normal states it is called *wild*.

The best known commutative example of a wild monotone complete C^*-algebra is straightforward to construct. Let $B(\mathbb{R})$ be the commutative C^*-algebra of all bounded, complex valued Borel measurable functions on $\mathbb{R}$. Let $M(\mathbb{R})$ be the ideal of all functions h in $B(\mathbb{R})$ such that $\{t \in \mathbb{R} : h(t) \neq 0\}$ is a meagre subset of $\mathbb{R}$. (Let us recall that a set is meagre if it is contained in the union of countably many nowhere dense sets; a set is nowhere dense if its closure has empty interior.) Then the quotient algebra $B(\mathbb{R})/M(\mathbb{R})$ is a commutative monotone complete C^*-algebra which has no normal states and so is not a von Neumann algebra. It turns out that if we replace $\mathbb{R}$ by any complete separable metric space, without isolated points, and perform the same construction then we end up with the same commutative monotone complete C^*-algebra.

A monotone complete C^*-algebra, like a von Neumann algebra, is said to be a *factor* if its centre is trivial. In other words, factors are as far as possible from being commutative. Just as for von Neumann algebras, monotone complete C^*-factors can be divided into Type I, Type II_1,Type II_∞ and Type III. It turns out that all Type I factors are von Neumann algebras. So it is natural to ask: are *all* monotone complete C^*-factors, in fact, von Neumann algebras? The answer is "no" in general but to clarify the situation, we need some extra notions. Let H be a separable Hilbert space and $L(H)$ the bounded operators on H. A C^*-algebra A is said to be *separably representable* if there exists a $*$-isomorphism π from A into $L(H)$. It is known that if A is a monotone complete C^*-factor which is also a separably representable C^*-algebra then A must be a von Neumann algebra [179]. So where are the wild factors?

A (unital) C^*-algebra B is said to be *small* if there exists a unital complete isometry from B onto an operator system in $L(H)$, where H is separable. When an algebra is separably representable then it is small but the converse is false. In other words, there exist C^*-algebras which can be regarded as operator systems on a separable Hilbert space but which can only be represented as $*$-algebras of bounded operators on a larger Hilbert space. There do exist small Type III monotone complete C^*-factors which are not von Neumann algebras. In fact they exist in huge abundance. There are 2^c, where c is the cardinality of the real numbers. By contrast, there are only c small von Neumann algebras. (Each small von Neumann algebra is isomorphic to a $*$-subalgebra of $L(H)$ where the subalgebra is closed in the weak operator topology. In particular, each small von Neumann algebra is separably representable. This follows from [1].) Incidentally, if a small C^*-algebra is a wild factor then it is always of Type III.

One way to find a wild monotone complete C^*-factor is to start with a separable, simple, unital C^*-algebra and use a kind of "Dedekind cut" completion [173]. This approach will be discussed later. Another method is to associate a monotone complete C^*-algebra with a dynamical system. This "generic dynamics" approach is outlined below.

Monotone complete C^*-algebras are a generalisation of von Neumann algebras. The theory of the latter is now very well advanced. But it took many years before it was demonstrated that there were continuum many von Neumann factors of Type III [126], Type II_1 [100] and Type II_∞ [148]. Then the pioneering work of Connes, Takesaki and other giants of the subject transformed our knowledge of von Neumann algebras, see [8, 30, 96, 162]. By comparison, the theory of monotone complete C^*-algebras is in its infancy with many fundamental questions unanswered. But great progress has been made in recent years. In the early study of monotone complete C^*-algebras the emphasis was on showing how similar they were to von Neumann algebras. Nowadays we realise how different they can be.

In 2001 Hamana [66] made a major breakthrough which implied that there are 2^c small monotone complete C^*-factors. In 2007 [144] we found a way to classify monotone complete C^*-algebras. This is set out in Chap. 3.

In [144] we introduced a quasi-ordering between monotone complete C^*-algebras. From this quasi-ordering we defined an equivalence relation and used this to construct, in particular, a classification semigroup $\mathcal{W}$ for small monotone complete C^*-algebras. This semigroup is abelian, partially ordered, and has the Riesz decomposition property. For each small monotone complete C^*-algebra A we assign a "normality weight", $w(A) \in \mathcal{W}$. If A and B are algebras then $w(A) = w(B)$, precisely when these algebras are equivalent. It turns out that algebras which are very different can be equivalent. In particular, the von Neumann algebras are equivalent to each other and correspond to the zero element of the semigroup. It might have turned out that $\mathcal{W}$ is very small and fails to distinguish between more than a few algebras. This is not so; the cardinality of $\mathcal{W}$ is 2^c, where $c = 2^{\aleph_0}$.

A natural reaction by anyone familiar with K-theory, is to construct the Grothendieck group of $\mathcal{W}$. But this group is trivial because each element of the semigroup is idempotent. However this implies that $\mathcal{W}$ has a natural structure as a semi-lattice. Furthermore, the Riesz Decomposition Property for $\mathcal{W}$ ensures that the semi- lattice is distributive.

As we shall see later, one of the useful properties of $\mathcal{W}$ is that it can sometimes be used to replace problems about factors by problems about commutative algebras [144].

To each monotone complete C^*-algebra we can associate a *spectroid* invariant, ∂A [144]. Just as a spectrum is a set which encodes information about an operator, a spectroid encodes information about a monotone complete C^*-algebra. It turns out that if $wA = wB$ then A and B have the same spectroid. So the spectroid may be used as a tool for classifying elements of $\mathcal{W}$.

Kaplansky wished to capture the algebraic essence of von Neumann algebras and to do it, introduced AW^*-algebras [90–92]. An AW^*-algebra may be defined as a unital C^*-algebra in which every maximal abelian $*$-subalgebra is monotone complete [146]. Every monotone complete C^*-algebra is easily seen to be an AW^*-algebra. Nobody has ever seen an AW^*-algebra which is *NOT* monotone complete. It is strongly suspected that *EVERY* AW^*-algebra is monotone complete. But in full generality this is a difficult open problem. But many positive results are known. In particular, all "small" AW^*-factors are known to be monotone complete. Since our interest is strongly focused on small C^*-algebras we shall postpone a discussion of AW^*-algebras until Chap. 8. (But this can be read now, without working through all the earlier chapters.) They will appear on our list of open problems, some of which have been unsolved for over 60 years. For a scholarly account of the classical theory of AW^*-algebras the reader may consult [13].

Generic dynamics is used in an essential way in this book but we shall not introduce this tool until Chap. 6. So some readers may prefer to turn immediately to Chap. 2 and postpone reading the introduction to generic dynamics.

1.2 Generic Dynamics

An elegant account of generic dynamics was given by Weiss [165]; the term occurred earlier in [157]. In these articles, the underlying framework is a countable group of homeomorphisms acting on a complete separable metric space with no isolated points (a perfect Polish space). The key result of [157] was a strong uniqueness theorem. As a consequence, the wild factor discovered by Dyer [36] and the factor found by Takenouchi [159] were shown to be isomorphic.

We devote a chapter to aspects of generic dynamics useful for monotone complete C^*-algebra theory, including some recent discoveries [145]. This is an elementary exposition. In this book, generic dynamics is only developed as far as we need it for applications to C^*-algebras. But this does require us to consider generic dynamics on compact non-metrisable separable spaces; a topic which has been little explored and gives rise to interesting open questions.

Let G be a countable group. Unless we specify otherwise, G will always be assumed to be infinite. Let X be a Hausdorff topological space with no isolated points. Further suppose that X is a Baire space i.e. such that the only meagre open set is the empty set. In other words, the Baire Category Theorem holds for X. We shall also suppose that X is completely regular. (These conditions are satisfied if X is compact or homeomorphic to a complete separable metric space or, more generally, a G_δ-subset of a compact Hausdorff space or is the extreme boundary of a compact convex set in a locally convex Hausdorff topological vector space.) A subset Y of X is said to be *generic* if $X\backslash Y$ is meagre.

Let ε be an action of G on X as homeomorphisms of X.

In classical dynamics we would require the existence of a Borel measure on X which was G-invariant or quasi-invariant, and discard null sets. In topological dynamics, no measure is required and no sets are discarded. In generic dynamics, we discard meagre Borel sets.

We shall concentrate on the situation where, for some $x_0 \in X$, the orbit $\{\varepsilon_g(x_0) : g \in G\}$ is dense in X. Of course this cannot happen unless X is separable. (A topological space is *separable* if it has a countable dense subset. This is a weaker property than having a countable base.) Let S be the Stone space of the (complete) Boolean algebra of regular open sets of X. Then, see below, the action ε of G on X induces an action $\hat{\varepsilon}$ of G as homeomorphisms of S; which will also have a dense orbit.

When, as in [165] and [157], X is a perfect Polish space, then S is unique; it can be identified with the Stone space of the regular open sets of $\mathbb{R}$. But if we let X range over all separable compact subspaces of the separable space, $2^{\mathbb{R}}$, then we obtain 2^c essentially different S; where S is compact, separable and extremally disconnected. For each such S, $C(S)$ is a subalgebra of ℓ^∞.

Let E be the relation of orbit equivalence on S. That is, sEt, if, for some group element g, $\hat{\varepsilon}_g(s) = t$. Then we can construct a monotone complete C^*-algebra M_E from the orbit equivalence relation. When there is a free dense orbit, the algebra will be a factor with a maximal abelian $*$-subalgebra, A, which is isomorphic to $C(S)$. There is always a faithful, normal, conditional expectation from M_E onto A. It can be shown that $wM_E = wA$. So some classification questions about factors can be replaced by questions about commutative algebras. *When E and F are orbit equivalence relations which coincide on a dense G_δ-subset of S then M_E is isomorphic to M_F.*

For $f \in C(S)$, let $\gamma^g(f) = f \circ \hat{\varepsilon}_{g^{-1}}$. Then $g \mapsto \gamma^g$ is an action of G as automorphisms of $C(S)$. Then we can associate a monotone complete C^*-algebra $M(C(S), G)$, the *monotone cross-product* with this action (see Chap. 7). When the action $\hat{\varepsilon}$ is free, then $M(C(S), G)$ is naturally isomorphic to M_E. In other words, the monotone cross-product does not depend on the group, only on the orbit equivalence relation.

In this book we shall consider 2^c algebras $C(S)$. Each is a subalgebra of ℓ^∞ and each takes different values in the weight semigroup $\mathcal{W}$. (Here $c = 2^{\aleph_0}$, the cardinality of $\mathbb{R}$.)

For general S there is no uniqueness theorem but we do show the following. Let G be a countably infinite group. Let α be an action of G as homeomorphisms of S and suppose this action has at least one orbit which is dense and free. Then, modulo meagre sets, the orbit equivalence relation obtained can also be obtained by an action of $\bigoplus \mathbb{Z}_2$ as homeomorphisms of S. This should be contrasted with the situation in classical dynamics. (e.g. It is shown in [31] that any action by an amenable group is orbit equivalent to an action of $\mathbb{Z}$. But, in general, non-amenable groups give rise to orbit equivalence relations which do not come from actions of $\mathbb{Z}$.)

On each of 2^c, essentially different, compact extremally disconnected spaces we construct a natural action of $\bigoplus \mathbb{Z}_2$ with a free, dense orbit. Let Λ be a set of cardinality 2^c, where $c = 2^{\aleph_0}$. Then by applying generic dynamics, as in [144], we can find a family of monotone complete C^*-algebras $\{B_\lambda : \lambda \in \Lambda\}$ with the following properties. Each B_λ is a monotone complete factor of Type III, and also a small C^*-algebra. For $\lambda \neq \mu$, B_λ and B_μ have different spectroids and so $wB_\lambda \neq wB_\mu$ and, in particular, B_λ is not isomorphic to B_μ. Furthermore each B_λ is generated by an increasing sequence of full matrix algebras.

Chapter 2
Order Fundamentals

This chapter presents basic material which will be needed later. Among the topics discussed are order limits, monotone σ-complete C^*-algebras and commutative algebras.

Our aim is to present an account of monotone complete C^*-algebras which can be followed without requiring the reader to constantly look up results elsewhere. So, particularly in this first chapter, we give proofs of some basic results which appear in standard texts. We shall also state a number of results without proof but with an indication of where proofs can be found.

The books [161] and [121] are our sources for much of the canonical theory of C^*-algebras; we have used their guidance for some basic results. There are many other excellent books in this area, an interesting recent example is [15] as well as the classic [34]. An elegant succinct introduction to operator algebras is to be found in [67].

In the 80 years of its development, operator algebra theory has woven together the thoughts of many brilliant contributors. In such a vast subject it is no longer practical to keep track of every individual contribution. If we fail to attribute results to their original discoverers this is not a slight, not an insult, but evidence of how fundamentally enmeshed in the general theory their work has become. In particular, none of the results are claimed as our own unless we specifically say so.

The first three sections of this chapter are “Order structures and order convergence”, “Monotone σ-complete C^*-algebras” and “Commutative algebras”. They are basic to all that follows. The final section, “Matrix algebras over a monotone complete C^*-algebra”, is not needed until the later chapters.

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2.1 Order Structures and Order Convergence

We are familiar with the fact that a bounded set of real numbers has a least upper bound. Let P be a partially ordered set and S a subset of P. An *upper bound* for S is an $x \in P$ such that $a \leq x$ for each $a \in S$. We call y a *least upper bound* for S if y is an upper bound for S and, whenever x is an upper bound for S, then $y \leq x$. If a least upper bound exists then it is unique. Lower bound and greatest lower bound are defined analogously. We shall also use "supremum" and "least upper bound" interchangeably; they mean the same. Similarly with "infimum" and "greatest lower bound". If, for each x, y in P the set $\{x, y\}$ has a supremum and an infimum then P is a *lattice.*

When $S \subset P$ and S has a supremum s, we write $s = supS$ or $s = \bigvee S$. Both these notations are in common use and we shall make use of both of them.

We recall that S is *upward directed* if $a \in S$ and $b \in S$ implies there exists $c \in S$ such that $a \leq c$ and $b \leq c$. (Downward directed is defined similarly.)

Let A be any C^*-algebra (not necessarily with a unit element) and let A_{sa} be the (real) Banach space of self-adjoint elements of A. We recall that the *positive elements* of A are, by definition, those of the form zz^*. Let A^+ be the set of all positive elements of A. Then A^+ is a cone, that is, if x and y are in A^+ and if λ and μ are in $\mathbb{R}^+$ then $\lambda x + \mu y$ is in A^+. This cone A^+ is closed in the norm topology. Furthermore, $A^+ \cap -A^+ = \{0\}$ and $A_{sa} = A^+ - A^+$. So we can define a partial ordering on A_{sa} by $x \geq y$ precisely when $x - y \in A^+$. (We also use $y \leq x$ to mean $x \geq y$.) Then with this partial ordering, A_{sa} is a partially ordered Banach space with the real numbers as scalars.

Let us put $A = C(T)$, the algebra of complex valued continuous functions on a compact Hausdorff space T. Then A_{sa} can be identified with $C_{\mathbb{R}}(T)$, the real valued continuous functions. Then $f \geq g$ if $f(t) \geq g(t)$ for each $t \in T$. It is easy to see that this ordering makes A_{sa} into a lattice.

In fact this lattice property is equivalent to commutativity. Given a C^*-algebra A, Sherman's Theorem [152] tells us that A_{sa} is a (vector) lattice if, and only if, A is commutative. A striking theorem of Kadison [81] tells us that $L(H)$ is an anti-lattice. That is, given $x, y \in L(H)_{sa}$, the pair $\{x, y\}$ does not have a supremum unless $x \leq y$ or $y \leq x$. For these, and more general results, see [9, 29]. At first sight Kadison's result seems puzzling, since we know the projections in $L(H)$ form a lattice, with respect to the partial ordering induced by $\leq$. But this apparent paradox is easily resolved. Given projections p and q in $L(H)$ the set of projections above both of them has a smallest element; but, in general, the set of all self-adjoint elements above both of them does not have a smallest element.

Whenever J is a closed ideal of A then J^+ is a hereditary cone in A^+, in other words, if $x \in A$, $0 \leq x \leq b$ and $b \in J^+$ then $x \in J^+$. (Our main reference for the basic theory of C^*-algebras is Takesaki [161] but see, also, Pedersen [121] for his elegant account of order properties in C^*-algebras.)

Let U be a subset of A_{sa}. Then U is a norm bounded, upward directed set if and only if $-U = \{-a : a \in U\}$ is a norm bounded, downward directed set. Furthermore U has a least upper bound x if, and only if, $x - U = \{x - y : y \in U\}$ has infimum 0.

If (a_n) is a monotone increasing sequence in A_{sa} then, clearly, $\{a_n : n = 1, 2 \ldots\}$ is an upward directed set; when this directed set has a supremum a we say that the sequence has supremum a. Similarly, a monotone decreasing sequence (b_n) has infimum b, when the downward directed set $\{b_n : n = 1, 2 \ldots\}$ has infimum b.

As well as sequences we shall also make use of nets, see below.

The set $\{t \in \mathbb{R} : t < 1\}$ is an upper bounded set of real numbers but it is not bounded. In general, given an upper bounded set U in A we can pick $y_0 \in U$ and define $U_0 = \{y \in U : y \geq y_0\}$. Then U_0 is norm bounded.

If U is upward directed then U_0 has the same set of upper bounds as U. But if U is *not* upward directed then this need not be true. (For a trivial example, recall that $L(H)$ is an anti-lattice. So we can find self-adjoint a and b for which $\{a, b\}$ has no supremum. Put $U = \{a, b\}$ and $U_0 = \{y \in U : y \geq b\} = \{b\}$.)

Definition 2.1.1 A C^*-algebra A is *monotone complete* if each norm bounded, upward directed subset of A_{sa} has a least upper bound.

By using the map $a \mapsto -a$ it is easy to see that A is monotone complete if each norm bounded, downward directed subset of A_{sa} has a greatest lower bound.

Definition 2.1.2 A C^*-algebra A is *monotone σ-complete* if each norm bounded, monotone increasing sequence has a least upper bound.

It is immediate that A is *monotone σ-complete* if each norm bounded, monotone decreasing sequence has a greatest lower bound.

We shall see, later, that all monotone complete C^*-algebras have a unit element. This is not true for monotone σ-complete C^*-algebras. However, suppose that A is monotone σ-complete and does not posses a unit. Then A^1 the algebra formed by adjoining a unit, will be shown to be monotone σ-complete.

When working with C^*-algebras things go much more smoothly when they possess a unit. But, particularly for dealing with ideals, we need to extend parts of the theory to the non-unital situation. Most of the (minor) contortions which this requires are dealt with in this chapter.

Let A be monotone (σ-)complete and let $\mathcal{T}$ be a Hausdorff topology for A. Let us call $\mathcal{T}$ *sequentially order compatible* if, whenever (a_n) is a norm bounded, monotone increasing sequence with least upper bound a, then $a_n \to a$ in the $\mathcal{T}$-topology. In $L(H)$ the strong operator topology is sequentially order compatible. When A is any von Neumann algebra, with predual A_*, then the $\sigma(A, A_*)$ topology is (sequentially) order compatible. Does every monotone complete C^*-algebra have a Hausdorff, locally convex vector space topology which is sequentially order compatible? If the answer were "yes" we could replace order considerations by topological arguments.

But the answer is "no". There are commutative counter examples:

Example 2.1.3

(a) Let $B[0, 1]$ be the space of all bounded Borel measurable, complex valued functions on the unit interval. Let M be the set of all f in $B[0, 1]$ for which $\{\lambda \in [0, 1] : f(\lambda) \neq 0\}$ is a meagre set. Then, when the algebraic operations are defined pointwise and $B[0, 1]$ is equipped with the supremum norm, it is a commutative C^*-algebra. Also M is a closed ideal and the quotient $B[0, 1]/M$ will be shown to be monotone complete in Chap. 4. We have already remarked, and will prove later, that this algebra has no normal states. It is known that if $\mathcal{T}$ is a Hausdorff, locally convex vector space topology for $B[0, 1]/M$ then $\mathcal{T}$ is not sequentially order compatible. This follows from [44] because this work shows that if $B[0, 1]/M$ is equipped with a Hausdorff topology $\mathcal{S}$ which is sequentially order compatible then the map $(x, y) \mapsto x - y$ is not jointly continuous. Hence such an $\mathcal{S}$ cannot be a locally convex vector topology. However, as we shall see later, the Wright Representation Theorem [171] does show that each monotone σ-complete C^*-algebra is the *quotient* of a monotone σ-complete C^*-algebra which *does* posses a sequentially order compatible topology. This is illustrated by the above example.

(b) Let $Bnd(\mathbb{R})$ be the commutative C^*-algebra of all bounded complex valued functions on $\mathbb{R}$. Let J be the ideal consisting of all $f \in Bnd(\mathbb{R})$ for which $\{r \in \mathbb{R} : f(r) \neq 0\}$ is a countable set. Then J is monotone σ-complete. It is not monotone complete and does not possess a unit element.

(c) Let $C[0, 1]$ be the C^*-algebra of continuous complex valued functions on the closed unit interval. Then this algebra is neither monotone complete nor monotone σ-complete. Now let U be the set of all $f \in C[0, 1]$ such that f is real valued, $f(0) = 0$ and $0 \leq f(\lambda) < 1$ for each $\lambda \in [0, 1]$. Then U is upward directed and the function with the constant value 1 is the least upper bound of U in $C[0, 1]$. We have $||f - 1|| = 1$ for each $f \in U$. So there does not exist a sequence in U which converges in norm to the least upper bound of U.

When A is not assumed to be monotone complete it may still have some norm bounded, downward directed subset which has a greatest lower bound e.g. in Example 2.1.3(c) put $D = \{1 - f : f \in U\}$. Then D has 0 as its greatest lower bound.

When A is a C^*-algebra without a unit, we define A^1 to be the algebra formed by adjoining a unit. Then A is a maximal ideal of A^1. When A does have a unit we put $A^1 = A$.

In the rest of this section A is a C^*-algebra which is not assumed to possess a unit and is not required to be monotone complete, unless this is stated explicitly.

Notation 2.1.4 *In an algebra A we use: "$(a_n) \uparrow a$" as an abbreviation for: (a_n) is a monotone increasing sequence in A_{sa} with least upper bound a in A_{sa}. We also use $(x_n) \downarrow x$ to indicate that (x_n) is a monotone decreasing sequence in A_{sa} with infimum x in A_{sa}. When we write "$(a_n) \uparrow$" that will mean that the sequence is* *monotone increasing. We shall also use a similar notation for sequences in more general partially ordered sets.*

We shall sometimes find it convenient to abuse our notation, mildly, by referring to a monotonic sequence or directed set as being "in A" when "in A_{sa}" would be more correct; similarly we sometimes refer to a supremum or infimum as being "in A".

Lemma 2.1.5 *Let A be any C^*-algebra. Let $(a_n) \uparrow a$ and $(b_n) \uparrow b$. Then $(a_n+b_n) \uparrow a+b$.*

Proof Let x be an upper bound for $(a_n + b_n)$. Then

$$a_r + b_m \leq a_{r+m} + b_{r+m} \leq x.$$

So $a_r \leq x - b_m$. Fix m. Then $x - b_m$ is an upper bound for (a_r). Thus $a \leq x - b_m$.

So, for all m, $b_m \leq x - a$. Hence $b \leq x - a$. It now follows that $a + b$ is the least upper bound of $(a_n + b_n)$. □

Let Λ be a directed set. Let $(a_\lambda : \lambda \in \Lambda)$ be a net in A_{sa}. Then the net is increasing if $\lambda \leq \mu$ implies $a_\lambda \leq a_\mu$. We say the net has least upper bound a when its range $\{a_\lambda : \lambda \in \Lambda\}$ has a least upper bound a. It is clear that when $(a_\lambda : \lambda \in \Lambda)$ is increasing then $\{a_\lambda : \lambda \in \Lambda\}$ is an upward directed subset of A_{sa}. Similarly the net is decreasing if $\lambda \leq \mu$ implies $a_\lambda \geq a_\mu$ and it has infimum b if its range $\{a_\lambda : \lambda \in \Lambda\}$ has infimum b. When D is a downward directed set in A_{sa} then $(d : d \in D)$ is a decreasing net.

Lemma 2.1.6

(i) *Let $(a_\lambda : \lambda \in \Lambda)$ be an increasing net in A_{sa} with least upper bound a and $(b_\lambda : \lambda \in \Lambda)$ an increasing net in A_{sa} with least upper bound b. Then $(a_\lambda + b_\lambda : \lambda \in \Lambda)$ is an increasing net in A_{sa} with least upper bound $a + b$.*

(ii) *Let $(x_\lambda : \lambda \in \Lambda)$ be a decreasing net in A_{sa} with infimum x and $(y_\lambda : \lambda \in \Lambda)$ a decreasing net in A_{sa} with infimum y. Then $(x_\lambda + y_\lambda : \lambda \in \Lambda)$ is a decreasing net in A_{sa} with infimum $x + y$.*

Proof

(i) This is a straightforward modification of the proof of Lemma 2.1.5.

(ii) Put $x_\lambda = -a_\lambda$ and $y_\lambda = -b_\lambda$ and apply (i). □

Lemma 2.1.7 *Let A be any C^*-algebra. Let (x_n) be a monotone increasing sequence in A_{sa}. Suppose this sequence converges in norm to x. Then the sequence has a supremum and this supremum is x.*

Proof For $m \geq n$, $x_m - x_n \geq 0$. Because the positive cone is closed in the norm topology,

$$x - x_n = lim_{m\to\infty}(x_m - x_n) \geq 0.$$

Let b be an upper bound for the sequence. Then $b - x = lim_{n\to\infty}(b - x_n) \geq 0$. So x is the least upper bound of the sequence. □

The converse of this lemma is, of course, false. To see this, first take a separable, infinite dimensional, Hilbert space H. Then, in $L(H)$, take a monotone increasing sequence of finite rank projections converging to 1_H (the identity operator on H) in the strong operator topology. Then the sequence has 1_H as its supremum but the sequence is certainly not convergent in norm.

Corollary 2.1.8 *Let $\mathcal{T}$ be a Hausdorff locally convex vector topology for A_{sa} such that A^+ is closed in the $\mathcal{T}$ topology. Let $(x_\lambda : \lambda \in \Lambda)$ be an increasing net in A_{sa} such that $(x_\lambda : \lambda \in \Lambda)$ converges in the $\mathcal{T}$ topology to x. Then x is the least upper bound of $\{x_\lambda : \lambda \in \Lambda\}$.*

Proof Straightforward modification of the proof of the preceding lemma. □

We shall see that if S is an upward directed set in A_{sa} with supremum s, then, for any z in A, zSz^* is upward directed with supremum zsz^*. We also obtain a weaker, but useful, result which can be used when S is not a directed set.

We call a subset $S \subset A_{sa}$ *order bounded in A* if there exist a and b in A_{sa} such that

$$a \leq x \leq b$$

for each x in S. Clearly an order bounded set is bounded in norm. When A is unital, norm bounded sets are order bounded. But when A is not unital, norm bounded sets need not be order bounded. For example, let A be the compact operators on a separable, infinite dimensional Hilbert space and take S to be the self-adjoint (compact) operators in the unit ball of A.

In the following lemma, there is no requirement that S be a directed set. Part of the proof is based on [61].

Lemma 2.1.9 *Let J be a (closed two sided) ideal in A. Let S be a norm bounded subset of J_{sa} with least upper bound s in J. If $z \in A^1$ is invertible then zSz^* is a norm bounded subset of J with least upper bound zsz^* in J. Furthermore, if S is order bounded in J_{sa}, and $c \in J^+$ then cSc has supremum csc in J^+.*

Proof We observe that J is a closed ideal of A^1.

Let $L = s - S = \{s - y : y \in S\}$. Then L is a subset of J^+ which is bounded in norm and has infimum 0. Since J is an ideal, zLz^* is a subset of J. We show that, in J, zLz^* has an infimum and that this infimum is 0.

Since $a \mapsto a^*$ is an anti-automorphism of A^1, z^* is invertible and $(z^{-1})^* = (z^*)^{-1}$. The map T_z defined by $T_z(x) = zxz^*$ for $x \in A$ is an order-isomorphism of A_{sa} onto itself with $T_z{}^{-1} = T_{z^{-1}}$ and $T_z(J) = J$. Hence zLz^* has infimum 0.

Now suppose S is order bounded in J_{sa}. Then there exists $a \in J_{sa}$ which is an upper bound for L. Let d be any element of L. So, there exists $a \in J_{sa}$ such that $d \leq a$ for all $d \in L$.

Then

$$d^2 \leq ||d||d \leq ||d||a \leq ||a||a.$$

Let $c \in J^+$. We shall show that cLc has an infimum in J and that this infimum is 0.

Let $x \in J$ be a lower bound for cLc and let d be any element of L. So $x \leq cdc$.

Let ε be a positive real number. Then

$$\begin{aligned}(c+\varepsilon 1)d(c+\varepsilon 1) &= cdc + \varepsilon(cd+dc) + \varepsilon^2 d \\ &= cdc + \varepsilon((c+d)(c+d)^* - c^2 - d^2) + \varepsilon^2 d \\ &\geq cdc - \varepsilon(c^2 + d^2) \\ &\geq x - \varepsilon(c^2 + \|a\|a) \in J_{sa}.\end{aligned}$$

Hence it follows that

$$d \geq (c+\varepsilon 1)^{-1}(x - \varepsilon c^2 - \varepsilon\|a\|a)(c+\varepsilon 1)^{-1} \text{ in } J_{sa}$$

for all $d \in L$. So, we have

$$0 \geq x - \varepsilon c^2 - \varepsilon\|a\|a$$

for all ε and so $0 \geq x$ follows. So, $csc = \sup cSc$ in J_{sa}. □

Proposition 2.1.10 *Let J be a (closed two sided) ideal in A. Let S be an upward directed subset of J_{sa} with least upper bound s in J. Then, for any $z \in A^1$, we have zSz^* is an upward directed subset of J with least upper bound zsz^* in J.*

Proof Let $a_0 \in S$ then $S_0 = \{a \in S : a \geq a_0\}$ is upward directed, order bounded and with supremum s. So, without loss of generality, we may assume that S is order bounded in J_{sa}. We use the same notation as in the preceding lemma.

Let $L = s - S = \{s - y : y \in S\}$. Then L is a subset of J^+ which is downward directed with infimum 0. It suffices to show that, in J, zLz^* has an infimum and that this infimum is 0.

We may assume that $z \neq 0$ for otherwise there is nothing to prove. Let $x = z + 2||z||$ and $y = z - 2||z||$. Then x and y are invertible elements of A^1. By the preceding lemma, xLx and yLy both have infimum 0. We shall show that $(x+y)L(x+y)^*$ also has infimum 0. To see this, we argue as follows.

For any f and g in A we find, by expanding $(f-g)(f-g)^*$, that $fg^* + gf^* \leq ff^* + gg^*$.

Let $d \in L$. Put $f = xd^{1/2}$ and $g = yd^{1/2}$. Then $xdy^* + ydx^* \leq xdx^* + ydy^*$. So $(x+y)d(x+y)^* \leq 2xdx^* + 2ydy^*$.

Let c be any lower bound for $(x+y)L(x+y)^*$. Then $c \leq 2xdx^* + 2ydy^*$ for every $d \in L$.

In Lemma 2.1.6 (ii) put $\Lambda = L$ and consider the nets $(2xdx^* : d \in L)$ and $(2ydy^* : d \in L)$. Then $(2xdx^* + 2ydy^* : d \in L)$ has infimum 0. So $c \leq 0$. Hence zLz^*has infimum 0 and so zSz^* has supremum zsz^*. □

Suppose A does not possess a unit. Let S be an upward directed set with supremum s. When S is regarded as a subset of A^1does it still have a supremum in A^1_{sa} and, if it does, is it s? Fortunately the answer is "yes". See the proposition below.

In contrast to this result, an upward directed set in A_{sa} may have a supremum in A^1_{sa} but fail to have a supremum in A_{sa}. To see this, let H be a separable, infinite-dimensional, Hilbert space. Let A be the algebra of compact operators on H. Then take an increasing sequence of finite rank projections converging (in the strong operator topology) to 1, the identity operator on H. This sequence has no supremum in A but, in A^1, it has 1 as its supremum.

Proposition 2.1.11 *Let A be a C^*-algebra without a unit element. Let S be an upward directed set in A_{sa} with supremum s in A_{sa}. Then S has supremum s in A^1.*

Proof Because A is a closed ideal in A^1 the quotient map $q : A^1 \mapsto A^1/A$ is a $*$-homomorphism and hence a positive map.

Let $a + \lambda 1$ be any upper bound for S in A^1, where $a \in A$. Since q maps each element of A to zero, it follows that $\lambda \geq 0$.

For any $z \in A$, since A is an ideal of A^1, $z(a + \lambda 1)z^*$ is an upper bound for zSz^* in A. So

$$zsz^* \leq z(a + \lambda 1)z^* \leq zaz^* + \lambda ||z||^2 1.$$

Now let (z_α) be an approximate unit for A. Then $z_\alpha s z_\alpha^* \leq z_\alpha a z_\alpha^* + \lambda 1$. Also $||z_\alpha(a - s)z_\alpha^* - (a - s)|| \to 0$. Since the positive cone of A^1 is closed in the norm topology it follows that $0 \leq a - s + \lambda 1$. That is, $s \leq a + \lambda 1$. □

Corollary 2.1.12 *Let A be monotone σ-complete. Let $(a_n) \uparrow a$. Let $||a_n|| \leq 1$ and $a_n \geq 0$ for each n. Then $||a|| \leq 1$.*

Proof In A^1, the unit 1 is an upper bound for the sequence. So, from the proposition, $a \leq 1$. □

Lemma 2.1.13 *Let ϕ be a positive linear functional on a C^*-algebra A. Then ϕ is a bounded linear functional.*

Proof Let $A_1^+ = \{a \in A^+ : ||a|| \leq 1\}$. It suffices to show ϕ is bounded on A_1^+. Suppose this is false. Then for each n there is a_n in A_1^+ such that $n2^n < \phi(a_n)$. Using norm convergence, let $a = \sum_1^\infty \frac{1}{2^n} a_n$. Then, for each n,

$$n \leq \phi(\frac{1}{2^n} a_n) \leq \phi(a).$$

This is impossible. □

Let ϕ be a positive linear functional on a C^*-algebra. Then ϕ is said to be *faithful* if $x \geq 0$ and $\phi(x) = 0$ implies $x = 0$.

Theorem 2.1.14 *Let A be a monotone σ-complete C^*-algebra. Let ϕ be a faithful, positive linear functional on A. Then A is monotone complete. In particular, let D be an upward directed, norm bounded set in A_{sa}. Then there is a monotone increasing sequence in D whose supremum is the supremum of D.*

Proof It suffices to prove this when D is a subset of A_1^+, the intersection of the closed unit ball with the cone of positive elements.

Let $\overline{D}$ be the set of all $x \in A_1^+$ for which there is a monotone increasing sequence in D with x as its supremum.

Since D is upward directed, it is easy to see that $\overline{D}$ is also upward directed. Let (d_n) be any sequence in $\overline{D}$. Then, for each n we have $(x_r^{(n)}) \uparrow d_n$ where each $x_r^{(n)}$ is in D. Since D is upward directed we can find a monotone increasing sequence in D, (y_n) such that $y_n \geq x_r^{(k)}$ for $n \geq k$ and $n \geq r$.

Let $(y_n) \uparrow d$. Then $d \in \overline{D}$ and $d \geq x_r^{(k)}$ for all r and all k. So $d \geq d_n$ for all n.

By the preceding lemma, ϕ is bounded. Let $\lambda = sup\{\phi(y) : y \in \overline{D}\}$. For each n let $d_n \in \overline{D}$ such that $\phi(d_n) \geq \lambda - 1/n$. Then there exists $d \in \overline{D}$ with $\phi(d) \geq \lambda$. So $\phi(d) = \lambda$.

Now let $b \in \overline{D}$. Then, because the set is upward directed, we can find $c \in \overline{D}$ such that $c \geq b$ and $c \geq d$. Thus $\lambda \geq \phi(c) \geq \phi(d) = \lambda$. So $\phi(c - d) = 0$. Since ϕ is faithful, $c = d$. So $d \geq b$. So d is an upper bound for D. Since d is the least upper bound of an increasing sequence from D, it is the least upper bound of D. □

Proposition 2.1.15 *Let A be monotone complete. Then A has a unit element.*

Proof Let $\Gamma = \{a \in A^+ : ||a|| < 1\}$. Then, see p.11 [121], Γ is upward directed and an approximate unit. Since it is norm bounded it has a supremum e in A. Then $0 \leq e$. By Proposition 2.1.11 $e \leq 1$ in A^1. So $\|e\| \leq 1$. By spectral theory $e^2 \leq e$. So $z^* e^2 z \leq z^* e z \leq z^* z$ for all $z \in A$.

Let $x \in A_{sa}$. Since Γ is an approximate unit, and since

$$||x^2 - xax|| \leq ||x||||x - ax||$$

it follows that the net $(||x^2 - xax|| : a \in \Gamma)$ converges to 0. So the net $(xax : a \in \Gamma)$ converges in the norm topology to x^2

But, by Proposition 2.1.10, xex is the least upper bound of $x\Gamma x$. So, by Lemma 2.1.7, $xex = x^2$.

So we have

$$\begin{aligned} 0 \leq (x - ex)^*(x - ex) &= x^2 - xex - xex + xe^2x \\ &= -x^2 + xe^2x \leq -x^2 + xex = 0 \end{aligned}$$

which implies that $\|x - ex\|^2 = 0$, that is, $x = ex$. Taking adjoints gives $x = xe$. So, e is a unit element of A. □