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Module Theory, Extending Modules and Generalizations

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With the cooperation of Patrick F. Smith, University of Glasgow

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Dedicated to our families, teachers, and students

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Preface

A module is *extending* (or CS, C_1) if every complement submodule is a direct summand or, equivalently, every submodule is essential in a direct summand. The terms extending and *lifting* (dual to extending) were firstly used by Harada and Oshiro ([Har82], [Oshi84]).

The importance of extending modules and rings in Ring and Module Theory, and more generally in Algebra, became obvious in the 1990's, but not exclusively, through the impact of the publication of monographs of S.H. Mohammed and B.J. Müller [MM90] and of N.V. Dung, D.V. Huynh, P.F. Smith, and R. Wisbauer [DHSW94]. Since that time there has been a continuing interest in such rings and modules and their various generalizations which arose not only directly from the study of CS concept, but also from work concerning the dual notion to extending, namely, lifting. Many results obtained for extending modules and rings were transfered to lifting modules and rings [CLVW06]. At a first glance, the extending and generalized extending concepts appear to be too similar to expect many differences in their application to the structure theory of rings and modules. However, we have shown many "surprising" differences, as we indicate throughout this monograph. To this end, we classified generalized extending modules into two groups. The first group consists of generalized extending modules such that either for every submodule or a kind of special submodules, there exists a direct summand with the property that the direct sum of the aforementioned submodule with the direct summand is essentially contained in the module (inner type generalization). The second group consists of generalizations of extending modules which are based on a technical condition like a homomorphism into a direct summand or an equivalence relation in the lattice of submodules, etc. (outer type generalization). We also apply our former equivalence relation idea to the dual extending case. We believe that this last application will foster research on dual extending modules and related classes of modules, which in turn, will greatly broaden the scope of the theory.

The purpose of this monograph is to give an up-to-date presentation of known as well as new results on generalized extending modules and some complementary results on extending matrix rings, and also to provide standard background material, but with somewhat selective topics, on Ring and Module Theory. A number of open research problems are listed at the end of the book to generate interest in research on generalized extending and related duals. Each section includes exercises of varying degrees of difficulty for graduate students. To keep the book to a reasonable length, some results have been included as exercises in various sections and appropriate references have been included in remarks at the end of each chapters. We are very thankful to Patrick F. Smith and Gary F. Birkenmeier for encouragement and constructive comments and suggestions for improvements.

We are thankful to our colleagues who helped by proof-reading various parts of the book and provided technical advise, among them Yeliz Kara, Uğur Yücel, Ramazan Yaşar, Talha Arıkan, Hacer İlhan, Hatice Zeybek, Esma Dirican, Nazife Erkurşun Özcan, Aslı Pekcan, Sema Yayla, Mesut Şahin, and Seçil Tokgöz.

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Introduction

The first chapter covers introductory material to parts of ring and module theory. We deal with standard, yet somewhat selective background topics. We hope that the chapter will prove useful for a general audience. We add numerous exercises that provide supplementary results. The chapter does not contain all of the aspects (e.g., category-theoretical concepts) on rings and modules, being too narrowly focused for this, but it could serve as a basis for independent study in rings and modules following a basic introduction to modern algebra.

Chapter 2 deals with certain type of modules, including those possessing the C_2 and C_3 properties, the summand intersection property (SIP), and the summand sum property (SSP) conditions. The first two sections present basic properties of complement submodules and relative injective modules which are used in the rest of the book. Section 3 introduces lifting submodules. Here, given two modules A and X over a ring R, we say that a submodule N of A is a *lifting submodule for* X *in* A if for any $\varphi \in \mathrm{Hom}_R(N, X)$ there exists $\theta \in \mathrm{Hom}_R(A, X)$ such that the restriction of θ on N is equal to φ. So, we build up the class

$$\mathrm{Lift}_X(A) = \{N : N \leq A \text{ and } N \text{ is a lifting submodule for } X \text{ in } A\}.$$

For instance, any direct summand of A belongs to $\mathrm{Lift}_X(A)$. This new class of submodules is examined in detail. Section 4 introduces the notion of *ejectivity* and the class $\mathrm{Elift}_X(A)$, which are interesting in their own right. We say that X is A-*ejective* if, for each $K \leq A$ and each homomorphism $\varphi : K \to X$ there exist a homomorphism $\theta : A \to X$ and an essential submodule E of K such that $\theta(x) = \varphi(x)$ for all $x \in E$. In this case, K is said to be an *elifting submodule* for X in A. So we set $\mathrm{Elift}_X(A) = \{K : K \leq A$ and K is an elifting submodule for X in $A\}$. This class is also examined in detail. The last section looks at module properties which eventually give direct summands. Especially, C_2 and C_3 properties are characterized in terms of lifting submodules.

Chapter 3 collects several results on extending modules and rings and continuous, quasi-continuous modules which cannot be found in other monographs. The direct sum of two extending modules need not be extending, so Section 1 examines when a direct sum of an arbitrary number of extending modules is again an extending module. The section ends with a look at the rational hull and members of some distinguished class of submodules of an extending module which enjoy the extending property.

Section 2 mainly concerns extending modules over commutative domains, but also somewhat generalized versions of the main result to the noncommutative setting. To this end we give a fundamental decomposition theorem on extending modules which states that a torsion-free extending module over a commutative domain is a finite direct sum of injective modules and uniform modules. We apply this fact to obtain more information about extending modules over Dedekind domains or principal ideal domains. Moreover, we show that any nonsingular reduced extending module over a commutative ring with only a finite number of minimal prime ideals has a finite uniform dimension; and any nonsingular extending module over a semiprime Goldie ring is a finite direct sum of injective modules and uniform modules.

Section 3 considers generalized triangular matrix (also called formal triangular matrix) CS-rings and the split null extension (also called trivial extension) CS-rings. Chapter 3 ends with a section on the notions of continuous and quasi-continuous modules; in particular, it is shown that continuous and quasi-continuous modules can be characterized in terms of lifting homomorphisms from certain submodules to the module itself.

The opening section of Chapter 4 introduces class of a generalized extending modules which are called weak CS-modules. A module A is *weak* CS if every semisimple submodule of A is essential in a direct summand. Any direct sum of a semisimple submodule and an injective module is a weak CS-module, but such a module is not, in general, a CS-module, even over a Dedekind domain. Moreover, over a Dedekind domain, every module with finite uniform dimension is a weak CS-module. In fact, weak CS-modules share some of the properties of CS-modules. For example, if a weak CS-module satisfies the ascending chain condition on essential submodules, then it is a direct sum of a semisimple module and a Noetherian module.

Section 2 introduces C_{11}-modules. A module A *satisfies* C_{11} (or is a C_{11}-module) if every (complement) submodule has a complement which is a direct summand of A. Any direct sum of modules with C_{11} satisfies C_{11}. Furthermore, a module A satisfies C_{11} if and only if $A = Z_2(A) \oplus K$ for some (nonsingular) submodule K of A and $Z_2(A)$ and K both satisfy C_{11}. This result shows that the study of modules with property C_{11} reduces to the case of Goldie torsion modules and nonsingular modules. In contrast to extending modules, direct summands of a C_{11}-module need not be C_{11}, in general. The following example is quite interesting in its own right. Let $S = \mathbb{R}[x, y, z]$ and $R = S/Ss$, where $s = x^2 + y^2 + z^2 - 1$. Then the R-module $A = R \oplus R \oplus R$ satisfies C_{11} but contains an indecomposable direct summand K with uniform dimension 2 and K does not satisfy C_{11}. For instance, K corresponds to regular sections of the tangent bundle of the real 2-sphere S^2.

However, it is shown in Section 3 that the C_{11} condition with a conditional direct summand property ensure that C_{11} is inherited by direct summands. This section also provides a decomposition into uniform submodules for C_{11}-modules that satisfy chain conditions on left annihilators or essential submodules.

In Section 4, we continue the study of structural properties of C_{11}-modules and connections between the C_{11} condition and various other generalizations of CS condition. A module A is FI-*extending* if every fully invariant submodule of A is essential in a direct summand of A [BCFG01], [BMR02], [BPR02]. Since the C_{11} property lies strictly between the CS and FI-extending properties, it is natural to seek conditions which ensure that a C_{11}-module is CS, or that a FI-extending module is C_{11}. Such conditions are given in this section.

The focus in Section 5 is on extensions of C_{11}-rings and modules. We study the transference of the C_{11} condition from a given ring or module to various ring or module extensions. In particular, we show that if R is a right C_{11}-*ring* (i.e., R_R is a C_{11}-module), then the ring of column finite matrices of size Γ over R, the ring of m-by-m upper triangular matrices over R, and any right essential overring T of R are all right C_{11}-rings. We also provide necessary and sufficient conditions under which the generalized triangular matrix ring is a right C_{11}-ring. For a module A, we prove that all essential extensions of A satisfying C_{11} are essential extensions of C_{11}-modules constructed from A and certain subsets of idempotent elements of the ring of endomorphisms of the injective hull of A. Finally, we prove that if A is a C_{11}-module, then so is its rational hull.

Section 6 introduces a framework which encompasses most of the generalizations of the CS property and allows us to target specific sets of submodules of a module for application of the CS property. Let $\varnothing \neq \underline{C} \subseteq L(A)$, where $L(A)$ denotes the set of submodules of A. We say A is $\underline{C}$-*extending* if for each $X \in \underline{C}$ there exists a direct summand D of A such that X is essential in D. This concept was introduced by Oshiro in [Oshi83] with slightly different terminology and notation. Oshiro assumes that $\underline{C}$ is closed under isomorphisms and essential extensions. In [DS98], the authors introduce type 1 χ-extending and type 2 χ-extending modules, where χ is a class of modules containing the zero submodule and closed under isomorphisms. In contrast to [Oshi83] and [DS98], we do not implicitly assume that $\underline{C}$ is closed under isomorphisms or essential extensions. Our investigation focuses on the behavior of $\underline{C}$-extending modules with respect to direct sums and direct summands. We obtain various well-known results about extending modules and generalizations as corollaries of our results.

In Section 7 we continue our investigation on $\underline{C}$-extending modules with respect to essential extensions. This section explains how to construct essential extensions of a module A which are $\underline{C}$-extending by using a set of representatives of an equivalence relation $\gamma_{\underline{C}}$ on $\{e = e^2 \in \underline{E}_{E(A)}\}$, where $\underline{E}_{E(A)}$ denotes the endomorphisms of the injective hull of A; and characterize when the rational hull of A is $\underline{C}$-extending in terms of such a set of representatives. The section ends with several well-known types of $\underline{C}$-extending conditions transfer from the module to its rational hull.

Section 8 concerns $\underline{\mathrm{P}}\,\underline{\mathrm{I}}$-extending rings. A ring R is *right* $\underline{\mathrm{P}}\,\underline{\mathrm{I}}$-*extending* if every projection invariant right ideal of R is essential in a direct summand of R. This section provides the transfer of the $\underline{\mathrm{P}}\,\underline{\mathrm{I}}$-extending condition from ring R to its various ring extensions. More specifically, one derives when a generalized upper triangular matrix is right $\underline{\mathrm{P}}\,\underline{\mathrm{I}}$-extending. The last section of Chapter 4 looks at weak version of the C_{11} property. A module A is *weak* C_{11} if every semisimple submodule has a complement which is a direct summand of A. Weak C_{11}-modules are a proper generalization of both C_{11}-modules and weak CS-modules.

The first section of Chapter 5 concerns the class of C_{12}-modules. This class properly contains the class of C_{11}-modules. A module A *satisfies* C_{12} if, for every submodule N of A, there exist a direct summand K of A and a monomorphism $\alpha : N \to K$ such that $\alpha(N)$ is an essential submodule of K. This section presents structural properties of C_{12}-modules and relationships with the other generalized extending conditions. The section ends with results on a weak version of C_{12}-modules.

Section 2 looks at modules A such that every homomorphism from a complement submodule of A to A can be lifted to A. Although such modules share some of the properties of CS-modules, it is shown that they form a substantially bigger class.

Section 3 focuses on the class of CLS-modules, which is another proper generalization of the class of extending modules. A module A is CLS if every z-closed submodule of A is a direct summand of A. Here, if N is a submodule of A, we say that N is *z-closed* if $Z(A/N) = 0$, i.e., A/N is nonsingular.

In Section 4, we defined a module A to be G-*extending* (or *Goldie extending*) if for each submodule X of A there exists a direct summand D of A such that $X \cap D$ is essential in both X and D. We develop several characterizations of the G-extending condition and we locate the G-extending condition with respect to several generalizations of the extending condition. Moreover, we obtain various conditions for a direct sum of G-extending modules to be G-extending or for a direct summand of a G-extending module to be G-extending. Our results enable us to obtain a characterization of the G-extending Abelian groups.

Uniform modules are often considered to be fundamental objects in the structure theory of various classes of modules, e.g., the class of modules with finite uniform dimension, or the class of extending modules. Hence, for a given class, solving the following problems is important.

(1) Determine conditions which force a module from a given class to be a direct sum of uniform submodules.

(2) Determine when a direct sum of uniform modules is in a given class.

Section 5 begins with results providing answers to the first problem and continues with solutions to the second problem, where the given class of modules is the G-extending class. Section 5 finishes with results on G-extending essential extensions of a module.

Section 6 looks at the G-extending condition under various ring extensions. Amongst other results, it shows that if R_R is G-extending and S is a right essential overring, then S_R and S_S are G-extending.

Section 7 is mainly devoted to the characterization of G-extending modules over Dedekind domains and principal ideal domains. Furthermore, the finite direct sums and direct summands of G-extending modules over such domains are examined.

In Section 8, we obtain a characterization of the right G-extending generalized triangular matrix rings. This result and its corollaries improve and generalize the existing results on right extending generalized triangular matrix rings. Connections to operator theory and a characterization of the class of right extending right SI-rings are also established.

Chapter 6 is devoted to dual Goldie and ec-complement versions of the extending property. It is well known that lifting modules are dual to extending modules and there are many works on this subject in the literature.

Section 1 introduces Goldie*-supplemented modules, which are dual to Goldie extending modules. We investigate this new class of modules and the class of H-supplemented modules. These classes are located among various well-known classes of modules related to the class of lifting modules.

Section 2 deals with the classes of Goldie-rad-supplemented and rad-H-supplemented modules which are based on the β^{**} equivalence relation. A module A is *Goldie*-rad-*supplemented* if for every submodule N of A, there exists a rad-supplement submodule S in A such that $N\beta^{**}S$. A module A is rad-H-*supplemented* if for every submodule N of A, there exists a direct summand D of A such that $N\beta^{**}D$. Section 3 concerns ECS and EC_{11}-modules as weaker forms of extending and C_{11}-modules, respectively. A module A is an ECS-*module* if every ec-complement submodule of A is a direct summand. A module A is an EC_{11}-*module* if every ec-submodule of A has a complement which is a direct summand. Here, given a submodule N of A, we say that N is an ec-*complement* if N is a complement in A that contains essentially a cyclic submodule.

Chapter 7 formulates a number of open problems and questions. We think that our problems are legitimate and will provide a basis for further research which will in turn greatly broaden the scope of the theory. Here we should mention recent developments on the latticial counterparts of conditions C_i $(i = 1, 2, 3)$ and C_{1j} $(j = 1, 2)$ for modules and applications to Grothendieck categories, module categories equipped with a torsion theory [AIT16].

The book ends with an Appendix. Since we do not treat in detail the rings of quotients in the previous chapters, it is better to construct these kind of rings for graduate students in mathematics. We believe that this appendix will allow to build a bridge between our theory and notions treated in the monograph [BPR13].

A Partial List of Symbols

$\mathbb{N}$	The set of positive integers
$\mathbb{Z}$	The ring of integers
$\mathbb{Z}_n$ or $\mathbb{Z}/\mathbb{Z}n$ $(n > 1)$	The ring of integers modulo n
$\mathbb{Z}(p^\infty)$	The Prüfer p-group
$\prod \mathbb{Z}$	The Specker group
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\mathrm{M}_n(R)$	The $n \times n$ matrix ring over R
$T_n(R)$	The $n \times n$ upper triangular matrix ring over R
$\mathbf{I}\ (R)$	The set of all idempotents of R
$\mathrm{Cen}(R)$	The center of R
$\langle X \rangle_R$	The subring of R generated by $X \subseteq R$
$Z({}_RA)$ or $Z(A)$	The singular submodule of ${}_RA$
$Z_2({}_RA)$ or $Z_2(A)$	The second singular submodule of ${}_RA$
$\mathrm{soc}(A)$ or $\mathrm{soc}({}_RA)$	The socle of ${}_RA$
$\mathrm{rad}(A)$ or $\mathrm{rad}({}_RA)$	The Jacobson radical of ${}_RA$
A^n or $A^{(n)}$	The direct sum of n copies of A
$J(R)$	The Jacobson radical of R
u-dim	Uniform dimension
$\mathrm{S}_l(R)$	The set of all left semicentral idempotents of R
$\mathrm{S}_r(R)$	The set of all right semicentral idempotents of R
$E({}_RA)$ or $E(A)$	The injective hull of ${}_RA$
$\widetilde{E}({}_RA)$ or $\widetilde{E}(A)$	The rational hull of ${}_RA$
$\mathcal{Q}(R)$	The maximal right ring of quotients of R
R-Mod	The category of left R-modules
acc (dcc)	The ascending (descending) chain condition
$N \trianglelefteq A$	N is a fully invariant submodule of A
$N \leq_e A$	N is an essential (large) submodule of A
$N \leq_c A$	N is a complement submodule of A

$N \leq_d A$	N is a direct summand of A
$N \leq_s A$	N is a small submodule of A
$N \overset{cs}{\hookrightarrow} L$	$N \subseteq L$ is cosmall in A
Δ	$\{f \in \mathrm{End}A \mid \ker f \leq_e A\}$
$CFM_\Gamma(R)$	The $\Gamma \times \Gamma$ column finite matrix ring over R
$CRFM_\Gamma(R)$	The $\Gamma \times \Gamma$ column and row finite matrix ring over R
$\|I\|$	The cardinal of I

Chapter 1

Introducing Modules

In this chapter we introduce modules, explore some of their general properties and then go on to look at particular types of modules. Why are modules so important? There are two basic reasons. In the first place, the concept of a module is a very general one and examples abound, including some very familiar ones. Secondly, the theory of rings is intimately connected with that of modules and it is inconceivable to try to study the former without some reference to the latter. Without further ado let us begin with the definition of a module.

1.1 Modules

Let A be an Abelian group with binary operation $+$. Let End$A$ denote the collection of endomorphisms θ of A, i.e., $\theta : A \to A$ satisfies

$$\theta(a+b) = \theta(a) + \theta(b) \quad (a, b \in A).$$

Define addition and multiplication in End$A$ by

$$(\theta + \phi)(a) = \theta(a) + \phi(a)$$
$$(\theta\phi)(a) = \theta(\phi(a))$$

for all $\theta, \phi \in \text{End}A$, $a \in A$. With these definitions it can be checked that End$A$ is a ring, called the *ring of endomorphisms of* A, with zero element the *zero mapping* $z_A : A \to A$ given by $z_A(a) = 0$ $(a \in A)$ and identity element the *identity mapping* $i_A : A \to A$ given by $i_A(a) = a$ $(a \in A)$. For example, suppose that A is an infinite cyclic group generated by an element a. For each $n \in \mathbb{Z}$ define $\theta_n \in \text{End}A$ by $\theta_n(ka) = nka$ $(k \in \mathbb{Z})$. It is easy to verify that

(i) $\text{End}A = \{\theta_n : n \in \mathbb{Z}\}$, and
(ii) $\theta_n + \theta_m = \theta_{n+m}$ and $\theta_n\theta_m = \theta_{nm}$ $(n, m \in \mathbb{Z})$.

Thus the mapping $f : \mathbb{Z} \to \text{End}A$ defined by $f(n) = \theta_n$ $(n \in \mathbb{Z})$ is a ring isomorphism, i.e., $\text{End}A \cong \mathbb{Z}$.

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Now let R be a ring. *Throughout, without further notice, R will always denote a ring with identity 1.* An Abelian group A is called a *left R-module* if there exists a homomorphism $f : R \to \mathrm{End}A$. In particular, f satisfies $f(1) = i_A$. A homomorphism $f : R \to \mathrm{End}A$ is sometimes called a *representation of R as a ring of endomorphisms of A*. In particular, A is a left ($\mathrm{End}A$)-module.

Let A be a left R-module and $f : R \to \mathrm{End}A$ the associated homomorphism. Given any elements $r \in R$ and $a \in A$ we define the "product" ra as

$$ra = f(r)(a).$$

Note the following elementary properties:

$$\begin{array}{rl} \text{(i)} & ra \in A, \\ \text{(ii)} & r(a_1 + a_2) = ra_1 + ra_2, \\ \text{(iii)} & (r_1 + r_2)a = r_1 a + r_2 a, \\ \text{(iv)} & (r_1 r_2)a = r_1(r_2 a), \text{ and} \\ \text{(v)} & 1a = a \end{array} \tag{1.1}$$

for all $a, a_1, a_2 \in A$ and $r, r_1, r_2 \in R$. Conversely, suppose that A is an Abelian group and R a ring for which the product ra can be defined for all $r \in R$, $a \in A$, in such a way that (1.1) holds. For each r in R define $\theta_r \in \mathrm{End}A$ by $\theta_r(a) = ra$ ($a \in A$). Define $g : R \to \mathrm{End}A$ by $g(r) = \theta_r$ ($r \in R$). Then g is clearly a homomorphism and thus A is a left R-module. To summarise, an Abelian group A is a left R-module if and only if the product ra can be defined for all $r \in R$ and $a \in A$ that (1.1) holds.

The product in the last paragraph is very reminiscent of something so familiar, namely, the product of a vector by a scalar in a vector space, and for this reason we shall refer to it as *multiplication by a scalar*. In fact it is now clear that if R is a field, then left R-modules are precisely vector spaces over R. On the other hand, every Abelian group A is a $\mathbb{Z}$-module with multiplication by scalars given by

$$na = \begin{cases} a + \cdots + a \quad \text{(n times)}, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n = 0, \\ (-a) + \cdots + (-a) \quad \text{(n times)}, & \text{if } -n \in \mathbb{N}, \end{cases}$$

for all $a \in A$. Vector spaces and Abelian groups are good examples of modules to keep in mind, and it will be helpful to consider, each time a new definition is introduced in the sequel, what it means for these two classes of examples. Indeed a great deal of what follows is motivated by them.

Before proceeding any further, let us make several remarks. Given a left R-module A, we know that there is a homomorphism $f : R \to \mathrm{End}A$ but of course in general there may be many such homomorphisms. For example, if h is an endomorphism of R then $fh : R \to \mathrm{End}A$ is a homomorphism. For each homomorphism from R to $\mathrm{End}A$ there is a corresponding multiplication by scalars defined as above, so that, in general A will have many such products. However, the important thing to remember is that A has one such product and we tend to forget that it has others. Secondly, if S is a subring of R, then by considering the appropriate restriction of f we see at once that A is a left S-module. More generally, if T is a ring and $g : T \to R$ a homomorphism, then A is a left T-module. Thus the same Abelian group A will be a left module for many different rings.

If A is a left R-module, then there is a homomorphism $f : R \to \mathrm{End}A$ and as always in this kind of situations we are interested in how much information about R is lost under this homomorphism. The normal way to measure this loss is by considering the kernel $\ker f$ of f, which is a very important ideal as far as the module A is concerned. The ideal $\ker f$ is called the *annihilator of A in R*, and is denoted by $\mathrm{ann}_R A$. Why is it given this name? Recall that

$$\mathrm{ann}_R A = \{r \in R : f(r) = z_A\} = \{r \in R : ra = 0 \ (a \in A)\},$$

so that each element r in $\mathrm{ann}_R A$ "annihilates" each element a in A in the sense that $ra = 0$. Now let I be an ideal of R such that $I \subseteq \mathrm{ann}_R A$. Then $f : R \to \mathrm{End}A$ induces a homomorphism $f_1 : R/I \to \mathrm{End}A$ so that A is a non-trivial left (R/I)-module. Note further that in terms of the multiplication by scalars defined above we have

$$(r + I)a = ra \quad (r \in R, \ a \in A).$$

Borrowing from the language of representation theory, a left R-module A is called *faithful* if $\mathrm{ann}_R A = 0$, i.e., with the above notation $f : R \to \mathrm{End}A$ is a monomorphism, and so a faithful representation of R.

Right modules can be defined in analogy with left modules. An Abelian group A is called a *right R-module* if there exists an anti-homomorphism $g : R \to \mathrm{End}A$. This just means that g preserves addition and

$$g(r_1 r_2) = g(r_2) g(r_1)$$

for all $r_1, r_2 \in R$. In practice this allows one to define the product ar for all $a \in A$, $r \in R$ so that the analogous properties to (1.1) are satisfied. If S and T are rings, then by a *left S-, right T-bimodule* is meant an Abelian group A such that A is a left S-module, a right T-module, and

$$s(at) = (sa)t$$

for all $a \in A$, $s \in S$ and $t \in T$. We often denote the fact that an Abelian group is a left R-module, a right R-module, or a left S-, right T-bimodule simply by

$${}_R A, \ A_R, \ {}_S A_T,$$

respectively. Let S and T be subrings of a ring R and I an ideal of R. Then I is a left S-, right T-bimodule, i.e., ${}_S I_T$, and in particular ${}_R R_R$. Moreover, left ideals of

R are left R-modules and right ideals of R are right R-modules, where in all of these cases the product by a scalar on either side is given by the multiplication of R.

Let $n \in \mathbb{N}$ and $M_n(R)$ denote the ring consisting of all $n \times n$ matrices with entries in R, addition and multiplication of matrices being given by the usual rules. If R is a field, then the ring $M_n(R)$ is familiar. In the sequel matrix rings over rings other than fields will be considered in various places. At this point merely note that ${}_RM_n(R)_R$. For, if $m \in M_n(R)$ is a matrix with (i, j)th entry m_{ij} for $1 \leq i, j \leq n$ and $r \in R$, define rm and mr to be the matrices with (i, j)th entry rm_{ij} and $m_{ij}r$ for $1 \leq i, j \leq n$, respectively, and check that ${}_RM_n(R)_R$.

Now we introduce a method of producing modules from given ones. Let Λ be any non-empty index set (i.e., Λ is just a set of labels). Let R be a ring and A_λ $(\lambda \in \Lambda)$ a collection of left R-modules. *The direct product* $A = \prod_{\lambda\in\Lambda} A_\lambda$ is defined as follows. The set A consists of all lists $\{a_\lambda\}$, where $a_\lambda \in A_\lambda$ $(\lambda \in \Lambda)$, and addition and multiplication by a scalar are defined by

$$\{a_\lambda\} + \{b_\lambda\} = \{a_\lambda + b_\lambda\},$$

and

$$r\{a_\lambda\} = \{ra_\lambda\},$$

for all $a_\lambda, b_\lambda \in A_\lambda$ $(\lambda \in \Lambda)$, $r \in R$. It is not hard to check that A is also a left R-module. For each $\lambda \in \Lambda$ the module A_λ is called a *direct factor of* A. In the special case when there exists ${}_RC$ such that $A_\lambda = C$ $(\lambda \in \Lambda)$, then A is written as C^Λ. Perhaps it is worth spelling out what this definition means when Λ is a finite set. Suppose that k is a positive integer and A_i is a left R-module for each $1 \leq i \leq k$ (so we are taking $\Lambda = \{1, \ldots, k\}$). Then

$$A = \{(a_1, \ldots, a_k) \ : \ a_i \in A_i \ (1 \leq i \leq k)\},$$

with equality, addition, and multiplication being given by the rules:

$$\begin{aligned} (a_1, \ldots, a_k) &= (b_1, \ldots, b_k) \text{ if and only if } a_i = b_i \ (1 \leq i \leq k), \\ (a_1, \ldots, a_k) + (b_1, \ldots, b_k) &= (a_1 + b_1, \ldots, a_k + b_k), \text{ and} \\ r(a_1, \ldots, a_k) &= (ra_1, \ldots, ra_k), \end{aligned}$$

for all $a_i, b_i \in A_i$ $(1 \leq i \leq k)$ and $r \in R$. In case $A_i = C$ $(1 \leq i \leq k)$, A is written as C^k.

Let R_λ $(\lambda \in \Lambda)$ be a non-empty collection of rings and $R = \prod_{\lambda\in\Lambda} R_\lambda$ their direct product as $\mathbb{Z}$-modules. The $\mathbb{Z}$-module R can be given the structure of a ring by defining multiplication as

$$\{r_\lambda\} \cdot \{s_\lambda\} = \{r_\lambda s_\lambda\}$$

for all $r_\lambda, s_\lambda \in R_\lambda$ $(\lambda \in \Lambda)$. The ring R is called the *direct product of the rings* R_λ $(\lambda \in \Lambda)$. Note that R has zero element $\{0_\lambda\}$ and identity element $\{1_\lambda\}$, where 0_λ is the zero and 1_λ the identity of R_λ $(\lambda \in \Lambda)$.

Notation. The symbol 0 will have a number of different roles in the sequel. For example, 0 will denote the zero element of a ring or of a module. In addition, the subset $\{0\}$ of a ring or module will also be denoted by 0. No confusion should arise since the context should make it clear which meaning to attach to 0.

Exercises

1.1. Given ${}_RA$, prove that

(i) $0a = 0$ and $r0 = 0$ for all $a \in A$ and $r \in R$,
(ii) $(-r)a = -(ra) = r(-a)$ for all $r \in R$, $a \in A$, and
(iii) $r(na) = n(ra)$ for all $r \in R$, $n \in \mathbb{Z}$, and $a \in A$.

1.2. Let R be a commutative ring (i.e., $rs = sr$ for all $r, s \in R$) and A an Abelian group. Prove that the following statements are equivalent:

(i) A is a left R-module,
(ii) A is a right R-module,
(iii) A is a left R-, right R-bimodule.

(In this case A is called simply an *R-module.*)

1.3. Prove that if ${}_RA$, then ${}_RA_{\mathbb{Z}}$.

1.4. Let I be an ideal of a ring R and A a left (R/I)-module. Prove that

(i) A is a left R-module such that $I \subseteq \operatorname{ann}_R A$, and
(ii) $\operatorname{ann}_{R/I} A = (\operatorname{ann}_R A)/I$.
Given ${}_RB$, deduce that B is a faithful left (R/I)-module if and only if $I = \operatorname{ann}_R B$.

1.5. Give an example of a ring R and a subring S of R such that S is not a left R-module. Give an example of a ring T for which there exists a left T-module A which is not a right T-module.

1.6. Let R and S be rings and ${}_RA_S$. Let T denote the set of "matrices"of the form

$$\begin{bmatrix} r & a \\ 0 & s \end{bmatrix},$$

where $r \in R$, $a \in A$, and $s \in S$. Define

$$\begin{bmatrix} r & a \\ 0 & s \end{bmatrix} = \begin{bmatrix} r' & a' \\ 0 & s' \end{bmatrix} \text{ if } r = r',\ a = a',\ \text{and } s = s',$$

$$\begin{bmatrix} r & a \\ 0 & s \end{bmatrix} + \begin{bmatrix} r' & a' \\ 0 & s' \end{bmatrix} = \begin{bmatrix} r + r' & a + a' \\ 0 & s + s' \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} r & a \\ 0 & s \end{bmatrix} \begin{bmatrix} r' & a' \\ 0 & s' \end{bmatrix} = \begin{bmatrix} rr' & ra' + as' \\ 0 & ss' \end{bmatrix},$$

for all $r, r' \in R$, $s, s' \in S$, and $a, a' \in A$. Prove that with respect to these definitions T is a ring. The ring T is usually denoted by

$$\begin{bmatrix} R & A \\ 0 & S \end{bmatrix}.$$

1.7. Let R be a ring and ${}_RA_R$. Prove that the set of "matrices"

$$\begin{bmatrix} r & a \\ 0 & r \end{bmatrix},$$

where $r \in R$ and $a \in A$, forms a ring which is usually denoted by

$$\begin{bmatrix} R & & A \\ & \diagdown & \\ 0 & & R \end{bmatrix}.$$

Further, prove that if R is a commutative ring and A is an R-module, then the ring

$$\begin{bmatrix} R & & A \\ & \diagdown & \\ 0 & & R \end{bmatrix}$$

is also commutative.

1.2 Homomorphisms

Recall that in the theory of vector spaces the key idea is that of a linear mapping. The corresponding concept in module theory is called a homomorphism, and this will be defined next. Let ${}_RA$ and ${}_RB$. A mapping $\theta : A \to B$ is called an R-*homomorphism* (or simply a *homomorphism* when there is no ambiguity about the ring being considered) provided

$$\begin{aligned} \theta(a_1 + a_2) &= \theta(a_1) + \theta(a_2), \text{ and} \\ \theta(ra) &= r\theta(a) \end{aligned}$$

for all $a, a_1, a_2 \in A$, $r \in R$. The first of these properties shows that θ is an Abelian group homomorphism in particular, and thus we know at once from the properties of such homomorphisms that

$$\begin{aligned} &\text{(i)} \quad \theta(0_A) = 0_B, \\ &\text{(ii)} \quad \theta(-a) = -\theta(a), \text{ and} \\ &\text{(iii)} \quad \theta(na) = n\theta(a), \end{aligned} \tag{1.2}$$

for all $a \in A$, $n \in \mathbb{Z}$, where $0_A, 0_B$ denote the zero elements of A and B, respectively. In particular, (1.2) shows that $\mathbb{Z}$-homomorphisms are precisely Abelian group homomorphisms. Moreover, (1.2) also gives that if a_1 and a_2 are elements of A such that $\theta(a_1) = \theta(a_2)$, then $\theta(a_1 - a_2) = 0_B$. The *kernel* $\ker\theta$ and the *image* $\operatorname{im}\theta$ of θ are defined as follows:

$$\ker\theta = \{a \in A : \theta(a) = 0_B\},$$

and

$$\operatorname{im}\theta = \{\theta(a) : a \in A\}.$$

The homomorphism θ is called a *monomorphism* provided it is one-to-one, i.e., $\theta(a_1) \neq \theta(a_2)$ for all $a_1 \neq a_2$ in A, equivalently $\ker\theta = \{0_A\}$. On the other hand, θ is called an *epimorphism* provided it is onto, i.e., for each element b in B there exists an element a in A such that $b = \theta(a)$, equivalently $B = \operatorname{im}\theta$. If there is an epimorphism $\theta : A \to B$, then B is sometimes called a *homomorphic image of* A. The mapping θ is called an *isomorphism* if it is both a monomorphism and an epimorphism, and in this case A and B are called *isomorphic*, B is called an *isomorphic copy of* A, and we write $A \cong B$. If $\theta : A \to B$ is an isomorphism, then θ is a bijection in particular, and hence has an inverse mapping $\theta^{-1} : B \to A$, which is also an R-homomorphism, as a moment's thought will show. If $\theta : A \to B$ is a monomorphism, then θ induces an isomorphism $A \cong \operatorname{im}\theta$, and for this reason θ will sometimes be called an *embedding* and we say that A *is embedded in* B.

Given Abelian groups A, B, the collection of all $\mathbb{Z}$-homomorphisms $\theta : A \to B$ will be denoted by $\operatorname{Hom}(A, B)$. If $\theta, \phi \in \operatorname{Hom}(A, B)$ then their sum $\theta + \phi$ is the mapping from A to B defined by

$$(\theta + \phi)(a) = \theta(a) + \phi(a) \quad (a \in A).$$

It is easy to check that with this definition of addition $\operatorname{Hom}(A, B)$ is an Abelian group. The zero of the group $\operatorname{Hom}(A, B)$ is the mapping which maps each element of A to the zero of B, and will also be denoted by 0. Now suppose that ${}_RA$ and ${}_RB$. Then it can be checked that the collection $\operatorname{Hom}_R(A, B)$ of all R-homomorphisms $\theta : A \to B$ is a subgroup of $\operatorname{Hom}(A, B)$. Note that in this notation $\operatorname{Hom}_{\mathbb{Z}}(A, B) = \operatorname{Hom}(A, B)$. Sometimes $\operatorname{Hom}_R(A, B)$, or even $\operatorname{Hom}(A, B)$, is a left or right R-module, as the following result shows (see also Exercises 1.9 and 1.24).

Proposition 1.1. *Let A be a right R-module and B an Abelian group. Then $\operatorname{Hom}(A, B)$ is a left R-module.*

Proof. Given $r \in R$ and $\theta \in \operatorname{Hom}(A, B)$, define the mapping $r.\theta : A \to B$ by

$$(r.\theta)(a) = \theta(ar) \ (a \in A).$$

Then $r.\theta \in \operatorname{Hom}(A, B)$. If $r_1, r_2 \in R$ and $\theta \in \operatorname{Hom}(A, B)$ then, for any $a \in A$,

$$\begin{aligned}[(r_1 + r_2).\theta](a) &= \theta(a(r_1 + r_2)) = \theta(ar_1 + ar_2) = \theta(ar_1) + \theta(ar_2) \\ &= (r_1.\theta)(a) + (r_2.\theta)(a) = (r_1.\theta + r_2.\theta)(a).\end{aligned}$$

It follows that $(r_1 + r_2).\theta = r_1.\theta + r_2.\theta$ and hence (1.1)(iii) is satisfied. The remaining properties in (1.1) can be verified in a similar way for this product, and it follows that $\mathrm{Hom}(A, B)$ is a left R-module. $\square$

For any ${}_RA$, a homomorphism $\theta : A \to A$ is called an *R-endomorphism* (or simply *endomorphism*) *of* A. Denote the collection of all R-endomorphisms of A by $\mathrm{End}_R(A)$. Note that

$$\mathrm{End}_{\mathbb{Z}}(A) = \mathrm{End}A \ \text{ and } \ \mathrm{End}_R(A) = \mathrm{Hom}_R(A, A).$$

Moreover, $\mathrm{End}_R(A)$ is a subring of $\mathrm{End}A$. An endomorphism θ of A is called an *automorphism* provided it is a bijection, i.e., automorphisms are endomorphisms which are also isomorphisms.

Given left R-modules A, B, C, D, let $\theta \in \mathrm{Hom}_R(A, B)$, $\phi \in \mathrm{Hom}_R(B, C)$ and $\psi \in \mathrm{Hom}_R(C, D)$. Define $\phi\theta : A \to C$ by

$$\phi\theta(a) = \phi(\theta(a)) \quad (a \in A).$$

It can easily be checked that $\phi\theta \in \mathrm{Hom}_R(A, C)$ and that $\psi(\phi\theta) = (\psi\phi)\theta$, so that the mapping $\psi(\phi\theta)$ is simply written $\psi\phi\theta$. There is a $\mathbb{Z}$-homomorphism $\theta^\star : \mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(C, B)$, defined by

$$\theta^\star(\alpha) = \theta\alpha \quad (\alpha \in \mathrm{Hom}_R(C, A));$$

likewise, there is a $\mathbb{Z}$-homomorphism $\theta_\star : \mathrm{Hom}_R(B, C) \to \mathrm{Hom}_R(A, C)$ defined by

$$\theta_\star(\beta) = \beta\theta \quad (\beta \in \mathrm{Hom}_R(B, C)).$$

Let A_λ $(\lambda \in \Lambda)$ be a non-empty collection of left R-modules and A their direct product $\prod_{\lambda\in\Lambda} A_\lambda$. For each $\mu \in \Lambda$ there exists an R-epimorphism $\pi_\mu : A \to A_\mu$ (called *the canonical* or *natural projection*) defined by

$$\pi_\mu(\{a_\lambda\}) = a_\mu$$

for all $a_\lambda \in A_\lambda$ $(\lambda \in \Lambda)$. The mapping π_μ is sometimes called the *μ-projection.* On the other hand, for each $\nu \in \Lambda$ and $b \in A_\nu$ let $\iota_\lambda(b)$ denote the element $\{b_\lambda\}$ of A defined by $b_\lambda = b$ if $\lambda = \nu$ and $b_\lambda = 0$ otherwise. Then ι_ν is an R-monomorphism (called *the canonical* or *natural injection*) for each $\nu \in \Lambda$. Note that

(i) $\pi_\mu\iota_\nu = 0$ for all $\mu \neq \nu$ in Λ, and
(ii) $\pi_\mu\iota_\mu = i_{A_\mu}$ $(\mu \in \Lambda)$.

In this context, in the sequel any mappings π_μ and ι_μ introduced without comment should be understood to be these R-homomorphisms.

Given left R-modules A, B, C, to say that

$$A \xrightarrow{\theta} B \xrightarrow{\phi} C$$

is an *exact sequence* means that θ and ϕ are R-homomorphisms such that

$$\operatorname{im}\theta = \ker\phi.$$

Note, in particular, that if $A = 0$, then ϕ is a monomorphism and if $C = 0$ then θ is an epimorphism. More generally, an exact sequence consists of a collection of left R-modules A_n $(n \in \mathbb{Z})$ and R-homomorphisms $\theta_n : A_n \to A_{n+1}$ such that $\operatorname{im}\theta_n = \ker\theta_{n+1}$ $(n \in \mathbb{Z})$ and this is denoted by

$$\cdots \longrightarrow A_{-2} \xrightarrow{\theta_{-2}} A_{-1} \xrightarrow{\theta_{-1}} A_0 \xrightarrow{\theta_0} A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots \text{ exact.} \tag{1.3}$$

If $A_m = 0$ for all $m \geq n+1$, then (1.3) is written

$$\cdots \longrightarrow A_{-2} \xrightarrow{\theta_{-2}} A_{-1} \xrightarrow{\theta_{-1}} A_0 \xrightarrow{\theta_0} A_1 \xrightarrow{\theta_1} \cdots \longrightarrow A_n \xrightarrow{\theta_n} 0 \qquad \text{exact.}$$

An exact sequence

$$0 \longrightarrow A \xrightarrow{\theta} B \xrightarrow{\phi} C \longrightarrow 0 \tag{1.4}$$

is called a *short exact sequence.* A short exact sequence (1.4) it is said to *split* or to be *split exact* provided there exists $\psi \in \operatorname{Hom}_R(C, B)$ such that $\phi\psi = i_C$.

A set-up of the type

$$\begin{tikzcd}
A \arrow[r, "\theta"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
C \arrow[r, "\phi"] & D
\end{tikzcd}$$

where A, B, C, D are all left R-modules and $\theta, \phi, \alpha, \beta$ R-homomorphisms is called a *diagram of R-modules.* The diagram is said to *commute* or to be *commutative* provided $\beta\theta = \phi\alpha$. Using such diagrams as building blocks, bigger diagrams of R-modules can be produced, e.g.,

$$\begin{tikzcd}
 & B' \arrow[r, "\phi'"] \arrow[d, "\beta'"'] & C' \arrow[r] \arrow[d, "\gamma'"'] & 0 & \text{exact} \\
A \arrow[r, "\theta"] \arrow[d, "\alpha"'] & B \arrow[r, "\phi"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 & \text{exact} \\
A'' \arrow[r, "\theta''"] \arrow[d] & B'' \arrow[r, "\phi''"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 & \\
0 & 0 & 0 & & \\
\text{exact} & \text{exact} & \text{exact} & &
\end{tikzcd} \tag{1.5}$$

where A, B, C, B', C', A'', B'', C'' are left R-modules and α, β, γ, β', γ', θ, ϕ, ϕ', θ'', ϕ'' R-homomorphisms. To say that such a bigger diagram commutes is to mean