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Directions in Number Theory

Proceedings of the 2014 WIN3 Workshop

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Preface

This volume is a compilation of research and survey papers in number theory, written by members of the *Women in Numbers* (WIN) network, principally by the collaborative research groups formed at *Women in Numbers 3*, a conference at the Banff International Research Station in Banff, Alberta, on April 21–25, 2014.

The WIN conference series began in 2008, with the aim of strengthening the research careers of female number theorists. The series introduced a novel research-mentorship model: women at all career stages, from graduate students to senior members of the community, joined forces to work in focused research groups on cutting-edge projects designed and led by experienced researchers. This model had tremendous success, branching out not only to *WINE (Women in Numbers Europe)* but also to *Algebraic Combinatorixx*, *WIT (Women in Topology)*, and others. The Association for Women in Mathematics (AWM), funded by the National Science Foundation, is now supporting this research-mentorship model under the umbrella of the *Research Collaboration Conferences for Women* initiative.

The goals for *Women In Numbers 3* were to establish ambitious new collaborations between women in number theory, to train junior participants about topics of current importance, and to continue to build a vibrant community of women in number theory. The majority of the week was devoted to research activities. Before the conference, the participants were organized into nine project groups by research interest and asked to learn background for their project topics. This led to more productive on-site research conversations and the groups were able to share preliminary results on the last day. The workshop also included a lecture series about arithmetic of curves, including elliptic curves, modular curves, and Shimura curves.

Forty-two women attended the WIN3 workshop, which was organized by the last three editors of this volume. This included 15 senior and mid-level faculty, 15 junior faculty and postdocs, and 12 graduate students. This volume is the fourth proceedings to come out of the WIN conference series. It is also the first in the series published by Springer for AWM.

The editors invited WIN3 research groups and members of the larger WIN3 community to submit articles in 2014. After a thorough referee process by external experts, we accepted 10 papers for the volume. One interesting attribute of the collection is the interplay between deep theory and intricate computation. The papers span a wide range of research areas: arithmetic geometry, analytic number theory, algebraic number theory, and applications to coding and cryptography. In this preface, we point out a few connections between the papers.

A major theme of the volume is the study of rational points on varieties via cohomological methods. Three papers on this theme are about rational points over number fields. The paper *Insufficiency of the Brauer-Manin obstruction for rational points on Enriques surfaces* (Balestrieri et al.) is about the failure of the Hasse principle for surfaces. In the paper *Shadow lines in the arithmetic of elliptic curves* (Balakrishnan et al.), the authors use information about analytic ranks and Tate-Shafarevich groups to develop an algorithm for computing anticyclotomic p-adic heights and shadow lines cast by rational points on elliptic curves over imaginary quadratic fields. In the paper *Galois action on the homology of Fermat curves* (Davis et al.), the authors use topology and the étale fundamental group to study obstructions for points on Fermat curves defined over cyclotomic fields.

The paper *Zeta functions of a class of Artin-Schreier curves with many automorphisms over finite fields* (Bouw et al.) is a bridge between several of the disparate topics. It fits in the vein of studying rational points via cohomological methods, because the ℓ-adic cohomology provides information about points on curves defined over finite fields. It connects to the topic of applications to coding theory and cryptography, because the class of Artin-Schreier curves produces large families of supersingular curves useful for error-correcting codes. Similarly, the paper *Hypergeometric series, truncated hypergeometric series, and Gaussian hypergeometric functions* (Deines et al.) draws together several topics. The hypergeometric varieties are higher-dimensional analogues of Legendre curves and the authors obtain information about the number of points on these varieties defined over finite fields. This paper also connects to the more analytic papers in the volume.

There are two other papers with an analytic and geometric focus. The paper *A generalization of S. Zhang's local Gross-Zagier formula for* GL_2 (Maurischat) is about Hecke operators and contains a fundamental lemma for some relative trace formulae. The paper *p-adic q-expansion principles on unitary Shimura varieties* (Caraiani et al.) has results about vanishing theorems for p-adic automorphic forms on unitary groups of arbitrary signature.

The final three papers are about applications of algebraic number theory. The paper *Kneser-Hecke-operators for codes over finite chain rings* (Feaver et al.) is about theta series for lattices for codes over finite fields and an analogue for Hecke operators in this context. In *Ring-LWE cryptography for the number theorist* (Elias et al.), the authors give a survey about attacks on the ring and polynomial learning with errors problems and discuss connections with open problems about algebraic number fields. Finally, the volume ends with a survey about arithmetic statistics in algebraic number theory, *Asymptotics for number fields and class groups* (Wood). This survey is an extended version of Wood's lecture notes for the Arizona Winter School in 2014, on the topic of counting number fields and the distribution of class groups.

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It was a pleasure to work with BIRS to organize the WIN3 conference and with Springer to prepare this volume. We would like to thank the following sponsoring organizations for their generous financial support of the workshop: Banff International Research Station, Clay Math Institute, Microsoft Research, Pacific Institute for the Mathematical Sciences, and the Number Theory Foundation. We would also like to thank the many referees, whose intelligence and effort helped the authors improve the papers for this volume.

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Insufficiency of the Brauer–Manin Obstruction for Rational Points on Enriques Surfaces

Francesca Balestrieri, Jennifer Berg, Michelle Manes, Jennifer Park, and Bianca Viray

Abstract In Várilly-Alvarado and Viray (Adv. Math. 226(6):4884–4901, 2011), the authors constructed an Enriques surface X over $\mathbb{Q}$ with an étale-Brauer obstruction to the Hasse principle and no *algebraic* Brauer–Manin obstruction. In this paper, we show that the nontrivial Brauer class of $X_{\overline{\mathbb{Q}}}$ does not descend to $\mathbb{Q}$. Together with the results of Várilly-Alvarado and Viray (Adv. Math. 226(6):4884–4901, 2011), this proves that the Brauer–Manin obstruction is insufficient to explain all failures of the Hasse principle on Enriques surfaces.

The methods of this paper build on the ideas in Creutz and Viray (Math. Ann. 362(3–4):1169–1200, 2015; Manuscripta Math. 147(1–2): 139–167, 2015) and Ingalls et al., (Unramified Brauer classes on cyclic covers of the projective plane, Preprint): we study geometrically unramified Brauer classes on X via pullback of ramified Brauer classes on a rational surface. Notably, we develop techniques which work over fields which are not necessarily separably closed, in particular, over number fields.

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1 Introduction

Given a smooth, projective, geometrically integral variety X over a global field k, one may ask whether X has a k-rational point, that is, whether $X(k) \neq \emptyset$. Since k embeds into each of its completions, a necessary condition for $X(k) \neq \emptyset$ is that $X(\mathbb{A}_k) \neq \emptyset$. However, this condition is often not sufficient; varieties X with $X(\mathbb{A}_k) \neq \emptyset$ and $X(k) = \emptyset$ exist, and these are said to fail the Hasse principle.

In 1970, Manin [12] significantly advanced the study of failures of the Hasse principle by use of the Brauer group and class field theory. More precisely, he defined a subset $X(\mathbb{A}_k)^{\mathrm{Br}}$ of $X(\mathbb{A}_k)$, now known as the Brauer–Manin set, with the property that

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k).$$

Thus, we may think of an empty Brauer–Manin set as an obstruction to the existence of rational points.

In 1999, Skorobogatov [14] defined a refinement of the Brauer–Manin set, the étale-Brauer set $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$, which still contains $X(k)$. He proved that this new obstruction can be stronger than the Brauer–Manin obstruction, by constructing a bielliptic surface $X/\mathbb{Q}$ such that $X(\mathbb{A}_{\mathbb{Q}})^{\text{ét},\mathrm{Br}} = \emptyset$ and $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$.

Bielliptic surfaces have a number of geometric properties in common with Enriques surfaces: both have Kodaira dimension 0 and nontrivial étale covers. This raises the natural question of whether the étale-Brauer obstruction is stronger than the Brauer–Manin obstruction *for Enriques surfaces*. Harari and Skorobogatov took up this question in 2005; they constructed an Enriques surface $X/\mathbb{Q}$ whose étale-Brauer set was strictly smaller than the Brauer–Manin set [9], thereby showing that the Brauer–Manin obstruction is insufficient to explain all failures of weak approximation[1] on Enriques surfaces. Their surface, however, had a $\mathbb{Q}$-rational point, so it did not fail the Hasse principle.

The main result of this paper is the analogue of Harari and Skorobogatov's result for the Hasse principle. Precisely, we prove

Theorem 1.1. *The Brauer–Manin obstruction is insufficient to explain all failures of the Hasse principle on Enriques surfaces.*

[1]A smooth projective variety X satisfies weak approximation if $X(k)$ is dense in $X(\mathbb{A}_k)$ in the adelic topology.

This theorem builds on work by Várilly-Alvarado and Viray. To explain the connection, we must first provide more information about the Brauer group. For any variety X/k, we have the following filtration of the Brauer group:

$$\operatorname{Br}_0 X := \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X) \subset \operatorname{Br}_1 X := \ker(\operatorname{Br} X \to \operatorname{Br} X_{k^{\mathrm{sep}}}) \subset \operatorname{Br} X = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m).$$

Elements in $\operatorname{Br}_0 X$ are said to be **constant**, elements in $\operatorname{Br}_1 X$ are said to be **algebraic**, and the remaining elements are said to be **transcendental**. The Brauer–Manin set $X(\mathbb{A}_k)^{\operatorname{Br}}$ depends only on the quotient $\operatorname{Br} X/\operatorname{Br}_0 X$ (this follows from the fundamental exact sequence of class field theory, see [15, Sect. 5.2] for more details). As transcendental Brauer elements have historically been difficult to study, one sometimes instead considers the (possibly larger) **algebraic Brauer–Manin set** $X(\mathbb{A}_k)^{\operatorname{Br}_1}$, defined in terms of the subquotient $\operatorname{Br}_1 X/\operatorname{Br}_0 X$.

We now recall the main result of [16].

Theorem ([16, Theorem 1.1]). *There exists an Enriques surface $X/\mathbb{Q}$ such that*

$$X(\mathbb{A}_{\mathbb{Q}})^{\text{ét},\operatorname{Br}} = \emptyset \quad \text{and} \quad X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}_1} \neq \emptyset.$$

The proof of [16, Theorem 1.1] is constructive. Precisely, for any $\mathbf{a} = (a,b,c) \in \mathbb{Z}^3$ with

$$abc(5a+5b+c)(20a+5b+2c)(4a^2+b^2)(c^2-100ab)(c^2+5bc+10ac+25ab) \neq 0, \tag{1}$$

the authors consider $Y_{\mathbf{a}} \subset \mathbb{P}^5$, the smooth degree-8 K3 surface given by

$$\begin{aligned} v_0v_1 + 5v_2^2 &= w_0^2, \\ (v_0+v_1)(v_0+2v_1) &= w_0^2 - 5w_1^2, \\ av_0^2 + bv_1^2 + cv_2^2 &= w_2^2. \end{aligned}$$

The involution $\sigma\colon \mathbb{P}^5 \to \mathbb{P}^5, (v_0 : v_1 : v_2 : w_0 : w_1 : w_2) \mapsto (-v_0 : -v_1 : -v_2 : w_0 : w_1 : w_2)$ has no fixed points on $Y_{\mathbf{a}}$ so the quotient $X_{\mathbf{a}} := Y_{\mathbf{a}}/\sigma$ is an Enriques surface.

Theorem ([16, Theorem 1.2]). *Let $\mathbf{a} = (a,b,c) \in \mathbb{Z}^3_{>0}$ satisfy the following conditions:*

(1) *for all prime numbers $p \mid (5a+5b+c)$, 5 is not a square modulo p,*
(2) *for all prime numbers $p \mid (20a+5b+2c)$, 10 is not a square modulo p,*
(3) *the quadratic form $av_0^2 + bv_1^2 + cv_2^2 + w_2^2$ is anisotropic over $\mathbb{Q}_3$,*
(4) *the integer $-bc$ is not a square modulo 5,*
(5) *the triplet (a,b,c) is congruent to $(5,6,6)$ modulo 7,*
(6) *the triplet (a,b,c) is congruent to $(1,1,2)$ modulo 11,*
(7) *$Y_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$,*

(8) *the triplet* (a, b, c) *is* Galois general *(meaning that a certain number field defined in terms of a, b, c is as large as possible).*

Then

$$X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\text{ét,Br}} = \emptyset \quad \textit{and} \quad X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \neq \emptyset.$$

Várilly-Alvarado and Viray deduce [16, Theorem 1.1] from [16, Theorem 1.2] by showing that the triplet $\mathbf{a} = (12, 111, 13)$ satisfies conditions (1)–(8). Henceforth, when we refer to "conditions" by number, we mean the conditions given in the theorem above.

In [16], the authors left open the question of a transcendental obstruction to the Hasse principle for the surfaces $X_{\mathbf{a}}$, due to the "difficulty [. . .] in finding an explicit representative for [the nontrivial] Brauer class of $[\overline{X}_{\mathbf{a}}]$." Recent work of Creutz and Viray [4, 5], and Ingalls et al. [10] makes this problem more tractable. Building on techniques from [4, 5, 10], we prove

Theorem 1.2. *If* $\mathbf{a} = (a, b, c) \in \mathbb{Z}^3_{>0}$ *satisfies conditions* (5)*,* (6)*, and* (8)*, then* $\operatorname{Br} X_{\mathbf{a}} = \operatorname{Br}_1 X_{\mathbf{a}}$. *In particular, if* $\mathbf{a}$ *satisfies conditions* (1)–(8)*, then*

$$X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\text{ét,Br}} = \emptyset \quad \text{and} \quad X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset.$$

1.1 Strategy and Outline

Theorem 1.1 and the second statement of Theorem 1.2 both follow immediately from the first statement of Theorem 1.2 and [16], since the triplet $\mathbf{a} = (12, 111, 13)$ satisfies conditions (1)–(8) [16, Lemma 6.1 and Proof of Theorem 1.1]. Thus, we reduce to proving the first statement of Theorem 1.2.

For any variety X over a field k, the quotient $\operatorname{Br} X/\operatorname{Br}_1 X$ injects into $(\operatorname{Br} X_{k^{\mathrm{sep}}})^{\operatorname{Gal}(k^{\mathrm{sep}}/k)}$. In Skorobogatov's pioneering paper [14], his construction $X/\mathbb{Q}$ had the additional property that $(\operatorname{Br} X_{\overline{\mathbb{Q}}})^{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = 0$, so $\operatorname{Br} X = \operatorname{Br}_1 X$. Unfortunately, this strategy cannot be applied to an Enriques surface X, as $\operatorname{Br} X_{k^{\mathrm{sep}}} \cong \mathbb{Z}/2\mathbb{Z}$ [9, p. 3223] and hence the unique nontrivial element is always fixed by the Galois action.

Instead, we will find a Galois extension $K_1/\mathbb{Q}$ and an open set $U' \subset X_{\mathbf{a}}$ such that

(1) the geometrically unramified subgroup $\operatorname{Br}^{\text{g.unr.}} U'_{K_1} \subset \operatorname{Br} U'_{K_1}$ (i.e., the subgroup of elements in $\operatorname{Br} U'_{K_1}$ which are contained in $\operatorname{Br} \overline{X}_{\mathbf{a}} \subset \operatorname{Br} \overline{U'}$ upon base change to $\overline{\mathbb{Q}}$) surjects onto $\operatorname{Br} \overline{X}_{\mathbf{a}}$, and
(2) $(\operatorname{Br}^{\text{g.unr.}} U'_{K_1}/\operatorname{Br} K_1)^{\operatorname{Gal}(K_1/\mathbb{Q})}$ is contained in $\operatorname{Br}_1 U'_{K_1}/\operatorname{Br} K_1$.

The key step is proving (2) *without* necessarily having central simple $\mathbf{k}(U'_{K_1})$-algebra representatives for all of the elements of $\operatorname{Br}^{\text{g.unr.}} U'_{K_1}/\operatorname{Br} K_1$.

Our approach follows the philosophy laid out in [4, 5, 10]: we study geometrically unramified Brauer classes on U'_{K_1} via pullback of ramified Brauer classes on a simpler surface S', of which U' is a double cover. However, in contrast to the work of [4, 5, 10], we carry this out over a field that is not necessarily separably closed. In particular, our methods can be carried out over a number field. As we expect this approach to be of independent interest, we build up some general results in Sect. 2 which can be applied to a double cover of a rational ruled surface, assuming mild conditions on the branch locus.

Remark 1.3. For convenience, we carry out the above strategy on the K3 surface $Y_{\mathbf{a}}$ instead of on the Enriques surface $X_{\mathbf{a}}$. We then descend the results to $X_{\mathbf{a}}$.

Starting in Sect. 3, we restrict our attention to the specific varieties $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$. After recalling relevant results from [16], we construct double cover maps $\pi\colon Y_{\mathbf{a}} \to S$ and $\tilde{\pi}\colon X_{\mathbf{a}} \to \widetilde{S}$, where S and $\widetilde{S}$ are ruled surfaces, and we study the geometry of these morphisms. These maps allow us to apply the results of [4] to construct, in Sect. 4, an explicit central simple $\mathbf{k}(\overline{X}_{\mathbf{a}})$-algebra representative $\mathcal{A}$ of the nontrivial Brauer class of $\overline{X}_{\mathbf{a}}$. This representative $\mathcal{A}$ will necessarily be defined over a number field K_1, be unramified over an open set U'_{K_1}, and be geometrically unramified. Furthermore, the number field K_1 and the open set U' can be explicitly computed from the representative $\mathcal{A}$.

Section 5 uses the results from Sect. 2 to study the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/K_1)$ on $\operatorname{Br}^{\text{g.unr.}} U'_{K_1}/\operatorname{Br} K_1$ and hence prove Theorem 1.2. Namely, by repeated application of the commutative diagram in Theorem 2.2, we demonstrate that no σ-invariant transcendental Brauer class can exist for $Y_{\mathbf{a}}$. Indeed, if such a class existed, the explicit central simple algebra from Sect. 4 would relate it to a function $\tilde{\ell}$ fixed by the Galois action. However a direct computation (given in the Appendix) shows that $\tilde{\ell}$ must be moved by some Galois action, providing the required contradiction.

1.2 General Notation

Throughout, k will be a field with characteristic not equal to 2, with fixed separable closure $\bar{k}$. For any k-scheme X and field extension k'/k, we write $X_{k'}$ for the base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ and $\overline{X}$ for the base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$. If X is integral, we write $\mathbf{k}(X)$ for the function field of X. We also denote the absolute Galois group of k by $G_k = \operatorname{Gal}(\bar{k}/k)$. For any k-variety W, we use the term splitting field (of W) to mean the smallest Galois extension of k over which every geometrically irreducible component of W is defined.

The Picard group of X is $\operatorname{Pic} X := \operatorname{Div} X/\operatorname{Princ} X$, where $\operatorname{Div} X$ is the group of Weil divisors on X and $\operatorname{Princ} X$ is the group of principal divisors on X; when X is projective, Pic X is representable by a scheme, called the Picard scheme [7, Cor. 6.6, p. 232–17]. If X is projective, let Pic X denote the subgroup of Pic X that maps to the connected component of the identity in the Picard scheme of X then; the

Néron–Severi group of X is $\mathrm{NS}\,X := \mathrm{Pic}\,X/\,\mathrm{Pic}^0\,X$. For a divisor $D \in \mathrm{Div}\,X$, we write $[D]$ for its equivalence class in $\mathrm{Pic}\,X$. When X is a curve, the Jacobian of X satisfies $\mathrm{Jac}\,X = \mathrm{Pic}^0\,X$.

For a k-scheme Y, we write $\mathrm{Br}\,Y$ for the étale cohomology group $\mathrm{Br}\,Y := \mathrm{H}^2_{\text{ét}}(Y, \mathbb{G}_m)$. If Y is projective, we additionally consider the geometrically unramified subgroup $\mathrm{Br}^{\text{g.unr.}}\,\mathbf{k}(Y) \subset \mathrm{Br}\,\mathbf{k}(Y)$ consisting of those Brauer classes which are contained in $\mathrm{Br}\,\overline{Y}$ upon base change to $\overline{k}$. For an open subscheme $U \subset Y$, we have $\mathrm{Br}^{\text{g.unr.}}\,U := \mathrm{Br}\,U \cap \mathrm{Br}^{\text{g.unr.}}\,\mathbf{k}(Y)$. If A is an étale k-algebra, then we write $\mathrm{Br}\,A$ for $\mathrm{Br}(\mathrm{Spec}\,A)$. Given invertible elements a and b in such an A, we define the quaternion algebra $(a, b) := A[i, j]/\langle i^2 = a, j^2 = b, ij = -ji\rangle$. We will identify the algebra (a, b) with its class in $\mathrm{Br}\,A$.

Now assume that Y is smooth and quasi-projective. Then the following sequence is exact:

$$0 \to \mathrm{Br}\,Y[2] \to \mathrm{Br}\,\mathbf{k}(Y)[2] \xrightarrow{\oplus_y \partial_y} \bigoplus_y \mathrm{H}^1(\mathbf{k}(y), \mathbb{Z}/2\mathbb{Z}), \tag{2}$$

where the sum is taken over the set of all codimension-1 points y on Y [8, Theorem 6.1]. As $\mathrm{Br}\,\mathbf{k}(Y)[2]$ is generated by quaternion algebras, we will only describe the residue map ∂_y on quaternion algebras: for any $a, b \in \mathbf{k}(Y)^\times$, we have

$$\partial_y\left((a, b)\right) = (-1)^{v_y(a)v_y(b)} a^{v_y(b)} b^{-v_y(a)} \in \mathbf{k}(y)^\times/\mathbf{k}(y)^{\times 2} \cong \mathrm{H}^1\left(\mathbf{k}(y), \mathbb{Z}/2\mathbb{Z}\right),$$

where v_y denotes the valuation corresponding to y; as $\mathbf{k}(y)^\times/\mathbf{k}(y)^{\times 2} \cong \mathrm{H}^1$ $(\mathbf{k}(y), Z/2Z)$, we move freely between additive and multiplicative notation when computing residues, depending on the context.

2 Brauer Classes on Double Covers Arising Via Pullback

Let $\pi^0\colon Y^0 \to S^0$ be a double cover of a smooth, projective, rational, geometrically ruled surface $\varpi\colon S^0 \to \mathbb{P}^1_t$ defined over k and let $B^0 \subset S^0$ denote the branch locus of π^0. (Throughout, $\mathbb{P}^1_t$ is shorthand for $\mathbb{P}^1_{[t_0:t_1]}$, with $t := t_0/t_1$.) We assume that B^0 is reduced, geometrically irreducible, and has at worst ADE singularities. The canonical resolution [1, Theorem 7.2] $\nu\colon Y \to Y^0$ of $\pi^0\colon Y^0 \to S^0$ has a 2-to-1 k-morphism $\pi\colon Y \to S$ to a smooth rational *generically* ruled surface S; the branch curve $B \subset S$ of π is a smooth proper model of B^0. In summary, we have the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\ \pi\ } & S \supset B \\ \big\downarrow{\scriptstyle \nu} & & \big\downarrow{\scriptstyle \nu_S} \\ Y^0 & \xrightarrow{\ \pi^0\ } & S^0 \supset B^0 \end{array}$$

Since B^0 is geometrically irreducible, $\mathrm{Pic}^0\,Y$ is trivial by [4, Corollary 6.3] and so we may conflate $\mathrm{Pic}\,Y$ and $\mathrm{NS}\,Y$.

The generic fiber of $\varpi \circ \pi^0$ is a double cover $C \to \mathbb{P}^1_{k(t)} \to \operatorname{Spec} k(t)$. Since $k(t)$ is infinite, by changing coordinates if necessary, we may assume that the double cover is unramified above $\infty \in S^0_{k(t)}$. Then C has a model

$$y^2 = c' h(x),$$

for some $c' \in k(t)$ and $h \in k(t)[x]$ square-free, monic, and with $\deg(h) = 2g(C)+2$, where $g(C)$ denotes the genus of C. Note that $\mathbf{k}(B) = \mathbf{k}(B^0) \cong k(t)[\theta]/(h(\theta))$; we write α for the image of θ in $\mathbf{k}(B)$.

As S^0 is rational and geometrically ruled, $\operatorname{Pic} S^0 \cong \mathbb{Z}^2$ and is generated by a fiber S^0_∞ and a section $\mathfrak{S}$, which we may take to be the closure of $x = \infty$. Since $\nu_S\colon S \to S^0$ is a birational map, $\operatorname{Pic} \overline{S}$ is generated by the strict transforms of $\mathfrak{S}$ and S^0_∞, and the curves $E_1, \ldots, E_n$ that are contracted by the map $S \to S^0$. We will often abuse notation and conflate $\mathfrak{S}$ and S^0_∞ with their strict transforms. In any case, by $\mathfrak{S}$, B, E_i, or S^0_∞, we always mean the actual divisors and *not* the divisor classes in the Picard group.

Let

$$\mathcal{E} = \left\{\mathfrak{S}, S^0_\infty, E_1, \ldots, E_n\right\}$$

denote the aforementioned set of $n+2$ generators and define

$$V := S \setminus \left(B \cup \bigcup_{E \in \mathcal{E}} E\right) \subset S.$$

Possibly after replacing k with a finite extension, we may assume that all elements of $\mathcal{E}$ are defined over k and, in particular, that $\operatorname{Pic} S = \operatorname{Pic} \overline{S}$. Since ν_S is defined over k, we additionally have that S is k-rational and so $\operatorname{Br} S = \operatorname{Br} S^0 = \operatorname{Br} k$.

For any $\ell \in \mathbf{k}(B)^\times$, we define

$$\mathcal{A}_\ell := \operatorname{Cor}_{\mathbf{k}(B)(x)/k(t,x)}\left((\ell, x-\alpha)\right) \in \operatorname{Br} \mathbf{k}(S).$$

We will be particularly concerned with $\mathcal{A}_\ell$ when ℓ is contained in the subgroup

$$\begin{aligned}\mathbf{k}(B)^\times_{\mathcal{E}} &:= \left\{\ell \in \mathbf{k}(B)^\times : \operatorname{div}(\ell) \in \operatorname{im}(\mathbb{Z}^{\mathcal{E}} \to \operatorname{Div}(S) \to \operatorname{Div}(B) \otimes \mathbb{Z}/2\mathbb{Z})\right\}\\ &= \left\{\ell \in \mathbf{k}(B)^\times : \nu_* \operatorname{div}(\ell) \in \langle \mathfrak{S}, S^0_\infty\rangle \subset \operatorname{Div}(B^0) \otimes \mathbb{Z}/2\mathbb{Z}\right\}.\end{aligned}$$

By [4, Proof of Theorem 5.2], this subgroup is exactly the set of functions ℓ such that $\pi^*\mathcal{A}_\ell$ is geometrically unramified. Note that $\mathbf{k}(B)^\times_{\mathcal{E}}$ contains $k^\times \mathbf{k}(B)^{\times 2}$.

Let

$$U := Y \setminus \left(\bigcup_{E \in \mathcal{E}} \pi^{-1}(E)\right) \subset Y.$$

The goal of this section is to prove the following two theorems:

Theorem 2.1. *Let k' be any Galois extension of k. Then we have the following exact sequence of* $\mathrm{Gal}(k'/k)$*-modules:*

$$0 \to \frac{\operatorname{Pic} Y_{k'}}{\pi^* \operatorname{Pic} S + 2 \operatorname{Pic} Y_{k'}} \xrightarrow{j} \frac{\mathbf{k}(B_{k'})^\times_{\mathcal{E}}}{k'^\times \mathbf{k}(B_{k'})^{\times 2}} \xrightarrow{\beta} \left(\frac{\mathrm{Br}^{\text{g. unr.}} U_{k'}}{\operatorname{Br} k'}\right)[2], \tag{3}$$

where j is as in Sect. 2.3 and β is as in Sect. 2.2. Furthermore, if k' is separably closed, then the last map surjects onto $\operatorname{Br} Y[2]$.

Theorem 2.2. *We retain the notation from Theorem 2.1. If* $\operatorname{Br} k' \to \operatorname{Br} \mathbf{k}(S_{k'})$ *is injective and* $\operatorname{Pic} \overline{U}[2] = 0$*, then there is a commutative diagram of* $\mathrm{Gal}(k'/k)$*-modules with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & \dfrac{\operatorname{Pic} Y_{k'}}{\pi^* \operatorname{Pic} S + 2\operatorname{Pic} Y_{k'}} & \xrightarrow{j} & j(\operatorname{Pic} Y_{k'}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \left(\dfrac{\operatorname{Pic} \overline{Y}}{\pi^* \operatorname{Pic} S + 2 \operatorname{Pic} \overline{Y}}\right)^{G_{k'}} & \xrightarrow{j} & \dfrac{\mathbf{k}(B_{k'})_{\mathcal{E}}}{k'^\times \mathbf{k}(B_{k'})^{\times 2}} & \xrightarrow{\beta} & \dfrac{\mathrm{Br}^{\text{g.unr.}} U_{k'}}{\mathrm{Br}_1 U_{k'}} & & \\
 & & \downarrow{\scriptstyle \beta\circ j} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & \dfrac{\mathrm{Br}_1 U_{k'}}{\operatorname{Br} k'} & \longrightarrow & \dfrac{\mathrm{Br}^{\text{g.unr.}} U_{k'}}{\operatorname{Br} k'} & \longrightarrow & \dfrac{\mathrm{Br}^{\text{g.unr.}} U_{k'}}{\mathrm{Br}_1 U_{k'}} & \longrightarrow & 0.
\end{array}$$

The structure of the section is as follows. In Sect. 2.1, we prove some preliminary results about the residues of $\mathcal{A}_\ell$; these are used in Sect. 2.2 to define the map β. Next, in Sect. 2.3, we define j and prove that it is injective. In Sect. 2.4, we characterize the elements of $\operatorname{Br} V$ that pull back to constant algebras under π^*. In Sect. 2.5, we combine the results from the earlier sections to prove Theorem 2.1, and, finally, in Sect. 2.6, we prove Theorem 2.2.

2.1 Residues of $\mathcal{A}_\ell$

In order to define the homomorphism β, we will need to know certain properties about the residues of $\mathcal{A}_\ell$ at various divisors of S^0. We first compute residues associated to horizontal divisors.

Lemma 2.3. *Let $\ell \in \mathbf{k}(B)^\times$, and let F be an irreducible horizontal curve in S^0, i.e., a curve that maps dominantly onto $\mathbb{P}^1_t$.*

(1) *If $F \neq B, \mathfrak{S}$, then* $\partial_F(\mathcal{A}_\ell) = 1 \in \mathbf{k}(F)^\times/\mathbf{k}(F)^{\times 2}$.
(2) $\partial_B(\mathcal{A}_\ell) = [\ell] \in \mathbf{k}(B)^\times/\mathbf{k}(B)^{\times 2}$.

Proof. The arguments in this proof follow those in [5, Proofs of Theorem 1.1 and Prop. 2.3]; as the situation is not identical, we restate the arguments here for the reader's convenience.

Let v be the valuation on $\mathbf{k}(S^0)$ associated to F. By [5, Lemma 2.1], we have

$$\partial_F(\mathcal{A}_\ell) = \prod_{w|v} \mathrm{Norm}_{\mathbf{k}(w)/\mathbf{k}(v)} \left((-1)^{w(\ell)w(x-\alpha)} \ell^{w(x-\alpha)} (x-\alpha)^{-w(\ell)}\right), \tag{4}$$

where w runs over all valuations on $\mathbf{k}(B \times_{\mathbb{P}^1_t} S^0)$ extending v. As F is a horizontal divisor, $v|_{k(t)}$ is trivial and hence $w|_{\mathbf{k}(B)}$ is trivial for all $w|v$. Therefore, (4) simplifies to $\prod_{w|v} \mathrm{Norm}_{\mathbf{k}(w)/\mathbf{k}(v)}(\ell^{w(x-\alpha)})$.

By definition of α, $\mathrm{Norm}_{\mathbf{k}(B)(x)/k(t)(x)}(x-\alpha) = h(x)$. Thus, $w(x-\alpha) = 0$ for all $w|v$ if $v(h(x)) = 0$, or equivalently, if $F \neq B, \mathfrak{S}$. This completes the proof of (1).

Now assume that $F = B$. We know that $h(x)$ factors as $(x-\alpha)h_1(x)$ over $\mathbf{k}(B)(x)$, where $h_1 \in k(t)[x]$ is possibly reducible. Hence, $x-\alpha$ determines a valuation $w_{x-\alpha}$ on $\mathbf{k}(B)(x)$ lying over v; similarly, the other irreducible factors of h_1 also determine valuations lying over v. Notice that since $h(x)$ is separable (as B is reduced), we have that $h_1(\alpha) \neq 0$, and hence that $w(x-\alpha) = 0$ for any valuation w over v corresponding to the irreducible factors of $h_1(x)$. Thus, (4) simplifies to

$$\prod_{w|v} \mathrm{Norm}_{\mathbf{k}(w)/\mathbf{k}(v)}(\ell^{w(x-\alpha)}) = \mathrm{Norm}_{\mathbf{k}(w_{x-\alpha})/\mathbf{k}(v)}(\ell) = \ell,$$

as required. □

Now we compute the residues associated to vertical divisors.

Lemma 2.4. *Let $\ell \in \mathbf{k}(B)^\times_{\mathcal{E}}$, $t_0 \in \mathbb{A}^1_t \subset \mathbb{P}^1_t$ be a closed point, and $F = S^0_{t_0}$. Then,*

$$\partial_F(\mathcal{A}_\ell) \in \mathrm{im}\left(\frac{\mathbf{k}(t_0)^\times}{\mathbf{k}(t_0)^{\times 2}} \to \frac{\mathbf{k}(F)^\times}{\mathbf{k}(F)^{\times 2}}\right).$$

Remark 2.5. If k is separably closed, then $\mathbf{k}(t_0)^{\times 2} = \mathbf{k}(t_0)^\times$ and the result follows from [5, Prop. 3.1].

Proof. It suffices to show that $\partial_F(\mathcal{A}_\ell) \in \mathbf{k}(F)^{\times 2}\mathbf{k}(t_0)^\times$. We repeat [4, Proof of Prop. 3.1] while keeping track of scalars to accommodate the fact that k is not necessarily separably closed.

By [5, Lemma 2.1], we have

$$\partial_F(\mathcal{A}_\ell) = \prod_{\substack{F' \subset S^0 \times_{\mathbb{P}^1_t} B \\ F' \mapsto F \text{ dominantly}}} \mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}((-1)^{w'(x-\alpha)w'(\ell)} \ell^{w'(x-\alpha)} (x-\alpha)^{-w'(\ell)}), \tag{5}$$

where F' is an irreducible curve and w' denotes the valuation associated to F'. The surface $S^0 \times_{\mathbb{P}^1_t} B$ is a geometrically ruled surface over B, so the irreducible curves F' are in one-to-one correspondence with points $b' \in B$ mapping to t_0. Furthermore, $\mathbf{k}(F') = \mathbf{k}(b')(x)$ and $\mathbf{k}(F) = \mathbf{k}(t_0)(x)$, so $\mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}$ is induced from $\mathrm{Norm}_{\mathbf{k}(b')/\mathbf{k}(t_0)}$. Thus, we may rewrite (5) as

$$\partial_F(\mathcal{A}_\ell) = \prod_{b' \in B, b' \mapsto t_0} \mathrm{Norm}_{\mathbf{k}(b')/\mathbf{k}(t_0)}((-1)^{w'(x-\alpha)w'(\ell)} \ell^{w'(x-\alpha)} (x-\alpha)^{-w'(\ell)}). \tag{6}$$

By [4, Lemma 3.3], there exists an open set $W \subset \mathbb{A}^1$ containing t_0 and constants $d \in \mathbf{k}(t)^\times$, $e \in \mathbf{k}(t)$ such that

$$S_W^0 \to \mathbb{P}_k^1 \times W, \quad s \mapsto (dx(s) + e, \varpi(s))$$

is an isomorphism. In particular, $dx + e$ is a horizontal function on S_W^0. Consider the following equality:

$$\begin{aligned}\mathrm{Cor}_{\mathbf{k}(B)(x)/\mathbf{k}(S^0)}((dx + e - (d\alpha + e), \ell)) &= \mathcal{A}_\ell + \mathrm{Cor}_{\mathbf{k}(B)(x)/\mathbf{k}(S^0)}((d, \ell)) \\ &= \mathcal{A}_\ell + (d, \mathrm{Norm}(\ell)).\end{aligned}$$

Since $(d, \mathrm{Norm}(\ell)) \in \varpi^* \operatorname{Br} k(t)$, we have

$$\partial_F(\mathcal{A}_\ell) \in \partial_F\left(\mathrm{Cor}_{\mathbf{k}(B)(x)/\mathbf{k}(S^0)}((dx + e - (d\alpha + e), \ell))\right) \mathbf{k}(t_0)^\times.$$

Thus, we may assume that x is a horizontal function, in particular, that x has no zeros or poles along F, and that it restricts to a non-constant function along F. It is then immediate that $w'(x - \alpha) \leq 0$, and that the inequality is strict if and only if $w'(\alpha) < 0$, which in turn happens if and only if b' lies over $B_{t_0}^0 \cap \mathfrak{S}$.

We first consider the factor of (6) that corresponds to points that do not lie over $B_{t_0}^0 \cap \mathfrak{S}$. If b' does not lie over $B_{t_0}^0 \cap \mathfrak{S}$, then (as stated above) $w'(x - \alpha) = 0$, where w' denotes the valuation associated to b'. Therefore, the corresponding factor of (6) simplifies to

$$\prod_{b' \in B \setminus \nu^{-1}(B^0 \cap \mathfrak{S}), b' \mapsto t_0} \mathrm{Norm}_{\mathbf{k}(b')/\mathbf{k}(t_0)}\left((x - \alpha(b'))^{-w'(\ell)}\right).$$

By definition, $\ell \in \mathbf{k}(B)_{\mathcal{E}}^\times$ implies that for all $b'' \in B^0 \setminus (B^0 \cap \mathfrak{S})$, $\sum_{b' \in B, b' \mapsto b''} w'(\ell) \equiv 0 \bmod 2$. Since $\alpha(b')$ depends only on the image of b' in B^0, this shows that the above factor is contained in $\mathbf{k}(F)^{\times 2}$.

Now consider the case that b' lies over $B_{t_0}^0 \cap \mathfrak{S}$. We claim that, since $w'(x) = 0$,

$$\mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}\left((-1)^{w'(x-\alpha)w'(\ell)} \ell^{w'(x-\alpha)} (x-\alpha)^{-w'(\ell)}\right) \tag{7}$$

reduces to a constant in $\mathbf{k}(F)$. Indeed, if $w'(\ell) = 0$, then we obtain $\ell^{w'(x-\alpha)}$, which reduces (after taking $\mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}$) to an element of $\mathbf{k}(t_0)^\times$. If $w'(\ell) \neq 0$, let $\pi_{F'}$ be a uniformizer for F'. Since $w'(x) = 0 > w'(\alpha)$, we have

$$\left(\frac{\ell}{\pi_{F'}^{w'(\ell)}}\right)^{w'(x-\alpha)}\left(\frac{x-\alpha}{\pi_{F'}^{w'(x-\alpha)}}\right)^{-w(\ell)}=\left(\frac{\ell}{\pi_{F'}^{w'(\ell)}}\right)^{w'(x-\alpha)}\left(\frac{-\alpha}{\pi_{F'}^{w'(x-\alpha)}}\right)^{-w(\ell)} \mod \pi_{F'}$$

and so (7) reduces (again, after taking $\mathrm{Norm}_{\mathbf{k}(F')/\mathbf{k}(F)}$) to an element in $\mathbf{k}(t_0)^\times$. Thus, every factor of (6) corresponding to points b' lying over $B^0_{t_0} \cap \mathfrak{S}$ is contained in $\mathbf{k}(t_0)^\times$, and every other factor is an element of $\mathbf{k}(F)^{\times 2}$. This completes the proof. □

2.2 The Morphism β

Proposition 2.6. *Let $\ell \in \mathbf{k}(B)^\times$. There exists an element $\mathcal{A}' = \mathcal{A}'(\ell) \in \operatorname{Br} k(t)$, unique modulo $\operatorname{Br} k$, such that*

$$\mathcal{A}_\ell + \varpi^*\mathcal{A}' \in \operatorname{Br} V.$$

This induces a well-defined homomorphism

$$\beta\colon \frac{\mathbf{k}(B)^\times_{\mathcal{E}}}{k^\times \mathbf{k}(B)^{\times 2}} \to \frac{\operatorname{Br}^{\text{g.unr.}} U}{\operatorname{Br} k}[2], \quad \ell \mapsto \pi^*\left(\mathcal{A}_\ell + \varpi^*\mathcal{A}'\right),$$

which is surjective if k is separably closed.

Proof. Recall that $V = S \setminus \left(B \cup \bigcup_{E\in\mathcal{E}} E\right) \subset S$. Therefore, as a subgroup of $\operatorname{Br}\mathbf{k}(S) = \operatorname{Br}\mathbf{k}(S^0)$, $\operatorname{Br} V$ is equal to $\operatorname{Br}\left(S^0 \setminus (\mathfrak{S} \cup S^0_\infty \cup B)\right)$, since the Brauer group of a surface is unchanged under removal of a codimension 2 closed subscheme [8, Theorem 6.1]. Thus, to prove the first statement, it suffices to show that there exists an element $\mathcal{A}' \in \operatorname{Br} k(t)$, unique up to constant algebras, such that $\partial_F(\mathcal{A}_\ell) = \partial_F(\varpi^*\mathcal{A}')$ for all irreducible curves $F \subset S^0$ with $F \neq \mathfrak{S}, S^0_\infty, B$.

If F is any horizontal curve, i.e., F maps dominantly to $\mathbb{P}^1_t$, then $\partial_F(\varpi^*\mathcal{A}') = 1$ for all $\mathcal{A}' \in \operatorname{Br} k(t)$. If we further assume that $F \neq \mathfrak{S}, B$, then Lemma 2.3 gives $\partial_F(\mathcal{A}_\ell) = 1$. Thus, for all $\mathcal{A}' \in \operatorname{Br} k(t)$, we have $\partial_F(\mathcal{A}_\ell) = \partial_F(\varpi^*\mathcal{A}')$ for all horizontal curves $F \neq \mathfrak{S}, B$.

Now we turn our attention to the vertical curves. Recall Faddeev's exact sequence [6, Corollary 6.4.6]:

$$0 \to \operatorname{Br} k \to \operatorname{Br} k(t) \xrightarrow{\oplus\partial_{t_0}} \bigoplus_{t_0\in\mathbb{P}^1_t} \mathrm{H}^1(G_{\mathbf{k}(t_0)}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sum_{t_0}\mathrm{Cor}_{\mathbf{k}(t_0)/k}} \mathrm{H}^1(G_k, \mathbb{Q}/\mathbb{Z}) \to 0. \tag{8}$$

Since the residue field at $t_0 = \infty$ is equal to k, this sequence implies that for any element $(r_{t_0}) \in \oplus_{t_0\in\mathbb{A}^1}\mathbf{k}(t_0)^\times/\mathbf{k}(t_0)^{\times 2}$, there exists a Brauer class in $\mathcal{A}' \in \operatorname{Br} k(t)$, unique modulo elements of $\operatorname{Br} k$, such that $\partial_{t_0}(\mathcal{A}') = r_{t_0}$ for all closed points $t_0 \in \mathbb{A}^1$. By Lemma 2.4, for all $t_0 \in \mathbb{A}^1$, we have $\partial_F(\mathcal{A}_\ell) \in \operatorname{im}\left(\mathbf{k}(t_0)^\times/\mathbf{k}(t_0)^{\times 2} \to \mathbf{k}(F)^\times/\mathbf{k}(F)^{\times 2}\right)$, where $F = S^0_{t_0}$. Hence, there exists an $\mathcal{A}' \in \operatorname{Br} k(t)$, unique modulo $\operatorname{Br} k$, such that $\partial_F(\varpi^*\mathcal{A}') = \partial_F(\mathcal{A}_\ell)$ for all $F \neq \mathfrak{S}, B, S^0_\infty$, as desired.

It remains to prove the second statement. The first statement immediately implies the existence of a well-defined homomorphism

$$\frac{\mathbf{k}(B)^\times_{\mathcal{E}}}{\mathbf{k}(B)^{\times 2}} \to \frac{\operatorname{Br} \pi^{-1}(V)}{\operatorname{Br} k}[2], \quad \ell \mapsto \pi^*\left(\mathcal{A}_\ell + \varpi^*\mathcal{A}'\right).$$

In order to complete the proof, we must prove that

(1) $\pi^*\left(\mathcal{A}_d + \varpi^*\mathcal{A}'\right) \in \operatorname{Br} k$ if $d \in k^\times$,
(2) the image lands in $\operatorname{Br}^{\text{g.unr.}} U/\operatorname{Br} k$, and
(3) the image is equal to $\operatorname{Br} Y[2]$ if k is separably closed.

We begin with (1). Let $d \in k^\times$. Then

$$\begin{aligned}\mathcal{A}_d = \operatorname{Cor}_{\mathbf{k}(B)(x)/k(t,x)}(d, x-\alpha) &= (d, \operatorname{Norm}_{\mathbf{k}(B)(x)/k(t,x)}(x-\alpha)) \\ &= (d, h(x)) = (d, c'h(x)) + (d, c').\end{aligned}$$

Since $\sqrt{c'h(x)}$ generates $\mathbf{k}(Y^0)/\mathbf{k}(S^0)$, $\operatorname{div}(c'h(x)) = B + 2Z$ for some divisor Z on S^0. Thus, $(d, c'h(x))$ is unramified away from B; in particular, $(d, c'h(x)) \in \operatorname{Br} V$. Since $\mathcal{A}'$ is the unique element in $\operatorname{Br} k(t)/\operatorname{Br} k$ such that $\mathcal{A}_d + \varpi^*\mathcal{A}'$, then $\mathcal{A}' = (d, c') + \mathcal{B}$ for some $\mathcal{B} \in \operatorname{Br} k$. Hence,

$$\begin{aligned}\pi^*\left(\mathcal{A}_d + \varpi^*\mathcal{A}'\right) &= \pi^*\left((d, c'h(x)) + (d, c') + \varpi^*(d, c') + \varpi^*\mathcal{B}\right), \\ &= \pi^*(d, c'h(x)) + \pi^*\varpi^*\mathcal{B}.\end{aligned}$$

Furthermore, since $c'h(x)$ is a square in $\mathbf{k}(Y^0)$, then $\pi^*\left(\mathcal{A}_d + \varpi^*\mathcal{A}'\right) = \pi^*\varpi^*\mathcal{B} \in \operatorname{Br} k$, as desired.

Now we turn to (2) and (3). Since B is the branch locus of π and π is 2-to-1, any 2-torsion Brauer class in $\operatorname{im}(\pi^*\colon \operatorname{Br}\mathbf{k}(S) \to \operatorname{Br}\mathbf{k}(Y))$ is unramified at $\pi^{-1}(B)_{\text{red}}$. Thus, the image is contained in $\operatorname{Br} U/\operatorname{Br} k$. To prove that it is contained in $\operatorname{Br}^{\text{g.unr.}} U$, we must show that $\pi^*\left(\mathcal{A}_\ell + \varpi^*\mathcal{A}'\right)_{\overline{k}}$ is contained in $\operatorname{Br}\overline{Y}$. By Tsen's theorem, $\pi^*\left(\mathcal{A}_\ell + \varpi^*\mathcal{A}'\right)_{\overline{k}} = (\pi^*\mathcal{A}_\ell)_{\overline{k}}$. This element is contained in $\operatorname{Br}\overline{Y}$ by [4, Theorem I], which yields (2). In fact, [4, Theorem I] shows that $\operatorname{Br}\overline{Y}[2]$ is generated by $\pi^*\mathcal{A}_\ell$ where ℓ runs over the elements in $\mathbf{k}(B_{\overline{k}})_{\mathcal{E}}$, which proves (3). □

2.3 The Morphism j

In this section, we define the map j and prove that it is injective. The map j will be induced by the following homomorphism:

$$\begin{aligned} j'\colon \operatorname{Div}(Y \setminus \pi^{-1}(B)) &\to \mathbf{k}(B)^\times/k^\times \\ D &\mapsto \ell|_B \end{aligned}$$

where $\ell \in \mathbf{k}(S)^\times$ is such that $\mathrm{div}_S(\ell) = \pi_* D - m_1 E_1 - \cdots - m_n E_n - d\mathfrak{S} - eS^0_\infty$. (Recall that $E_1, \ldots, E_n$, $\mathfrak{S}$, and S^0_∞ form an integral basis for $\mathrm{Pic}\, S = \mathrm{Pic}\, \overline{S}$.)

Lemma 2.7. *The homomorphism j' induces a well-defined injective homomorphism*

$$j : \frac{\mathrm{Pic}\, Y}{\pi^* \mathrm{Pic}\, S + 2\, \mathrm{Pic}\, Y} \to \frac{\mathbf{k}(B)^\times_{\mathcal{E}}}{k^\times \mathbf{k}(B)^{\times 2}}.$$

Proof. For any divisor $D \in \mathrm{Div}\, Y \setminus \pi^{-1}(B)$, the projection formula [11, p.399] yields

$$2\pi_*(D \cap \pi^{-1}(B)_{\mathrm{red}}) = \pi_*(D \cap 2\pi^{-1}(B)_{\mathrm{red}}) = \pi_*(D \cap \pi^*(B)) = (\pi_* D) \cap B.$$

Thus, for any divisor $D \in \mathrm{Div}\, Y \setminus \pi^{-1}(B)$, we have that $[D \cap \pi^{-1}(B)_{\mathrm{red}}] \in \left(\frac{\mathrm{Pic}\, B}{\mathrm{im}\, \mathrm{Pic}\, S \to \mathrm{Pic}\, B}\right)[2]$. By the same argument as in proof of [10, Lemma 4.8], this induces a well-defined injective homomorphism

$$\frac{\mathrm{Pic}\, Y}{\pi^* \mathrm{Pic}\, S + 2\, \mathrm{Pic}\, Y} \to \left(\frac{\mathrm{Pic}\, B}{\mathrm{im}\, \mathrm{Pic}\, S \to \mathrm{Pic}\, B}\right)[2], \quad [D] \mapsto [D \cap \pi^{-1}(B)_{\mathrm{red}}]. \tag{9}$$

One can also check that there is a well-defined injective homomorphism

$$\left(\frac{\mathrm{Pic}\, B}{\mathrm{im}\, \mathrm{Pic}\, S \to \mathrm{Pic}\, B}\right)[2] \to \frac{\mathbf{k}(B)^\times_{\mathcal{E}}}{k^\times \mathbf{k}(B)^{\times 2}} \tag{10}$$

that sends a divisor D which represents a class in $\left(\frac{\mathrm{Pic}\, B}{\mathrm{im}\, \mathrm{Pic}\, S \to \mathrm{Pic}\, B}\right)[2]$ to a function ℓ such that $\mathrm{div}(\ell) = 2D + \sum_{C \in \mathrm{Pic}\, S} n_C C \cap B$. Since j is the composition of (9) and (10), this completes the proof that j is well-defined and injective. □

2.4 Brauer Classes on V That Become Constant Under π^*

Proposition 2.8. *If $\mathcal{A} \in \mathrm{Br}\, V$ is such that $\pi^* \mathcal{A} \in \mathrm{Br}\, k \subset \mathrm{Br}\, \mathbf{k}(Y)$, then there exists a divisor $D \in \mathrm{Div}\, Y$ such that $j([D]) = \partial_B(\mathcal{A})$ in $\mathbf{k}(B)^\times / k^\times \mathbf{k}(B)^{\times 2}$.*

Proof. Recall that $\mathbf{k}(Y_k) = \mathbf{k}(S_k)(\sqrt{c' h(x)})$. Thus, if $\pi^* \mathcal{A} \in \mathrm{Br}\, k$, then

$$\mathcal{A} = (c' h(x), G) + \mathcal{B} \tag{11}$$

for some $G \in \mathbf{k}(S_k)^\times$ and some $\mathcal{B} \in \mathrm{Br}\, k$. Since B is the branch locus of π, $v_B(c' h(x))$ must be odd. Therefore, without loss of generality, we may assume that B is not contained in the support of G; write

$$\mathrm{div}(G) = \sum_i n_i C_i + d(\mathfrak{S}) + e(S^0_\infty) + m_1 E_1 + \cdots + m_n E_n,$$

where C_i are k-irreducible curves of S distinct from $\mathfrak{S}$, S^0_∞, and $E_1, \ldots, E_n$.

Now we consider the residue of $\mathcal{A}$ at C_i. By (11), the residue of $\mathcal{A}$ at C_i is $[c'h(x)] \in \mathbf{k}(C_i)^\times/\mathbf{k}(C_i)^{\times 2}$. On the other hand, $\mathcal{A} \in \operatorname{Br} V$, so the residue is trivial at C_i. Together, these statements imply that $\pi^{-1}(C_i)$ consists of two irreducible components C_i' and C_i''. As this is true for all C_i, we have that $\operatorname{div}(G) = \pi_*(\sum_i n_i C_i') + m(\mathfrak{S}) + m_0(S^0_\infty) + m_1 E_1 + \cdots + m_n E_n$, and so $j'(\sum n_i C_i') = G|_B$ modulo $k^\times$. Since the residue of $\mathcal{A}$ at B is equal to $G|_B$, this completes the proof. □

2.5 Proof of Theorem 2.1

We note that much of this proof is very similar to proofs in [10, Lemmas 4.4 and 4.8].

We will first prove the sequence is exact, and then show that the maps are compatible with the Galois action. Since all assumed properties of k are preserved under field extension, we may, for the moment, assume that $k = k'$. Then Lemma 2.7 yields an injective homomorphism

$$j\colon \frac{\operatorname{Pic} Y_{k'}}{\pi^* \operatorname{Pic} S + 2 \operatorname{Pic} Y_{k'}} \longrightarrow \frac{\mathbf{k}(B_{k'})^\times_{\mathcal{E}}}{k'^\times \mathbf{k}(B_{k'})^{\times 2}},$$

and Proposition 2.6 yields a homomorphism

$$\beta\colon \frac{\mathbf{k}(B_{k'})^\times_{\mathcal{E}}}{k'^\times \mathbf{k}(B_{k'})^{\times 2}} \longrightarrow \left(\frac{\operatorname{Br}^{\text{g. unr.}} U_{k'}}{\operatorname{Br} k'}\right)[2],$$

which is surjective if k' is separably closed. We now show that $\operatorname{im}(j) = \ker(\beta)$.

Let $\ell \in \mathbf{k}(B_{k'})^\times_{\mathcal{E}}$ be such that $\beta(\ell) \in \operatorname{Br} k'$. Recall that β factors through $\operatorname{Br} V/\operatorname{Br} k$ by the map

$$\ell \mapsto \underbrace{\mathcal{A} := \mathcal{A}_\ell + \varpi^* \mathcal{A}'}_{\in \operatorname{Br} V/\operatorname{Br} k} \mapsto \pi^* \mathcal{A},$$

where $\mathcal{A}' \in \operatorname{Br} k'(t)$ is as in Proposition 2.6. By assumption, $\pi^*\mathcal{A} \in \operatorname{Br} k$, thus, by Proposition 2.8, there is some $D \in \operatorname{Div} Y_k$ such that $j([D]) = \partial_B(\mathcal{A}) = \partial_B(\mathcal{A}_\ell)\partial_B(\varpi^*\mathcal{A}') \bmod k^\times$. However, $\partial_B(\varpi^*\mathcal{A}') = 1$ since B is a horizontal divisor, and $\partial_B(\mathcal{A}_\ell) = [\ell]$ by Lemma 2.3. Hence, $\ell \in \operatorname{im}(j)$, and so $\operatorname{im}(j) \supset \ker(\beta)$.

For the opposite inclusion, it suffices to prove that $\beta(j([D])) \in \operatorname{Br} k'$ for any prime divisor $D \in \operatorname{Div}(Y_{k'} \setminus \pi^{-1}(B))$. Let $\ell = j'(D)$; recall that ℓ is the restriction to B of a function $\ell_S \in \mathbf{k}(S_{k'})$ such that $\operatorname{div}(\ell_S) = \pi_* D - m_1 E_1 - \cdots - m_n E_n - d\mathfrak{S} - eS^0_\infty$. As above, let $\mathcal{A} := \mathcal{A}_\ell + \varpi^*\mathcal{A}'$. We claim that

$$\mathcal{A} = (c'h(x), \ell_S) + \mathcal{B} \quad \in \operatorname{Br} \mathbf{k}(S_{k'}) = \operatorname{Br} \mathbf{k}(S^0_{k'})$$

for some $\mathcal{B} \in \operatorname{Br} k'$. Since $c'h(x) \in \mathbf{k}(Y_{k'})^{\times 2}$, this equality implies that $\pi^*(\mathcal{A}) = \pi^*\mathcal{B} \in \operatorname{Br} k'$. To prove the claim, we will compare residues of $\mathcal{A}$ and $(c'h(x), \ell_S)$ on S^0. Repeated application of Faddeev's exact sequence [6, Corollary 6.4.6] shows