Bettina Albers · Mieczysław Kuczma
Editors

Continuous Media with Microstructure 2

Springer

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The 2nd conference on Continuous Media with Microstructure (CMwM2015) *was held on March 2–5, 2015 in Łagów, Poland, in memory of*

Professor Krzysztof Wilmański

who regrettably passed away on 26/08/2012.

We dedicate this book of CMwM2015 *contributions to him.*

Preface

This book is a collection of papers dedicated to the memory of Prof. Dr. Krzysztof Wilmański. It contains the written form of many contributions to the 2nd International Conference on *Continuous Media with Microstructure* held in Łagów, Poland, March 2–5, 2015 (Fig. 1).

CMwM2015, also announced as an ECCOMAS Special Interest Conference, was organized by the Polish Academy of Sciences, Poznan University of Technology, Berlin University of Technology, and the Polish Association for Computational Mechanics. Many friends and colleagues of Prof. Krzysztof Wilmański eagerly accepted the invitation of the conference chairpersons Bettina Albers (at that time: TU Berlin) and Mieczysław Kuczma (Poznan UT). Professor Krzysztof Wilmański regrettably passed away on August 26, 2012 but would have celebrated his 75th birthday on March 1, 2015. The 1st conference CMwM took place in Zielona Góra, Poland, in 2010 to celebrate the 70th birthday of Prof. Wilmański. At this occasion he received the first part of the book *Continuous Media with Microstructure*. That book contains further information about him, especially a photo, his curriculum, and his publication list (Reference: Albers, B. (ed.):

Fig. 1 Participants of the 2nd International Conference on Continuous Media with Microstructure

Continuous Media with Microstructure. Collection in Honor of Krzysztof Wilmanski, Springer, Berlin, Heidelberg, 2010, ISBN 978-3-642-11444-1).

CMwM2015 was a conference with an intimate atmosphere, attended by nearly 40 scientists from Brazil, Czech Republic, Estonia, Georgia, Germany, Italy, Poland, Russia, and the USA, who gave 35 presentations.

The general lectures were delivered by

- Tadeusz Burczyński on *Intelligent optimization of media with microstructure*,
- Carlo Giovanni Lai on *Measurement of damping ratio spectra in soils from the exact solution of Kramers-Krönig equations of linear viscoelasticity*,
- I-Shih Liu on *A mixture theory of porous media and some problems of poroelasticity*,
- Martin Ostoja-Starzewski on *Continuum mechanics beyond the second law of thermodynamics*,
- Jörg Schröder on *A* FE^2*-homogenization scheme for the analysis of product properties of two-phase magnetoelectric composites*, and
- David M.J. Smeulders on *Electrokinetic experiments in porous media for energy applications*.

The contributions to the book concern various aspects of extension of classical continuum models and of engineering applications of continuum theories. In particular, the contributions deal with the following subjects:

- continuum mechanics,
- thermodynamics,
- porous and granular media,
- engineering applications.

We would like to kindly thank both the participants of *CMwM2015* and the contributors to the current book for the nice cooperation and for their commitment. Furthermore, we are grateful for technical work, especially the transformation of some contributions into LaTeX, by Benedikt Preugschat. Last but not least we appreciate very much the pleasant collaboration with Springer, especially with Christoph Baumann who accepted the publication of this book.

Essen Bettina Albers
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October 2015

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Krzysztof Wilmanski (1940–2012)

Ingo Müller

When I first met Krzysztof Wilmanski in 1977 he was one of the bright young scientists in the Institute of Fundamental Technological Research of the Polish Academy of Sciences. This was the time of the cold war and it was not altogether easy for us westerners to meet colleagues from beyond the iron curtain. But among all people from the east it was still easiest to meet Polish scientists. Because, indeed, the wise elder scientists at the helm of the Polish Academy—among them Professors Nowacki, Olszak, Fiszdon, and Sawczuk—held some influence in political circles. And they knew that good science requires free and easy communication between scientists. Also they believed in mechanics as an essential part of the natural sciences.

Therefore they sent their young mechanicians abroad, to the east and to the west. Still, it was easier for them go east. Thus, long before I met him, Krzyszt of Wilmanski had been in Moscow, where he tried to join Professor Sedov's research group. Somehow that did not work out well. He was disappointed and left after a few months. Next he went to Baltimore where at the time—in the late 1960s—Professors Truesdell and Ericksen conducted a lively group of graduate student and post-doctoral fellows at the Johns Hopkins University. He was well received there and worked successfully. Thus in a manner of speaking he finished his scientific education there, an education which had started in civil engineering.

Strangely, although I had been at Johns Hopkins before Krzysztof and again after he had left, we never met there. However, we had been exposed to the same unique scientific atmosphere, created by professors Ericksen and Truesdell at the height of Rational thermodynamics; and so, —even without actually meeting—, we became members of a loosely knit group which, much later, some unfriendly person dubbed the "Johns Hopkins gang".

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After Johns Hopkins Krzysztof went to Iraq. There was a program, haggled out between the Polish government and some oil-rich left-leaning Arabian states in the near east, by which Polish professors could obtain a well-paid teaching job—in Iraq, or Syria, or Libya—for 1 or 2 years. So, Krzysztof went to Iraq for 2 years. Scientifically that stay was little fruitful, later he would complain that he met a mix of ignorance and arrogance in his host faculty. But the job was profitable money-wise. And so on a side trip to Saudi Arabia Krzysztof acquired a nice new red BMW.

All of this was before Krzysztof and I met.

When we did meet I was a young professor in Paderborn, Germany, recently returned from the United States. The occasion was the founding of ISIMM, the International Society for the Interaction of Mechanics and Mathematics which took place in Kozubnik in the south of Poland. That society was to further facilitate scientific relations between the East and the West and many of the grand old men of mechanics were there. Well, with the exception of the Russians; they feared a conflict of interest with IUTAM, the International Union of Theoretical and Applied Mechanics which was the forum for east-west interaction officially recognized by the Soviet government. But everybody else who was somebody in mechanics and applied mathematics was present in Kozubnik: Sneddon, Chadwick and Spencer from Great Britain, Fichera, Graffi and Grioli from Italy, Kirchgässner and Kröner from Germany, i.e. West Germany. And of course all the Polish tycoons of science were there. Krzysztof Wilmanski and myself were in the junior crowd, along with Robin Knops, Costas Dafermos, Carlo Cercignani and many others. Little did we think at the time that we should in the future become officers of the Society; and yet that happened: Between the years 2000 and 2004 Krzysztof and I were president and secretary of ISIMM.

On the last day of the meeting, a Saturday morning, a decrepit Russian-built bus was to carry the participants back to Warsaw airport. However, the technological decline of the East was already far advanced and so the bus wouldn't start; an essential part of the ignition system was broken. Henryk Zorski, Krzysztof Wilmanski and I myself were watching when the driver gave up his efforts and announced a 3-h delay, because he had to send to Kattowitz for a spare part. Naively I asked Zorski whether he thought that the needed part would be available in Kattowitz. "Oh, no", said Henryk, always the cynic, "nowhere in Poland but there will be a spare bus."

I myself was spared the bus ride, because Krzysztof offered to take me in his BMW. Looking back I now realize that the long ride in close companionship was to soften me up, so that I should support Krzysztof's application for a Humboldt scholarship which he was preparing. Really that effort on his part was quite unneccessary. I would have supported that application anyhow, after all: One member of the Johns Hopkins gang and the other. But the trip as such was not uneventful. Indeed, the pot holes of Southern Poland yanked a shock absorber clean out of the body of the BMW. There was a tremendous noise in the back and when we reached Krakow it became imperative to find a workshop—at noon on a Saturday in a socialist economy! I considered that hopeless. However, Krzysztof was unfazed and thus he proved not only a keen scientist but also an ingenious organizer of an everyday calamity. He developed a train of thought by which the dairies of the town of necessity are on a 24 h schedule, —including Saturdays—, that they need lorries for the collection and distribution of milk and that the lorries, given their state of repair, need a dairy-owned workshop for repairs. So, we inquired about dairies, found the appertaining workshop and there was a young mechanic who welded the shock absorber back into the place where it belonged.

Time had been lost, though, and so Krzysztof regretted that he could not show me the tomb of Marshall Pilsudski whom he admired very much for having chased out the Russians from some small eastern part of Poland in 1918, or so, and for a short while. Krzysztof was a fervent Polish patriot and remained that throughout his life, even after—much later—he became a German citizen. In fact sometimes he surprised me with his Polish view of events in German history. Actually, if the truth were known there seem to be few events in the histories of the two countries which are viewed in the same light from the two sides of the Polish-German border; excepting only, of course, Russian attempts for domination.

Anyway, the application for the Humboldt scholarship was successful and so, early in 1979, Krzysztof showed up in Paderborn. This started a year of intensive and successful collaboration between the two of us, mostly on shape memory alloys. And that was before anybody else in continuum mechanics recognized the importance of these materials for the understanding of large deformations in solids. So, when I moved to Berlin in that same year, Krzysztof came with me and we continued our work. For me and, I believe, for Krzysztof as well this was a highly satisfactory period of joint learning and research, which we would have liked to continue. But politics intervened, cold war politics.

As some of you may know, the Humboldt foundation gives stipends for no more than 1 year. But, upon application this period can be extended for up to another year. So, naturally Krzysztof and I applied and we got the extension. But cold war tactics was against us. Krzysztof was in trouble with the Polish authorities when he applied for keeping his passport. At that time the official communist doctrine about Germany, dictated by the Russians, was that there were three Germanies: West Germany, DDR and Westberlin. So Krzysztof in joining me in Berlin had violated the rules laid down in his passport which allowed him to visit West Germany only. He had to be punished and he was not allowed to stay on in Berlin. Friends back in Paderborn offered him an office but, although they meant well, they worked on different subjects. So our collaboration suffered; all of this was in the days before e-mail which might have helped us a lot.

And so began Krzysztof's veritable scientific odyssey in Germany and between Germany and Poland: Paderborn, Berlin, Warsaw, Berlin, Hamburg, Berlin, Essen. This odyssey lasted 14 years instead of the requisite ten. During his Warsaw period the two of us organized a Euromech conference on shape memory in Jabłonna near Warsaw. In Hamburg Krzysztof became an expert on plasticity, in Essen he developed an improved theory of porous and granular media and in the end in Berlin, at the Weierstrass Institute, he concentrated his research efforts on wave propagation in soils, in close collaboration with civil engineers: Wet soils for off-shore wind turbines and dry soils for tunneling.

A high point in his itinerant life, perhaps, was Krzysztof's invitation to a year's stay at the Wissenschaftskolleg Berlin. That august institution—the Institute of Advanced Studies—had heretofore never invited natural scientists—it thrives on political and social "sciences". However, in 1984 I got the unexpected possibility to invite two persons from the natural sciences and they were Ronald Rivlin and Krzysztof Wilmanski. The three of us spent at least 1 day a week together in intensive discussion.

This was the time when a disaster concerning the stability of rheological fluids had overwhelmed Rational Thermodynamics, the theory of Coleman and Noll: According to that theory the free energy should have a maximum in equilibrium when everybody in thermodynamics knew for a century that it has to have a minimum. Rivlin was overjoyed, because he disliked Coleman and Noll for their close association with Truesdell. I myself did not care much, since I worked on Extended Thermodynamics which was untouched by the disaster. And Krzysztof suggested that we look into the problem and perhaps understand its reason. So we studied the papers by Dunn and Fosdick and by Joseph on fluids of nth grade and we came to the conclusion that it is not legitimate to approximate the constitutive functional of the history of some field by a few time derivatives of the field at the present time. The procedure leads to instability. Rivlin did not appreciate the result since it cast doubt on the usefulness of the Rivlin-Ericksen tensors, with which he was strongly associated. And Fosdick—when asked—was also unenthusiastic, because he maintained that stability would miraculously reappear far from equilibrium; a clear case of wishful thinking. So Krzysztof and I were frustrated; we did write a paper and published it in Rheologica Acta to show that the free energy indeed has a minimum in equilibrium as it should be. And there was some interest, —at least we received a lot of reprint requests. But the paper did not really catch on. And, if the truth were known, our arguments lacked the systematic clarity necessary to be convincing. Indeed the problem of a proper thermodynamic theory of non-Newtonian fluids remains unsolved to this day.

Years later, when Roger Fosdick celebrated his 60th birthday, Krzysztof and I addressed the problem again from the point of view of the kinetic theory of rheological fluids. Somewhat maliciously we offered that study for Roger's Festschrift. But then it turned out that neither of us could attend the anniversary meeting and so our paper remained unpublished except as a preprint report of the Weierstrass Institute.

So let me pull our main conclusion out of oblivion in this present eulogy for Krzysztof: Considering a solution of Hookean dumbells—a standard model of rheology—we derived a differential equation between the deviatoric stress and the deviatoric velocity gradient, viz.[1]

$$\left(1+\frac{1}{2}\frac{\zeta}{\lambda}\frac{\delta}{\delta t}\right)t_{<pq>}=\eta_0\left(1+\frac{\eta_s}{\eta_0}\frac{1}{2}\frac{\zeta}{\lambda}\frac{\delta}{\delta t}\right)\frac{\partial u_{<p}}{\partial x_{q>}} \tag{1}$$

λ is the elastic constant of the dumbell spring. ζ is the Stokes friction coefficient of a dumbell mass in the solvent and η_s is the viscosity of the solvent. η_0 is defined as $\eta_s+\frac{5}{6}nkT\frac{\zeta}{\lambda}$ with n as the number density of dumbells. The equation is known as the Giesekus equation in rheology, but our derivation was marginally more systematic than Giesekus's so that there was a tiny little bit of originality.

All the coefficients are positive so that in regard to stability the equation is fine: If the velocity gradient vanishes, the stress will exponentially approach zero and if the stress vanishes, the velocity gradient relaxes to zero.

So far so good. The argument, —based on reliable molecular considerations—, shows us what the constitutive relation between the stress and the velocity gradient should look like in a rheological fluid. And this does not have the form assumed by Rational thermodynamics. Indeed, in Rational Thermodynamics the stress should be alone on the left-hand-side and it should be given by the velocity gradient and its time derivatives. Such a form may be obtained by shifting the operator $1+\frac{1}{2}\frac{\zeta}{\lambda}\frac{\delta}{\delta t}$ from the numerator on the left-hand-side to the denominator of the right-hand-side and then approximating it—rather daringly—as follows

$$\frac{1}{1+\frac{1}{2}\frac{\zeta}{\lambda}\frac{\delta}{\delta t}}\approx 1-\frac{1}{2}\frac{\zeta}{\lambda}\frac{\delta}{\delta t}. \tag{2}$$

In this manner we obtain

$$t_{<ij>}\approx\eta_0\left(1+\left(\frac{\eta_s}{\eta_0}-1\right)\frac{1}{2}\frac{\zeta}{\lambda}\frac{\delta}{\delta t}\right)\frac{\partial u_{<i}}{\partial x_{j>}}. \tag{3}$$

Now, however, this equation leads to instability. Indeed, if the stress vanishes, the velocity gradient grows exponentially (!) which makes no sense. Thus Krzysztof and I showed in our paper where the instability comes from and how it should be avoided. The fallacy lies in the daring approximation which is inherent in the a priori assumption that the stress should be determined by the velocity gradient and its rate of change.

[1]The complicated time derivatives are Oldroyt derivatives. But for the present brief review we may consider it as just an ordinary time derivative.

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Elementary thermodynamic and stochastic arguments on a non-Newtonian fluid

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submitted April 28th 1999

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Later, —due to the demands of subsequent jobs—Krzysztof left that interesting field and turned to more practical tasks, primarily in wave propagation in porous media as mentioned before. The primary objective was diagnostic, viz. the diagnosis of hidden irregularities in soils. For this work he finally found a secure and stimulating environment in the Weierstrass Institute in Berlin, where he spent the 10 years before his retirement in 2005.

But that was by no means the end of his scientific career. Krzysztof's passion for science and academia did not allow him to stay idle. So he accepted an appointment at the newly founded University of Zielona Góra in Poland where he spent the last years of his life in scientific research, teaching and administration. The faculty in Zielona Góra was lucky to have him, a man of vast experience gathered in many countries in a lifetime devoted to science.

At the end—still working and publishing and full of enthusiasm and energy—he had to succumb to the perfidious sudden attack of the disease that threatens all of us.

I have mourned him, and I am sure we all did, those of us who knew him.

Berlin, 05.05.2015 Ingo Müller

Part I
Continuum Mechanics

Virtual Power and Pseudobalance Equations for Generalized Continua

Gianpietro Del Piero

Abstract In this paper the balance equations of linear and angular momentum are deduced from some regularity properties of the system of contact actions and from the law of action and reaction. This approach provides a simple and unifying formulation of the theories of non polar and polar continua. It also allows for a direct deduction of the classical plate and beam theories as special Cosserat continua, obtained by dimensional reduction induced by appropriate geometrical constraints.

Deduction of Balance Equations

1. The traditional, generally accepted approach to Continuum Mechanics is based on Euler's balance laws of linear and angular momentum. During the second half of the past century, this approach was revisited a number of times. In 1963, W. Noll showed that the Euler laws are in fact a consequence of the postulate of the indifference of power [12]. Later, Gurtin and Martins [9] and Šilhavý [14, 15] came to the conclusion that the same laws, until then regarded as balance equations between distance and contact actions, are in fact regularity assumptions on the system of the contact actions alone.
   This conclusion also originated from an idea of Noll. In [13] he showed that, if a system of contact actions is skew-symmetric,[1] it is also additive on the boundaries of disjoint sets.[2] If this is the case, the contact action over the boundary of a part Π of the body, which may also be seen as a volume action,

[1] This assumption corresponds to Newton's law of action and reaction.
[2] That is, if Π and Π' are disjoint sets with a portion S of boundary in common and if $Q(S) = -Q(-S)$ is the contact action interchanged across S, the contact action on $\partial(\Pi \cup \Pi')$ is the sum of the contact actions on $\partial\Pi$ and on $\partial\Pi'$.

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$$F(\Pi) = -Q(\partial\Pi), \tag{1}$$

is additive on disjoint sets.[3] In the presence of sufficient regularity, a *surface density* s can be associated with Q and a *volume density* $b^\dagger$ can be associated with F, and the preceding equation can be given the form

$$\int_\Pi b^\dagger(x)\,\mathrm{d}V = -\int_{\partial\Pi} s(x, \partial\Pi)\,\mathrm{d}A. \tag{2}$$

A system of contact actions which admits both surface and volume densities is called a *Cauchy flux*, and Eq. (2) is called a *pseudobalance equation*. The reason for the name is that, though it looks like a balance equation, *this is not a balance equation, but only an identity between two different representations of the same contact action*.

The pseudobalance equation is all what is needed to prove the dependence of $s(x, \partial\Pi)$ on the normal n to $\partial\Pi$ at x,[4] and the linearity of this dependence.[5] That is, to prove that there is a linear transformation on the vectors, the *Cauchy stress tensor* T, such that

$$s = Tn. \tag{3}$$

Thus, rather than a consequence of the balance of linear momentum, the existence of the Cauchy tensor is a property enjoyed by all sufficiently regular skew-symmetric systems of contact actions. This reduces the importance of the role played by the Euler balance laws, which are usually considered as a fundamental postulate of mechanics. Indeed, to define a *classical continuum* it becomes convenient to take as primitive the concept of *external power*, which is an integral involving the inner products of the *external actions* of contact s and at distance b by a field v of *virtual velocities*[6]

$$P_{ext}(\Pi, v) = \int_\Pi b \cdot v\,\mathrm{d}V + \int_{\partial\Pi} s \cdot v\,\mathrm{d}A. \tag{4}$$

In particular, a *rigid virtual velocity field* is a vector field of the form

$$v(x) = a + \varpi \times x, \tag{5}$$

[3]The minus sign on the right is just matter of convenience.

[4]The dependence of s on the normal was conjectured by Cauchy, and was currently called the *Cauchy postulate*. Only in 1959 Noll proved that this conjecture is true, under the assumption that the internal actions have a local character [11]. Since then, the Cauchy postulate has become the *Noll theorem*.

[5]This is the *tetrahedron theorem* of Cauchy.

[6]Alternatively, one can take as primitives the concept of virtual velocity and the existence of two types of actions, distance and contact.

with a and ϖ arbitrary constant vector fields. Assuming the *indifference of power* under rigid virtual velocity fields,

$$P_{ext}(\Pi, a) = 0, \qquad P_{ext}(\Pi, \varpi \times x) = 0, \tag{6}$$

the balance laws of Euler

$$\int_{\Pi} b \, \mathrm{d}V + \int_{\partial\Pi} s \, \mathrm{d}A = 0, \quad \int_{\Pi} x \times b \, \mathrm{d}V + \int_{\partial\Pi} x \times s \, \mathrm{d}A = 0, \tag{7}$$

easily follow. With the aid of the relation (3) and of the divergence theorem, the surface integral in (4) can be transformed into a volume integral, called the *internal power*

$$P_{int}(\Pi, v) = \int_{\Pi} \big((b + \operatorname{div} T) \cdot v + T \cdot \nabla v\big) \, \mathrm{d}V. \tag{8}$$

The indifference conditions (6) applied to this integral yield the *local forms* of the balance equations

$$\operatorname{div} T + b = 0, \qquad T = T^T. \tag{9}$$

Equating the two expressions (4), (8) of the power, the *equation of virtual power*

$$P_{ext}(\Pi, v) = P_{int}(\Pi, v) \tag{10}$$

is obtained. This is not an equation between two different powers, as it is frequently asserted,[7] but only an identity between two different expressions of the same power.

Substituting the local forms (9) into the internal power (8), a *reduced form* for the power is obtained

$$P_{red}(\Pi, v) = \int_{\Pi} T \cdot \nabla v^S \, \mathrm{d}V. \tag{11}$$

This reduced form characterizes T as the unique *active internal action* present in a classical continuum, and ∇v^S as the corresponding *generalized deformation velocity*.

2. The framework introduced above is easily extended to the generalized continua. A *generalized continuum* is a continuum whose description involves a finite array $\{\xi^\alpha\}$ of primary variables (*state variables*), which can be scalar, vectorial, or tensorial. Coupled with dual variables $\beta^\alpha, \sigma^\alpha$ of the same tensorial nature (*bulk and surface external actions*), they determine the external power[8]

[7] In fact, on this assumption is based of the "method of virtual power" developed by Germain [7, 8] and others.

[8] Here and in the following, repeated indices are summed.

$$P_{ext}(\Pi, v, v^\alpha) = \int_\Pi (b \cdot v + \beta^\alpha \cdot v^\alpha)\, \mathrm{d}V + \int_{\partial\Pi} (s \cdot v + \sigma^\alpha \cdot v^\alpha)\, \mathrm{d}A, \qquad (12)$$

in which the v^α are the *virtual velocities* of the state variables. If the contact actions σ^α are Cauchy fluxes, each of them has its own pseudobalance equation

$$\int_{\partial\Pi} \sigma^\alpha(x, \partial\Pi)\, \mathrm{d}A = -\int_\Omega \beta^{\alpha\dagger}(x)\, \mathrm{d}V, \qquad (13)$$

and from it, with the aid of Noll's and Cauchy's theorems, follows the existence of a linear transformation Σ^α such that

$$\sigma^\alpha = \Sigma^\alpha n. \qquad (14)$$

The divergence theorem then allows the passage from the external to the internal power

$$P_{int}(\Pi, v, v^\alpha) = \int_\Pi \big((b + \mathrm{div}\, T) \cdot v + T \cdot \nabla v + (\beta^\alpha + \mathrm{div}\, \Sigma^\alpha) \cdot v^\alpha + \Sigma^\alpha \cdot \nabla v^\alpha\big)\, \mathrm{d}V, \qquad (15)$$

from which the balance equations are deduced imposing the indifference to rigid virtual velocity fields. But, unlike in classical continua, the *rigid virtual velocities* are not uniquely defined, since their definition depends on the physical nature of the state variables. In what follows, we consider two classes of generalized continua, *polar* and *non-polar*, with different definitions of rigid virtual velocities. In a non-polar continuum, the state variables describe rearrangements of matter at the microscopic level. A rigid virtual velocity involves no rearrangements, that is, the corresponding virtual velocities v^α are zero. Therefore, the indifference conditions are

$$P_{ext}(\Pi, a, 0) = 0, \qquad P_{ext}(\Pi, Wx, 0) = 0, \qquad (16)$$

where W is the skew-symmetric tensor associated with the rotation vector ϖ, defined by the relation

$$Wa = \varpi \times a, \qquad (17)$$

for all vectors a. In a polar continuum, the state variables introduce further degrees of freedom for the deformation. Then a rigid rotation is a simultaneous rotation of the macroscopic deformation and of the state variable. In the case of tensorial variables,[9] the virtual velocities v^α are tensor fields, and the appropriate indifference conditions are

$$P_{ext}(\Pi, a, 0) = 0, \qquad P_{ext}(\Pi, Wx, W) = 0. \qquad (18)$$

[9]This case includes the *micromorphic continua* [6] and, in particular, the *micropolar continua*, also called *Cosserat continua*.

Thus, the polar and non-polar continua have the same translational indifference condition, but different rotational indifference conditions.

3. An example of a non-polar continuum is met in the theory of *gradient plasticity*. This theory is based on the Kröner-Lee decomposition

$$\nabla f = F^e F^p, \tag{19}$$

according to which the macroscopic deformation gradient ∇f is supposed to be the composition of a microscopic rearrangement F^p and of the local deformation F^e necessary to restore the macroscopic deformation ∇f. This decomposition defines a generalized continuum described by a single state variable, the tensor F^p. For it, the relation (14) has the form

$$S = \mathbb{T} n, \tag{20}$$

where the second-order tensor S is the contact action associated with F^p and the third-order tensor $\mathbb{T}$ is the corresponding internal action. Denoting by L^p the virtual velocity of F^p, the external and internal powers take the form

$$\begin{aligned} P_{ext}(\Pi, v, L^p) &= \int_\Pi (b \cdot v + B \cdot L^p)\, \mathrm{d}V + \int_{\partial\Pi} (s \cdot v + S \cdot L^p)\, \mathrm{d}A, \\ P_{int}(\Pi, v, L^p) &= \int_\Pi \big((b + \mathrm{div} T) \cdot v + T \cdot \nabla v \\ &\qquad + (B + \mathrm{div}\mathbb{T}) \cdot L^p + \mathbb{T} \cdot \nabla L^p\big)\, \mathrm{d}V. \end{aligned} \tag{21}$$

The indifference conditions (16) yield the same restrictions (9) of the classical continuum, and therefore the reduced form of the internal power is

$$P_{red}(\Pi, v, L^p) = \int_\Pi \big(T \cdot \nabla v^S + (B + \mathrm{div}\mathbb{T}) \cdot L^p + \mathbb{T} \cdot \nabla L^p\big)\, \mathrm{d}V. \tag{22}$$

In plasticity it assumed that the Cauchy stress T is a function of the elastic part F^e of the decomposition (19). From this decomposition follows the relation

$$\nabla v = L^e + L^p \tag{23}$$

between the corresponding virtual velocities. Thus, the reduced power takes the form

$$P_{red}(\Pi, v, L^p) = \int_\Pi \big(T \cdot D^e + T^p \cdot L^p + \mathbb{T} \cdot \nabla L^p\big)\, \mathrm{d}V, \tag{24}$$

where D^e is the symmetric part of L^e and T^p is the *plastic stress*

$$T^p = T + B + \mathrm{div}\mathbb{T}. \tag{25}$$

The last equation and the balance equation $(9)_1$ are the differential equations of the equilibrium problem of gradient plasticity.[10] The formulation of the problem is completed by a set of constitutive equations between the internal actions T, T^p, $\mathbb{T}$ and the corresponding generalized deformations, and by appropriate boundary conditions.

4. In a polar continuum, quite frequently the state variables are supposed to be vectorial, and in this case they are called the *directors*. The number of the directors depends on the nature of the continuum. For example, the *liquid crystals* have just one director, in *crystal plasticity* the number of the directors coincides with the number of the slip planes, and a *micromorphic continuuum* is characterized by a triple of linearly independent directors. Just as the macroscopic deformation of the body is described locally by the deformation gradient ∇f, the microscopic deformation of a micromorphic continuuum is described by a second-order tensor F^m, the *microscopic deformation gradient*. Thus, at each point of the continuum the microdeformation has the same geometric structure of the macrodeformation undergone by the whole body.[11]
In the microdeformation, the directors d^α are mapped into the vectors $F^m d^\alpha$. Denoting by

$$v^\alpha = L^m d^\alpha \tag{26}$$

the corresponding virtual velocity, substituting into (12), and setting

$$B = \beta^\alpha \otimes d^\alpha, \qquad S = \sigma^\alpha \otimes d^\alpha, \tag{27}$$

the external and internal powers (21) are re-obtained, with L^m in place of L^p. With the indifference conditions (18), the balance laws are

$$\mathrm{div}T + b = 0, \qquad T + T^m = (T + T^m)^T, \qquad T^m = B + \mathrm{div}\mathbb{T}. \tag{28}$$

That is, the symmetry of the Cauchy stress T required by the balance laws (9) of the non-polar continua is now replaced by the symmetry of the tensor $T + T^m$. As a consequence, in the integrand function of $(21)_2$ one has

$$\begin{aligned} T \cdot \nabla v + (B + \mathrm{div}\mathbb{T}) \cdot L^m &= T^S \cdot \nabla v^S + T^W \cdot \nabla v^W + T^m \cdot L^m \\ &= T^S \cdot \nabla v^S - T^{mW} \cdot \nabla v^W + T^m \cdot L^m = T^S \cdot \nabla v^S + T^m \cdot \mathcal{L}^m, \end{aligned} \tag{29}$$

[10]We emphasize that (25) is a consequence of the pseudobalance equation (13) and not a new balance equation. In the literature, it is named *balance of micromomentum, microforce balance, equilibrium equation for the macrostress tensor*, and is presented, at least tacitly, as a new axiom of mechanics.

[11]Ericksen and Truesdell [5], Mindlin [10] and Eringen [6].

where

$$\mathcal{L}^m = L^m - \nabla v^W \tag{30}$$

is the virtual velocity of the directors with respect to the body already deformed by the macroscopic deformation. The reduced power then takes the form

$$P_{red}(\Pi, v, L^m) = \int_\Pi \left(T^S \cdot \nabla v^S + T^m \cdot \mathcal{L}^m + \mathbb{T} \cdot \nabla L^m\right) \mathrm{d}V. \tag{31}$$

For a micromorphic continuum, the differential equations of the equilibrium problem are $(9)_1$ and $(28)_3$, and the constitutive equations are relations between the internal actions T^S, T^m, $\mathbb{T}$ and the generalized deformations ∇v^S, $\mathcal{L}^m$, ∇L^m. The Cauchy stress is not symmetric, and its skew-symmetric part T^W does not appear in the expression of the power. It plays the role of a reaction, determined by the relation $(28)_2$, $T^W = -T^{mW}$.

5. [12]In a micromorphic continuum, the deformation of the directors may be subject to geometrical constraints. For example, the *micropolar continua* are micromorphic continua for which the only deformation allowed to the orthonormal triple of directors is a rigid rotation, variable from point to point. Thus, if $R^m U^m$ is the polar decomposition of F^m, the constraint acting on a micropolar continuum is

$$F^m = R^m, \qquad U^m = I. \tag{32}$$

In this case the virtual velocity L^m reduces to its skew-symmetric part W^m, and in the external power we have

$$B \cdot L^m = B^W \cdot W^m = c \cdot \omega, \qquad S \cdot L^m = S^W \cdot W^m = m \cdot \omega, \tag{33}$$

with ω, $c/2$ and $m/2$ the vectors associated with W^m, B^W and S^W by the relation (17)

$$W^m a = \omega \times a, \qquad B^W a = \frac{1}{2} c \times a, \quad S^W a = \frac{1}{2} m \times a. \tag{34}$$

The external power then takes the form

$$\mathcal{P}_{ext}(\Pi, v, \omega) = \int_\Pi \left(b \cdot v + c \cdot \omega\right) \mathrm{d}V + \int_{\partial\Pi} \left(s \cdot v + m \cdot \omega\right) \mathrm{d}A, \tag{35}$$

with ω the vectorial measure of the virtual rotation of the directors, and c and m the *volume couple* and the *surface couple*. If s and m are Cauchy fluxes, they have the representations

[12]For reasons of brevity, from here on most of the statements are given without comments and proofs. More detailed treatments can be found in the paper [2] and in the forthcoming lecture notes [4]. For plate and beam theories, see [3].

$$s = Tn, \qquad m = Mn, \tag{36}$$

with T the Cauchy stress and M the *couple stress tensor*. With the aid of the divergence theorem one obtains the internal power

$$\mathcal{P}_{int}(\Pi, v, \omega) = \int_{\Pi} \big((b + \text{div}T) \cdot v + T \cdot \nabla v + (c + \text{div}M) \cdot \omega + M \cdot \nabla\omega\big)\, \text{d}V. \tag{37}$$

The indifference conditions now give

$$\text{div}T + b = 0, \qquad \text{div}M + c + 2t = 0, \tag{38}$$

with t the vector associated with the skew-symmetric part of T. Substitution into Eq. (37) yields the reduced form

$$\mathcal{P}_{red}(\Pi, v, \omega) = \int_{\Pi} \big(T^S \cdot \nabla v^S - 2\, t \cdot \varphi + M \cdot \nabla\omega\big)\, \text{d}V, \tag{39}$$

where

$$\varphi = \omega - \frac{1}{2}\text{curl}\, v \tag{40}$$

is the vector measure of the relative rotation $W^m - \nabla v^W$. The equilibrium problem now consists of the differential equations (38), of constitutive equations relating the internal actions T^S, t and M with the generalized deformations ∇v^S, φ and $\nabla\omega$, and of a set of boundary conditions.

The *constrained Cosserat continua* are obtained by imposing the supplementary constraint

$$\omega = \frac{1}{2}\text{curl}\, v, \tag{41}$$

which requires that the relative rotation φ be zero.[13] For such continua the indifference conditions still have the form (38), and the reduced power is

$$\mathcal{P}_{red}(\Pi, v) = \int_{\Pi} \big(T^S \cdot \nabla v^S + \frac{1}{2} M \cdot \nabla\text{curl}\, v\big)\, \text{d}V. \tag{42}$$

Here, T^S and M are the only active internal actions. The rotation ω formally disappears from the list of the geometric variables, though its effects are still present in the product $M \cdot \nabla\text{curl}\, v$.[14] As a consequence, t is not anymore an active internal action, and therefore it is not anymore determined by a constitutive equation. In

[13]This constraint corresponds to the *Cauchy-Born hypothesis*, according to which the directors follow the macroscopic deformation.

[14]The presence of a microstructure which does not appear explicitly in the expression of the power characterizes this continuum as a *continuum with latent microstructure* [1].

the equilibrium problem, t is eliminated from the differential equations (38) with the aid of the identity

$$\mathrm{div} T^W = -\mathrm{curl}\, t, \tag{43}$$

thanks to which the two equations merge in the single, higher-order equation

$$\mathrm{div} T^S + \frac{1}{2}\mathrm{curl}\,(\mathrm{div} M + c) + b = 0. \tag{44}$$

Of course, the boundary conditions must be re-formulated accordingly.

6. Other geometrical constraints lead to *dimensional reduction*, providing thereby the classical theories of plates and beams, viewed as two- and one-dimensional Cosserat continua. Assume that the body in its reference configuration has a cylindrical shape, and let $\{e, e^\alpha\}$ be an orthonormal triple of vectors, with e directed as the axis of the cylinder. The constraint

$$v(x) = v_3(x_1, x_2)\, e, \qquad \omega(x) = \omega_\alpha(x_1, x_2)\, e^\alpha, \qquad \alpha \in \{1,\ 2\}, \tag{45}$$

allows for a virtual velocity v parallel to e and for a virtual rotation ω about an axis orthogonal to e. It also requires that both v and ω be constant in the direction e. Under these constraints, the external power (35) reduces to

$$P_{ext}(\Gamma, v_3, \omega_\alpha) = \int_\Gamma (b_3 v_3 + c_\alpha \omega_\alpha)\, \mathrm{d}A + \int_{\partial\Gamma} (s_3 v_3 + m_\alpha \omega_\alpha)\, \mathrm{d}\ell, \tag{46}$$

where the volume element Π is replaced by its perpendicular projection Γ in the direction e, and $\mathrm{d}\ell$ is the line element on the boundary line $\partial\Gamma$.
In the relations (36), by effect of the constraints, the stress tensor T_{ij} degenerates into the vector of the *internal shearing forces* Q_α, and the couple-stress tensor M_{ij} degenerates into the 2×2 tensor of the *internal moments* $M_{\alpha\beta}$

$$s_3 = Q_\alpha n_\alpha, \qquad m_\alpha = M_{\alpha\beta} n_\beta. \tag{47}$$

Then the internal power becomes

$$\begin{aligned}\mathcal{P}_{int}(\Gamma, v_3, \omega_\alpha) = \int_\Gamma \big((q + Q_{\alpha,\alpha}) v_3 + Q_\alpha v_{3,\alpha} \\ + (c_\alpha + M_{\alpha\beta,\beta})\omega_\alpha + M_{\alpha\beta}\omega_{\alpha,\beta}\big)\, \mathrm{d}A,\end{aligned} \tag{48}$$

with the component b_3 of the body force now viewed as a transverse load q. The indifference conditions (18) provide the balance equations

$$Q_{\alpha,\alpha} + q = 0, \qquad M_{\alpha\beta,\beta} + c_\alpha + e_{\alpha\beta} Q_\beta = 0, \tag{49}$$