NUMERICAL METHODS SERIES

Finite Element Method

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0.1.1 General remarks

Modern technological advances challenge engineers to carry out increasingly complex and costly projects, which are subject to severe reliability and safety constraints. These projects cover domains such as space travel, aeronautics and nuclear applications, where reliability and safety are of crucial importance. Other projects are connected with the protection of the environment, for example control of thermal, acoustic or chemical pollution, water course management, management of groundwater and weather forecasting. For a proper understanding, analysts need mathematical models that allow them to simulate the behavior of complex physical systems. These models are then used during the design phase of the projects.

Engineering sciences (mechanics of solids and fluids, thermodynamics, etc.) are used to describe the behavior of physical systems in the form of partial differential equations. Today, the finite element method has become one of the most frequently used methods for solving such equations. This method requires intensive use of a computer, and can be applied to solve almost all problems encountered in practice: steady or transient problems in linear and nonlinear regions for one-, two- and three-dimensional domains. Moreover, it can be successfully adapted for use in the heterogeneous environments and domains of complex forms often encountered in practice by engineers.

The finite element method consists of using a simple approximation of unknown variables to transform partial differential equations into algebraic equations. It draws on the following three disciplines:

- engineering sciences to describe physical laws (partial differential equations);
- numerical methods for the elaboration and solution of algebraic equations;
- computing tools to carry out the necessary calculations efficiently using a computer.

0.1.2 HISTORICAL EVOLUTION OF THE METHOD

Structural mechanics allows us to analyze frames and trusses. The behavior of each truss or beam element is represented by an elementary stiffness matrix constructed using knowledge of the strength of materials. Using these matrices, we are able to construct a system of algebraic equations verifying the conditions of displacement continuity and balance of forces at the nodes. The solution of the system of equations corresponding to applied loads leads to the displacements of all nodes in the structure. The emergence of computers and the requirements of the aeronautical industry led to rapid developments in the field of structural mechanics in the 1950s. The concept of finite elements was introduced by Turner, Clough, Martin and Topp [TUR 56] in 1956: they represented an elastic two-dimensional domain by an assembly of triangular panels across which displacements are presumed to vary in a linear manner. The behavior of each panel is represented by an elementary stiffness matrix. Structural mechanics tools are then employed to obtain nodal displacements under different applied loads and boundary conditions.

We should also highlight the work carried out by Argyris and Kelsey [ARG 60], who employed the notion of energy in structural analysis. The basic ideas involved in the finite element method, however, appeared earlier, in an article published by Courant in 1943 [COU 43].

From 1960 onward, finite element method developed rapidly in a number of directions:

— The method was reformulated, based on energetic and variational considerations, in the general form of weighted residuals or weak formulations [ZIE 65; GRE 69; FIN 75; ARA 68].
— A number of authors created high-precision elements [FEL 66], curved elements and isoparametric elements [ERG 68; IRO 68].
— The finite element method was recognized as a general method of solution for partial differential equations. It thus came into use in solving nonlinear and transient problems of structures, as well as in other fields, such as soil and rock mechanics (geomechanics), fluid mechanics and thermodynamics [PRO 01–PRO 13].
— A mathematical basis for finite element method was established using concepts of functional analysis [PRO 14–PRO 15].

Since 1967, many books have been published on the finite element method [MON 01–MON 29]. We wish to highlight, in particular, the three editions of the book by Zienkiewicz [MON 02], which are available throughout the world. We may refer to Pironneau, Géradin, Imbert, Batoz-Dhatt and

Dhatt-Touzot, among others, for books available in French. An exhaustive list of reference works in the domain of finite elements may be easily obtained using an Internet search engine.

0.1.3 STATE OF THE ART

The finite element method (FEM) is nowadays widely used in industrial applications, including aeronautical, aerospace, automobile, naval and nuclear construction fields, and in applications of fluid mechanics, including tidal studies, sedimentary transportation, the study of thermal or chemical pollution phenomena and fluid-structure interactions. A number of general-purpose computer codes are available for industrial users of the finite element method, such as IDEAS, SAMCEF, NASTRAN, ABAQUS, FIDAP, MARC, ANSYS, ADINA, LSDYNA, ASTER and CASTEM.

In order for the finite element method to be effective in industrial applications, computer codes must be used to assist in the preparation of data and the interpretation of results. These pre- and post-processing tools are usually integrated into more general computer-aided design (CAD) software packages, such as IDEAS, CASTOR or CATIA.

0.2 Object and organization of the book

0.2.1 TEACHING THE FINITE ELEMENT METHOD

The finite element method is now widely taught at both undergraduate and postgraduate level. The teaching of the finite element method requires a multidisciplinary approach involving different aspects:

— understanding of the physical problem and intuitive knowledge of the nature of the solution being sought;
— representation of the physical phenomenon in the form of partial differential equations and weak "variational" or "integral" formulations;
— discretization techniques to produce a discrete or algebraic model;
— matricial organization of data;
— numerical methods for integration of functions with several variables solution of linear and nonlinear algebraic equations;
— computer programming tools for handling massive data files and for creating user friendly graphic interface.

It is hard to conceive a balanced formation in all these diverse disciplines. Moreover, suitable teaching software must be used that includes certain characteristics of general industrial codes. Finally the practical aspects of implementing the finite element method in computer codes must not be overlooked.

0.2.2 OBJECTIVES OF THE BOOK

This book attempts to simplify the teaching of the finite element method by smoothing out certain difficulties. It has been developed by engineers to solve engineering problems. Thus, the presentation of the book is primarily addressed to this audience. The mathematical knowledge required is limited to the domains of differential and matrix calculus.

This book is intended for readers who wish to understand the finite element method and apply it to solve engineering problems using a computer. Moreover, it should be of use to students and researchers in applied sciences and as well as to practicing engineers who wish to go beyond the basic level of knowledge implied by the use of "black box" programs.

0.2.3 ORGANIZATION OF THE BOOK

This volume is organized into six chapters, each providing a relatively independent presentation of various concepts of the finite element method as well as the corresponding numerical and programming techniques. Examples are provided for illustrative purposes, accompanied by simple programs written using Matlab©.

Chapter 1

This chapter presents the approximation of continuous functions over subdomains in terms of nodal values and introduces the concepts of interpolation functions, reference elements, geometrical transformations and approximation error.

Chapter 2

This chapter presents the interpolation functions for classical elements in one, two and three dimensions.

Chapter 3

This chapter gives a description of the weighted residual method that allows us to obtain weak formulations (known as integral formulations) associated with partial differential equations (known as strong formulations).

Chapter 4

This chapter presents the matrix formulation of the finite element method, which consists of discretizing the integral formulation from Chapter 3, using the approximations of unknown functions from Chapters 1 and 2. We introduce notions of elementary matrices and vectors, assembly and global matrices and vectors.

Chapter 5

This chapter describes the numerical methods needed to construct and solve the systems of algebraic equations formed in Chapter 4: numerical methods of integration, methods for the solution of linear and nonlinear algebraic systems domain, methods for integrating first- and second-order propagation systems in time domain and methods for calculating the eigenvalues and vectors.

Chapter 6

This chapter provides a brief overview of the finite difference and finite volume methods as well as the computing techniques that are characteristic of the finite element method using a simple program written in Matlab©.

Figure 0.1 shows the logical sequence of these chapters. Note that Chapters 1, 3 and 4 are devoted to the fundamental concepts underlying the finite element method, while Chapters 2 and 5 are intended as reference chapters, and finally Chapter 6 presents a simple program for illustrative purposes.

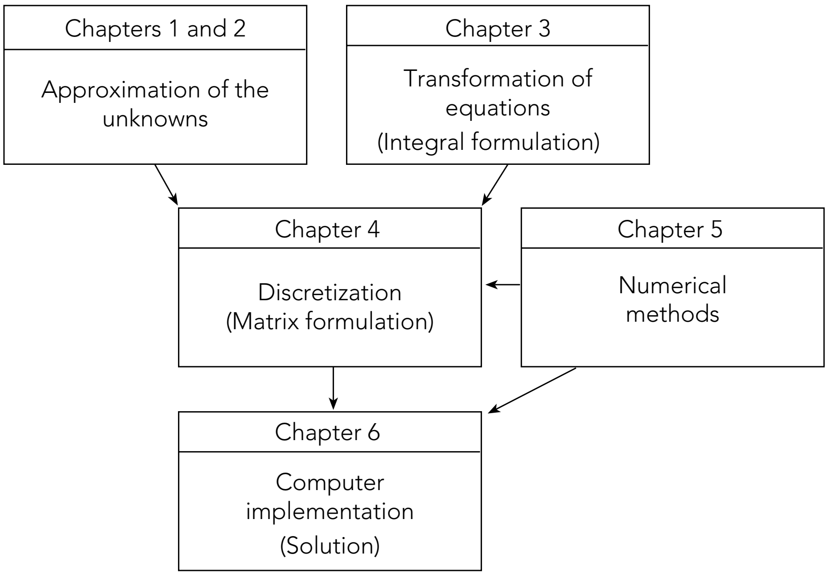


Figure 0.1. Logical flow of the chapters

0.3 Numerical modeling approach

0.3.1 General aspects

This section gives a brief introduction to the different concepts to be covered in the following chapters.

Numerical modeling is used to simulate the behavior of physical systems using computers. This involves:

— description, in engineering terms, of the physical system in question and the problem under study (**physical model**);

— translation of the engineering problem into a mathematical form (**mathematical model**);

— construction of a **numerical model** (or algebraic model) that can be solved using a computer, and which uses discretization methods such as the finite element method;

— development of a computer code to simulate the behavior of the physical system (**computer model**).

A variety of errors may be introduced into different models or during the passage from one model to another. Three main types of errors are encountered:

- Errors in the choice of the mathematical model, representing the difference between the exact solution to the mathematical model and the real behavior of the physical system.
- Discretization errors, representing the difference between the exact solution to the mathematical model and the exact solution to the numerical model.
- Computer-based errors due to the limited precision of the calculations carried out by the computer and, potentially, programming errors.

Modeling specialists should be able to master these errors so that the solution provided by the software is reasonably close to the real behavior of the physical system under study (*a priori* unknown). In practice, it is often necessary to carry out the steps described above more than once until a satisfactory solution is produced.

0.3.2 PHYSICAL MODEL

The description of a physical system includes:

- a representation of its geometry;
- the selection of unknown variables for which we wish to evaluate spatio-temporal variations;
- the physical laws governing the system's behavior;
- values for the physical properties that are assumed to be known;
- applied forces, boundary conditions and, where applicable, initial conditions for transient problems.

EXAMPLE 0.1. ***Thermal equilibrium of a truss (physical model)***

- *Geometry: rectilinear truss of length L and rectangular section A oriented in the direction x (Figure 0.2).*

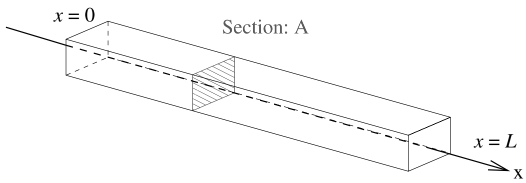


Figure 0.2. Rectilinear truss of constant section

— *Unknown variables:*

- *the temperature (in degrees Celsius or Kelvin),* $T(x)$*;*
- *the thermal flux component as a function of x (W/m²),* $q(x)$*;*

The problem represents steady state flow; variables are thus independent of time.

— *Physical laws:*

- *conservation of thermal flow as a function of x;*
- *Fourier's law of thermal behavior relating the temperature gradient to the flow.*

— *Physical properties: thermal conductivity, k (W/°C-m).*

— *Thermal load from the Joule effect (electrical current): f (W/m³).*

— *Boundary conditions:*

- *Imposed temperature:* $T(0) = T_0$.
- *Imposed flux:* $q(L) = q_L$, W/m^2.

Objectives

Obtain $T(x)$ and $q(x)$ that verify the physical laws and boundary conditions. One possible application would be to estimate heat loss through the wall of a dwelling for which we wish to improve the insulation, i.e. limit the thermal flow.

0.3.3 MATHEMATICAL MODEL

This model is obtained by expressing the laws of conservation and constitutive laws in the form of partial differential equations. As this problem is to be solved using the finite element method, it will also be necessary to give an integral formulation (or weak formulation).

EXAMPLE 0.2. Application to the thermal equilibrium of a truss (mathematical model)

— *Law of conservation of thermal flow written as a function of x for Example 0.1:*

$$\frac{d(qA)}{dx} - f_0 A = 0, \quad f_0 > 0\text{: } \textit{source of volumetric heat}$$

— *Fourier's constitutive law:* $q(x) = -k\dfrac{dT(x)}{dx}$.

— *Boundary conditions:*

$$T(x=0) = T_0, \qquad \qquad : \textit{Dirichlet}$$

$$q(x=L) = -k\left.\frac{dT}{dx}\right|_{x=L} = q_L > 0 \quad \textit{(loss)} : \textit{Neumann}$$

These relations may also be written in the form (for a constant section A):

$$\frac{d}{dx}\left(k\frac{dT(x)}{dx}\right) + f_0 = 0,$$

The exact solution to the mathematical model in the present case (where k is constant) is:

$$T(x) = T_0 - \frac{q_L}{k}x + \frac{f_0}{k}\left(Lx - \frac{x^2}{2}\right).$$

The integral form obtained using the weighted residual method is written as:

$$W = \int_0^L \frac{d\delta T}{dx} k \frac{dT}{dx} dx + W_{Neu} - \int_0^L \delta T . f_0 dx = 0,$$

with $W_{Neu} = \delta T(L).q_L$ *and* $\delta T(x=0) = 0.$

where $\delta T(x)$ *is any test function.*

The solution to the problem is the function T(x), such that W = 0 for all test functions.

0.3.4 NUMERICAL MODEL

The numerical model associated with the mathematical model is obtained using a discretization method, such as:

— the finite difference method, or

— the finite element method.

In this case, we will illustrate the numerical model using the finite difference method.

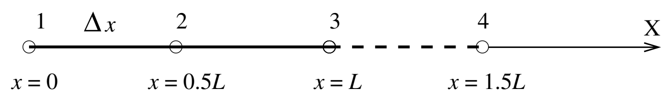
EXAMPLE 0.3. Application to the thermal equilibrium of a truss (numerical model based on the finite difference method)

For cases where "k" is constant, the differential equation governing the thermal equilibrium of the truss is written as:

$$k\frac{d^2T(x)}{dx^2} + f_0 = 0,$$

$$T(x=0) = 0 \quad and \quad k\frac{dT}{dx}\bigg|_L = -q_L.$$

Let us take a set of equidistant discretization points (known as nodes) across the domain. This may be illustrated using three equidistant nodes.



This set of nodes is known as a mesh. A fourth, "fictional" node has been added in order to give the same level of spatial precision for the boundary condition at $x = L$.

The equilibrium relationship is applied at each node, "i":

$$k\frac{d^2T(x)}{dx^2}\bigg|_i + f_0 = 0, \quad i = 1, 2, 3.$$

Let us associate an unknown variable with each node in the mesh, so that:

$$T(x_1 = 0) = T_1, \quad T(x_2 = \Delta x) = T_2,$$
$$T(x_3 = L) = T_3, \; T(x_4 = L + \Delta x) = T_4,$$

where $\Delta x = L/2$ *is the distance between two successive nodes.*

The finite difference method consists of rewriting the derivatives in discrete form, so that:

$$\left.\frac{d^2T}{dx^2}\right|_x = \left.\frac{d^2T}{dx^2}\right|_i \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2},$$
$$\left.\frac{dT}{dx}\right|_x = \left.\frac{dT}{dx}\right|_i \approx \frac{T_{i+1} - T_{i-1}}{2\Delta x}.$$

We thus obtain the discrete form of the thermal equilibrium equation at node "i":

$$k\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + f_0 = 0, \;\; i = 2,\ldots,3.$$

Its application to nodes 2 and 3, associated with the boundary condition on node 1, translates as:

$$T_1 = T_0,$$
$$T_1 - 2T_2 + T_3 + \frac{\Delta x^2 f_0}{k} = 0,$$
$$T_2 - 2T_3 + T_4 + \frac{\Delta x^2 f_0}{k} = 0.$$

The boundary condition for $x = L$ *gives*

$$\frac{T_4 - T_2}{2\Delta x} = \frac{-q_L}{k} \;\; \Rightarrow \;\; T_4 = T_2 - \frac{2\Delta x . q_L}{k}.$$

Organizing these relations in a matrix form leads to

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} T_0 \\ f_0 \Delta x^2 / k \\ 0.5 f_0 \Delta x^2 / k - q_L \Delta x / k \end{Bmatrix},$$

which gives

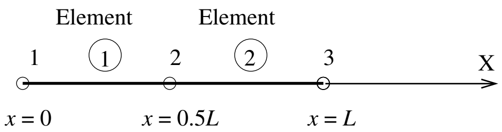
$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} T_0 + 1.5 f_0 \Delta x^2 / k - q_L \Delta x / k \\ T_0 + 2 f_0 \Delta x^2 / k - 2q_L \Delta x / k \end{Bmatrix}.$$

Remark

In this case (where f_0 is constant), we observe that the solution to the numerical model coincides with that of the mathematical model at the nodes.

EXAMPLE 0.4. ***Approach based on the finite element method***

The finite element method consists of constructing a discrete representation of the integral form W of Example 0.2. To do this, we first select a set of two elements as illustrated below.



The integral form is written as:

$$W = \sum_{e=1}^{2} W^e + W_{Neu} = 0,$$

where $W^e = \int_{x_i}^{x_{i+1}} \frac{d\delta T(x)}{dx} k \frac{dT(x)}{dx} dx - \int_{x_i}^{x_{i+1}} \delta T(x).f_0 dx$ *and* $W_{Neu} = \delta T(L).q_L$.

For each element, we choose a linear approximation of the solution function T(x) and the test function $\delta T(x)$ *J* with $\delta T(x=0)=0$.

For element 1: $L^e = x_2 - x_1 = \frac{L}{2}$.

$$T(x) = \left(\frac{x_2 - x}{L^e}\right) T_1 + \left(\frac{x - x_1}{L^e}\right) T_2, \quad x_1 = 0, \quad x_2 = L/2,$$

$$\delta T(x) = \left(\frac{x_2 - x}{L^e}\right) \delta T_1 + \left(\frac{x - x_1}{L^e}\right) \delta T_2.$$

where $T_1 = T_0$ *and* $\delta T_1 = 0$.

The elementary integral form associated with element 1 is then written as:

$$W^1 = <\delta T_1 \, \delta T_2> \left(\frac{k}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} - f_0 \frac{L^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right).$$

For element 2: $L^e = x_3 - x_2 = \frac{L}{2}$.

The approach taken is equivalent to that used for element 1. The elementary integral form is written as:

$$W^2 = <\delta T_2\, \delta T_3> \left(\frac{k}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - f_0 \frac{L^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right).$$

The flow term is expressed as $W_{Neu} = \delta T_3 . q_L$.

After assembly, the integral form W is written as:

$$W = <\delta T_1\, \delta T_2\, \delta T_3> \left(\frac{2k}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} - f_0 \frac{L}{4} \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ q_L \end{Bmatrix} \right) = 0.$$

Introducing the boundary condition at node 1, we obtain the following system:

$$\frac{2k}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = f_0 \frac{L}{4} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} - \begin{Bmatrix} -2kT_0 / L \\ q_L \end{Bmatrix},$$

leading to

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} T_0 + 1.5 f_0 \Delta x^2 / k - q_L \Delta x / k \\ T_0 + 2 f_0 \Delta x^2 / k - 2 q_L \Delta x / k \end{Bmatrix}.$$

Remark

In this particular case, the finite difference and finite element methods provide exactly identical solutions at the nodes.

0.3.5 COMPUTER MODEL

The two programs given in Figures 0.3 and 0.4 are written in the Matlab© programming language and cover Examples 0.3 and 0.4, respectively.

```
%----- initialization
clear all

%----- geometry
L=1;                          % length (m)
nnt=20;                       % number of nodes
dx=L/(nnt-1):                 % discretization dx

%----- properties
kd=2;                         % thermal conductivity
f0=50;                        % heat production per unit of length

%----- boundary conditions
T0=30;                        % Dirichlet at node 1
qL=10;                        % Neumann at node nnt

%----- construction of system of equations
vkg=zeros(nnt,nnt); % initialization of the vkg global matrix
vfg=zeros(nnt,1); % initialization of the vfg global vector
%----- node loop
if(nnt>2)
   for i=2:nnt-1
       vfg(i)=f0*dx^2/kd;
      vkg(i,[i-1, i, i+1])=[-1, 2, -1];
   end
end
%----- Dirichlet boundary condition (x=0)
vkg(1,1)=1; vfg(1)=T0;

%----- Neumann boundary condition (x=L)
vkg(nnt,[nnt-1 nnt])=[-1 1]; vfg(nnt)=0.5*f0*dx^2/kd-qL*dx/kd;
%----- solution
vsol=vkg\vfg;
%----- display numerical solution
plot([0:nnt-1]*dx,vsol)
```

Figure 0.3. 1D program using the finite difference method (Example 0.3)

```
%----- initialization
clear all
%----- geometry
L=1;                    % length (m)
nnt=20;                 % number of nodes
Le=L/(nnt-1):           % discretization dx=Le
%----- properties
kd=2;                   % thermal conductivity
f0=50;                  % heat production per unit of length
%----- boundary conditions
T0=30;                  % Dirichlet at node 1
qL=10;                  % Neumann at node nnt
%----- construction of system of equations
vkg=zeros(nnt,nnt); % initialization of vkg
vfg=zeros(nnt,1); % initialization of vfg
c=kd/Le;
%----- elementary matrix and vector
vke=[c, -c; -c c];
vfe=f0*Le/2*[1; 1]
%----- element loop
for ie=1:nelt
    vfg([ie ie+1])= vfg([ie ie+1])+vfe;
    vkg([ie ie+1], [ie ie+1])= vkg([ie ie+1], [ie ie+1])+vke;
end
%----- Dirichlet boundary condition (x=0)
vkg(1,:)=zeros(1,nnt); vkg(1, 1)=1; vfg(1)=T0
%----- Neumann boundary condition (x=L)
vkg(nnt)= vfg(nnt –qL;
%----- solution
vsol=vkg\vfg;
%----- display numerical solution and exact solution
plot([0:nnt-1]*Le,vsol)
hold on
x=0:L/100:L;
solexact=-0.5*f0/kd*x.^2+(f0*L-qL)/kd*x;
plot(x,solexact)
```

Figure 0.4. 1D program using the finite element method (Example 0.4)

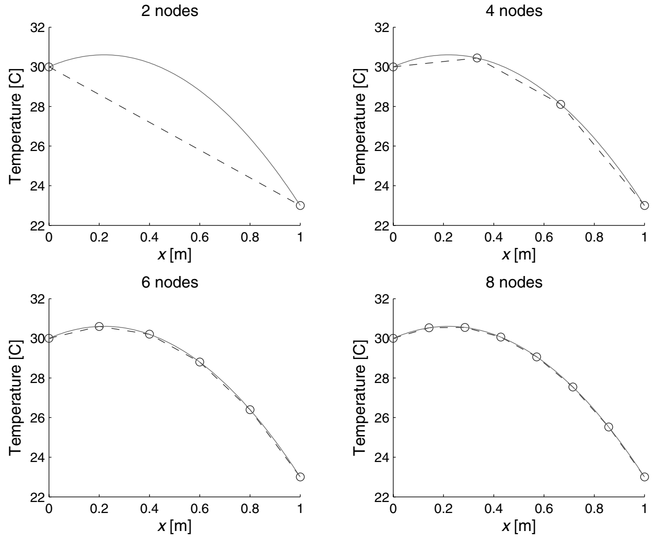


Figure 0.5. Exact solutions and solutions obtained using the finite element method

Figure 0.5 shows the results of the program based on the finite element method for different numbers of nodes. Each illustration shows the exact solution to the problem (continuous curve) and the numerical solution (dotted line).

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