ENVIRONMENTAL HYDRAULICS SERIES

Mathematical Models

Edited by Jean-Michel Tanguy

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Summary of Volume 5 Modeling Software

Environmental Hydraulics

volume 2

Mathematical Models

Edited by
Jean-Michel Tanguy

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Introduction [1]

This study of environmental hydraulics is made up of five volumes. The first two volumes are concerned with describing the principle processes in physical domains which can be observed and measured.

The first volume is dedicated to mathematical modeling for hydrology and fluvial hydraulics.

It is completed by a volume dedicated to mathematical modelization for oceanic hydraulics for models of transport and conceptual models.

This is followed by two volumes dedicated to numerical modelization. One is on the presentation of operational software and the two volumes are on the various applications of software in case studies.

This present volume draws mathematical theories which enable us to present these processes and simulate them. There is no one unified theory which describes with a unique system the many physical processes occurring within the field and this book.

Most scientific disciplines began with a period of experimentation (be it in hydraulics at the open surface or hydrogeology) in order to understand and research of laws of behavior and draw from these observable correlations.

In contrast there were a number of important theories on the mechanics of fluids which blossomed in the 18th century, notably Euler and his studies. Empirical and theoretical approaches did not go hand in hand until the end of the 19th century.

Today the majority of scientific disciplines we will look at in this book abandoned pure empirical methods and replaced them with theoretical ones. These deal with many different things, some made up from complex mathematics while others retain their roles as very simple models. What they have in common is that they all deal with natural environments adhering to the laws of conservation: mass, quantity of movement and energy.

Our objective in this book is to give the reader a vision which incorporates the many disciplines and laws on water.

The main thread of this book charts the development of numerical models in each scientific discipline. In effect there has been a pendulum movement between complexity and simplicity since the 1960s. With the coming of computer technology, the first objective was to simplify the complex 3D mathematical models as much as possible whilst making sure they could still be used in engineering. As time went on and methods progressed, there was a reverse movement away from the former simplified models and towards more sophisticated ones which now form the basis of the latest models. These days, models are continually being made ever more sophisticated. This iterative step is common to all scientific disciplines, and along the way, a multitude of more or less simplified mathematical models were created. We shall present these later on.

This book is made up of sixteen chapters:

Chapter 1: Reminders for the mechanical properties of fluids

Chapter 2: Navier-Stokes' Equations

Chapter 3: Atmospheric models

Chapter 4: Hydrologic Models

Chapter 5: Fluvial currentology and oceanic models

Chapter 6: Urban hydrologic models

Chapter 7: Tidal model and tide streams

Chapter 8: Swell generation and coastal current models

Chapter 9: Solid transport models and evolution of the seabed

Chapter 10: Oil spill models

Chapter 11: Conceptual, empirical and other models
Chapter 12: Reservoir models in hydrology
Chapter 13: Reservoir models in hydro-geology
Chapter 14: Formal neural network models
Chapter 15: Model coupling
Chapter 16: Different categories of hydrology models

Using a mathematical format, Chapter 1 presents the principles behind mechanical properties of fluids. It introduces the main rules of this discipline in a brief and theoretical manner which can studied quickly if desired. It recalls the theorem for quantity of movement, the fundamental notations of kinetic energy, enthalpy and the first principles of thermodynamics — Newton's Law concerning the forces in a moving fluid. The theoretical corpus we incorporate establishes the following presentations by discipline.

Chapter 2 deals with Navier-Stokes equations, allowing us to exactify the system of equations which form the basis of great developments in the mechanics of fluids. These are particularly used in their tridimensional form in meteorology as well as for underground flows and swells. It is likely in the next few years that this trend will move progressively towards fluvial hydraulics, water quality and sedimentology.

Naturally this chapter is followed with a presentation of atmospheric models (Chapter 3) which are used by national meteorological services. In order to establish the first models and resolve them by computer, a number of simplifications were necessary to create the barotrope[1] model from 1950 onwards. These consisted of one layer and zero divergence. The coming of calculators enabled more complicated models further removed from the previous simplified hypotheses. Models filtered with several layers of altitude (known as filtered barocline models baroclines[2]) came into being. It was not until recently that nonhydrostatic 3D models were created, allowing studies of planes with scales of several kilometers. As illustrated by Figure i.1, these models enabled weather forecasting and by consequence provided additional parameters for other models: rain and temperature for hydrologic models and wind and pressure to complete oceanic models looking at the state of the sea.

For hydrologists the task is not so simple. Apart from rain they also need to represent what is happening in the soils under the surface. These two types of flows come together. As we will later see, hydrological models are of a conceptual nature. On the other hand, flows in the soil are well understood from hydrogeological models (Chapter 4). These provide hydrologists and hydraulicists with flows and layers. Hydrogeologists use Navier-Stokes’ equations but also draw on Darcy’s law of macroscopic behavior. Even in this discipline developments were made progressively, first dealing with saturated then non-saturated flows. The latest are the most sophisticated, but still draw on the empirical laws in representing water retention and permeability.

After presenting hydrogeologic models, we will move on to models for fluvial and oceanic currentology (Chapter 5). In the same way as in meteorology, models began as simple as possible and gradually became more complex. 1D models drawn up by the research offices in the 1960s were replaced, bit by bit with 2D models. These models extended their domains towards the river’s upper reaches, as in the case of hydrology, but also towards the lower reaches and ocean where we meet our oceanographer colleagues who use similar, if 3D models. These models draw on empirical relations essentially to represent roughness. The closing models that were used to represent turbulence were again less than satisfactory.

We must then make a detour towards urban hydrogeology (Chapter 6) where we use the same models but in two different manners. Firstly to simulate flow in networks of drainage pipes, and secondly above ground on the roads and sidewalks where excess water flows as a result of rain and overflowing drains. This discipline converges with as hydrogeology as it is concerned with small pools which are strongly impermeable to complex fluid behavior and classic hydraulics. It therefore uses the two types of tools.

By reaching the estuaries, the waters of the rivers get in contact with the *tide* (Chapter 7) and some tidal currents. Oceans and seas are subjected to the movement of the Earth and to the influence of celestial bodies. The tide is considered as the consequence of a succession of actions which get linearly superposed: a hypothesis which remains valid in high waters, but which becomes quite wrong close to the shore. The various models used for the representation of this process are the models in shallow waters belonging to the Saint-Venant type. However, some recent and highly accurate measurements led to the detection of the irregularities and of the anomalies in the propagation of this phenomenon and especially in the vertical profiles of the rates. This suggests then that some 3D models will be developed in the near future.

To complete the consideration of the phenomena which occurs next to the shoreline, the *swell* phenomenon (Chapter 8) will be studied. This phenomenon is quite complex and hard to model. Made of several waves, the swell is quite sensitive to the bathymetry of the beds and to geometry of the coasts generated in high waters by the wind. It develops on large areas and is always changing due to diffraction and reflection phenomena. The first models limited to the refraction of the beds were designed in the 1960s in the research and development offices, but it was only in the 1980s that new and more complex models could be proposed, considering the diffraction processes of the structures and the reflection of the shoreline. Regarding the generation and the propagation of the swells, the models were still until recently quite empirical but thanks to the international WAM (*WAve Model*) group, some new second and third generation models appeared, so the empirical equations are used less often. Going back to the use of the complex theoretical models has occurred for the last few years in order to finely reproduce the transformation conditions of the swells close to the shoreline. However, some empirical equations are still used, like in the case of the modeling of breaking of waves.

All these fields dealt with the flowing process in the waters of the considered domains as being fixed in time and space. However, it is well known that this is not the truth: everything depends on how things are observed, both in time and space; waters loaded of sediments will become concentrated in the water currents, spreading to the rivers and are then thrown into the sea by the rivers. With a time scale of a few days, the violence of the flowing processes is quite striking during a flood, due to their consequences, such as the breaking of meanders or also by the movement, or even the destruction of some structures in the rivers. In some cases, beaches can get entirely swept away by coastal currents. A presentation of the *fluvial and maritime sedimentology* (Chapter 9) is then necessary. The corresponding models are based on the main conservation principles, but depend more on empirical laws: which is due to the evolution of the quality of the sediments during their transport, to the complex interactions with the bed, the banks and the coasts. After some quite unsatisfactory tests based on a 1D modeling process, 2D models did not lead to much improvement: a new generation of 3D hydrodynamic and sedimentological models is being developed and will be able to better understand the evolutions of the rivers and coastal areas in the future.

To complete this part, the transport models of some materials and especially of *oil spills* (Chapter 10), usually due to some accidental pollutions, will be presented in real time.

A presentation, limited to these physical processes, shows that most of them are quite similar to each other and that the theories used still remain in the frame of the main theories of fluid mechanics. Indeed, this is not really true because some physical mediums are so complex that these theoretical models could not be applied everywhere yet.

For instance, the *evapotranspiration* process (Chapter 11) is a fundamental process which regulates the interactions between the ground and the atmosphere. There actually are some very localized processes, which are necessary for any modeling, but which are analyzed locally by some empirical equations dealing with the exchange process between different mediums.

Another large category of models which ensure the conservation of water within the catchment areas is made of the *conceptual reservoir models in hydrology* (Chapter 12) and *in hydro-geology* (Chapter 13). These models do not deal with the infiltration or superficial flowing models very precisely, but they represent them on a very large scale. Without any knowledge regarding the real path of the water within the catchment areas, and without any adapted and global measuring systems, these models still have to be used. However, some recent developments have enabled us to show that some kinematical wave model from some hydraulic and hydro-geological models are beginning to give good results…

Finally, when the behavior of a physical model is too complex for the description of its own components (these empirical models cannot really represent the dynamics of the basin), the *neural network* model (Chapter 14) can be a really useful tool and can always be improved in new situations.

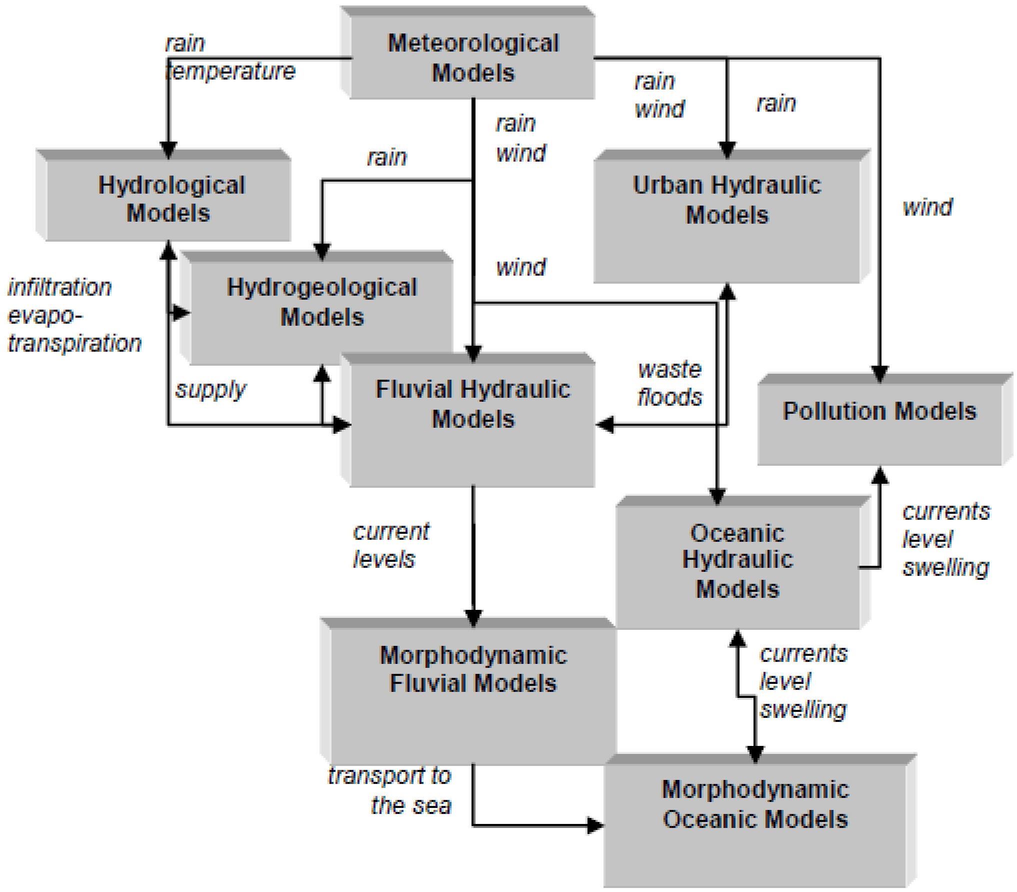
The last chapters deal with the *coupling process of these models* (Chapter 15) and with the different *categories of codes* (Chapter 16). When a river, an estuary, or a coastal area is studied, water is only one of the components which influence the environment: some behaviors integrate a lot, like pollution, morpho-dynamics, the impact of storms, tsunamis or low waters. Their interactions then have to be modeled. The chapter dealing with this topic specifies the principles and the main issues related to this topic.

Figure i.1 illustrates the various equations and interactions between the models. The upstream meteorological models, simulate the atmosphere and provide the time parameters to the hydrologic, hydraulic and hydro-geological models which deal with the flowing water. In addition, they interact with themselves in most of the natural situations. The third layer model leads to the representation of the flowing effects on the fluvial or maritime medium as well as of the issues of the pollution of water.

We encourage the reader to cover all of these developments and take note of the coherence that exists between the many disciplines. Many draw from the same theories, whilst specializing in specific areas of study. Whether it pools with complex soils, rivers with their changing beds and water levels, urban spaces with heterogeneous occupation of soils, coastal interference from swelling and sediment moving with the current — all these environments are yet to reveal their secrets.

Jean-Michel TANGUY

Figure i.1. *Diagram of links between models*



1Introduction written by Jean-Michel TANGUY.

1 Of a state or a forecasting model in which constant surface pressures (isobares) are parallel to these of constant density. The corresponding models are bidimensional.

Chapter 1

Reminders on the Mechanical Properties of Fluids [1]

1.1. Laws of conservation, principles and general theorems

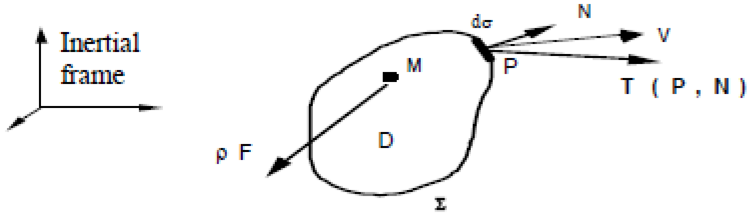
In this chapter, we will go back over the different theorems and principles of mechanics and thermodynamics and express them through Euler's variable using the rules defined in previous volumes for a material domain.

1.1.1. *Mass conservation, continuity equation*

1.1.1.1. *Mass conservation*

PRINCIPAL 1.1 (Figure 1.1). Mass in a material domain is conserved over the course of time.

Figure 1.1.



Taking *D* as a place for observation, noting that the material product for the mass of the domain is zero, we fully accept that the term for accumulation is balanced by the flow crossing the boundaries Σ.

We call $\underline{T}$ the surface effort at every point of Σ of perpendicular angle $\underline{N}$.

Note. As a rule, the perpendicular angle $\underline{N}$ will always be pulled toward the outside.

CLASSIFICATIONS. An integral as defined by volume is represented by $\int_D \varphi d\omega$, a surface integral $\int_D \varphi d\sigma$ and a vector $\underline{A}$.

Faithful to Liebniz' rule, the global equation is written as follows:

$$\frac{\partial}{\partial t}\int_D \rho d\omega + \int_\Sigma \rho(\underline{V} - \underline{W}) \bullet \underline{N}\ d\sigma = 0.$$

Liebniz' rule: if *D(t)* is a deformable domain we can write:

$$\int_D \frac{\partial \rho f_\psi}{\partial t} d\omega = \frac{\partial}{\partial t}\left(\int_D \rho f_\psi d\omega\right) - \int_\Sigma \rho f_\psi \underline{W} \bullet \underline{N}\ d\sigma,$$

$\underline{W}$ therefore represents the localized velocity of displacement for all or part of the interface (boundary or component of the boundary) for *D*.

We notice that on the level of a mobile surface, the local flow $(\underline{V} - \underline{W}) \bullet \underline{N}$ is zero by definition as the control's surface sets the boundaries for the domain. This signifies that even if the fluid runs over the surface with a relative velocity above zero, it will not cross the surface, where the domain *D* is fixed:

$$\int_D \frac{\partial \rho}{\partial t} d\omega + \int_\Sigma \rho \underline{V} \bullet \underline{N}\ d\sigma = 0.$$

$\int_D \frac{\partial \rho}{\partial t} d\omega$ represents the rate of accumulation (or loss) for mass in the domain.

$\int_\Sigma \rho \underline{V} \bullet \underline{N}\ d\sigma$ represents the flow of mass crossing the boundaries of the domain.

The conservation of mass for a domain is expressed as the void sum of a term of accumulation (or loss) of mass in the domain and as a fixed term representing flow of mass to the boundaries of the domain.

The term for flow is represented by $\int_D \underline{\nabla} \bullet \rho \underline{V} \, d\omega,$, using the following theorem.

Theorem for divergence

We will often have the need to pass between localized scripture to global scripture and vice versa. It is therefore important to be able to pass between integrals for volume and integrals for surface reciprocally. We therefore use the theorem of divergence: $\int_D \underline{\nabla} \bullet \underline{\underline{\Psi}} \, d\omega = \int_\Sigma \underline{\underline{\Psi}} \bullet \underline{N} \, d\sigma.$.

This expression shows us that the integral for volume of a greater divergence is equal to the surface flow of the same size.

The pseudo-vector *nabla* is written as

$$\underline{\nabla} = \left| \begin{array}{l} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{array} \right. .$$

It represents the gradient of the size we are considering. The point $\bullet$ represents the contracted product of two tensors (or the scalar product when applied to two vectors). The divergence is therefore equal to the scalar product of the operator *nabla* by the size being considered.

We can therefore consider that the divergence corresponds to the diffusion of a surface term on the inside of the liquid domain. In a more general way, every time we will meet a term for divergence in a localized equation, we will interpret it as the diffusion of an issued term from a surface action.

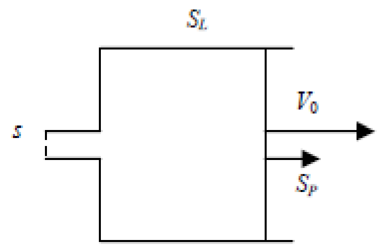
The theorem for divergence applies itself equally as well to vectors as to tensors:

$\underline{V}$ vector $\Rightarrow \int_D \nabla \bullet \underline{V}\ d\omega = \int_\Sigma \underline{V} \bullet \underline{N}\ d\sigma,$

$\underline{\underline{T}}$ tensor $\Rightarrow \int_D \nabla \bullet \underline{\underline{T}}\ d\omega = \int_\Sigma \underline{\underline{T}} \bullet \underline{N}\ d\sigma.$

A tensor is represented by $\underline{\underline{T}}$. It is said to be of second order if it is represented in the form of a 3 × 3 matrix. Its scalar product by a vector is a vector.

Figure 1.2.



EXAMPLE 1.1 (Figure 1.2). We consider a cylinder inside of which a piston moves at velocity V_0. Calculate the debit on entry of an incompressible fluid.

Let us write the conservation of mass for a mobile domain.

$$\frac{\partial}{\partial t}\int_D \rho d\omega + \int_\Sigma \rho(\underline{V} - \underline{W}) \bullet \underline{N}\ d\sigma = 0.$$

If we call *L(t)* the direction in which the piston moves, the volume D is $S_P L$. The term for accumulation is $\frac{\partial}{\partial t}\int_D \rho d\omega = \frac{\partial \rho S_P L}{\partial t} = \rho S_P V_0$. On the other hand, the flow for mass which reduces to the flow of entry is represented by $\int_\Sigma \rho(\underline{V} - \underline{W}) \bullet \underline{N}\ d\sigma = \int_s \rho \underline{V} \bullet \underline{N}\ d\sigma = -\rho V_s s$. By calling *V* the velocity on entry and considering the flow crossing the inner surfaces of the cylinder is zero such that the flow crossing the piston moves at the speed of fluid which wets it *(V = W at* the level of the piston, *W* = 0 for everywhere else).

We therefore deduce that the flow on entry is equal to $Q = V_S s = V_0 S_P$.

1.1.1.2. *Continuity equation*

The local equation which we call the continuity equation is written as $\frac{\partial \rho}{\partial t} + \nabla \bullet \rho \underline{V} = 0$, once we have taken account of the height of the domain of integration as equal zero.

The equation at a fixed Cartesian point with Einstein's grading is as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho V_j)}{\partial x_j}=\frac{\partial \rho}{\partial t}+V_j\frac{\partial \rho}{\partial x_j}+\rho\frac{\partial V_j}{\partial x_j}=\frac{d\rho}{dt}+\rho\frac{\partial V_j}{\partial x_j}=0.$$

For convenience we often use the notation of Einstein. In order to reduce the amount of writing, each time an index is doubled we have a sum on each index. We will note from now on by convention: $s = A_i B_i$ instead of $s = \sum^3{}_{i=1} A_i B_i$. The gradient of a vector is a tensor and is written as $(\underline{\underline{\nabla A}})_{ij} = \frac{\partial A_i}{\partial x_j}$, so that the gradient of a scalar is a vector $(\underline{\underline{\nabla A}})_{ij} = \frac{\partial A_i}{\partial x_j}$ The divergence of a vector represents the scalar product of pseudo-vector *nabla* by the vector being considered. It is written as follows: div $\mathrm{div}(\underline{A}) = \underline{\nabla} \bullet \underline{A} = \frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2}+\frac{\partial A_3}{\partial x_3}=\frac{\partial A_i}{\partial x_i}$ (not to be confused with the gradient of a vector which is its tensor).

1.1.1.3. *Incompressible fluid*

By definition a fluid is incompressible if $\frac{d\rho}{dt}=0$; that is to say if:

$$\underline{\nabla} \bullet \underline{V} = \frac{\partial V_j}{\partial x_j}=0.$$

The divergence of velocity is zero in the case of an incompressible fluid. We establish that the divergence represents the rate of compression/dilation in the domain, therefore its variation in volume. By consequence, an incompressible fluid will allow no variation in volume.

COMMENT 1.1. We are used to considering a fluid as incompressible if its volumic mass is constant. The previous definition encompasses incompressibility of a fluid in permanent movement such that the velocity is perpendicular to the gradient of the volumic mass. It is for example in cases where atmospheric conditions affect the flow where we can consider quasi-horizontal movements of incompressible air on a large scale. (The gradients of volumic mass are important but essentially vertical.)

1.1.2. *Theorem for the conservation of momentum*

1.1.2.1. *Assessment for the momentum*

THEOREM 1.1. *The product in relation to time for the momentum in a material domain is equal to the sum of exterior forces which act in the domain. This also concerns volumic forces (such as weight and electromagnetic forces) and surface forces*.

The densities of volumic forces $\underline{F}$ are densities of forces of distance which act in every part of the fluid. Usually it concerns the action of weight but we should also take into account electromagnetic forces.

We will now do a checklist of surface forces.

As by definition, pressure acts perpendicularly to the surface of the control, and we know that the effort of pressure is represented by $\underline{p} = -p\underline{N}$. . (The minus sign indicates that the pressure moves toward the interior of the domain.)

The forces of friction $\underline{T_f}$ depend on the orientation of the surface of contact. We can therefore consider that they are the product of a tensor of friction (represented by $\underline{\underline{\tau}}$) by local orientation of the normal at the surface. We write $\underline{T_f} = \underline{\underline{\tau}} \bullet \underline{N}$.

The total surface forces $\underline{T}$ therefore correspond to action on the surface of the domain. They also correspond to a tensor called a stress tensor, which is represented by $\underline{\underline{\sigma}}$ .They clearly break down the constraints in pressure and the friction:

$$\underline{T} = -p\underline{N} + \underline{\underline{\tau}} \bullet \underline{N} = (-p\underline{\underline{I}} + \underline{\underline{\tau}}) \bullet \underline{N} = \underline{\underline{\sigma}} \bullet \underline{N}.$$

We can also write:

$T_i = -pN_i + \tau_{ij} N_j = (-p\delta_{ij} + \tau_{ij}) N_j = \sigma_{ij} N_j$.

The effort of normal pressure on a section element is written as follows:

$\underline{p} = -p\underline{N} = -p\underline{\underline{I}} \bullet \underline{N}$.

$\underline{\underline{I}} = \left[\delta_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the unitary tensor or Kronecker's tensor for an element, where:

$$\frac{\partial}{\partial t} \int_D \rho \underline{V}\, d\omega + \int_\Sigma \rho \underline{V} (\underline{V} - \underline{W}) \bullet \underline{N}\, d\sigma = \int_D \rho \underline{F}\, d\omega + \int_\Sigma \underline{T}\, d\sigma$$

$$= \int_D \rho \underline{F}\, d\omega + \int_\Sigma -\mathrm{p}\underline{\mathrm{N}}\, d\sigma + \int_\Sigma \underline{\underline{\tau}} \bullet \underline{\mathrm{N}}\, d\sigma.$$

We find again that in the first member the term for accumulation is a temporal variation for the momentum in the domain. The term for flowing is the difference between the sum of quantities of momentum entering and leaving. This product is balanced by the action at every point of weight and/or the electromagnetic forces as well as by the forces of the surface (pressure and friction).

NOTE 1.1. The theorem for conservation of momentum is a vectoral equation. In other words, it must be projected on the three axes and correspond with the three scalar equations.

The application of the theorem for divergence allows the following equation:

$$\frac{\partial}{\partial t} \int_D \rho \underline{V}\, d\omega + \int_D \underline{\nabla} \bullet \rho \underline{V} (\underline{V} - \underline{W})\, d\omega$$

$$= \int_D \rho \underline{F}\, d\omega + \int_D \underline{\nabla} \bullet \underline{\underline{\tau}}\, d\omega$$

or for a fixed domain:

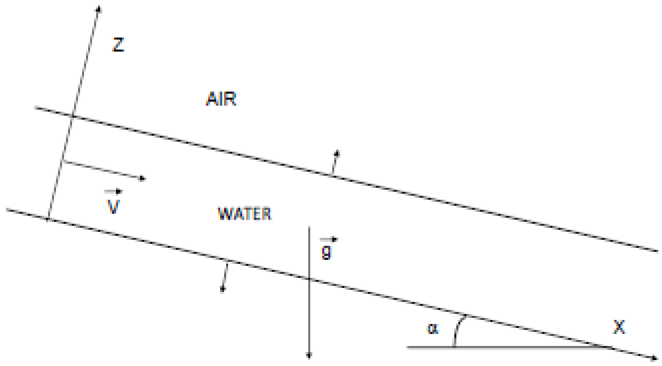
$$\int_D \frac{\partial}{\partial t} (\rho \underline{V})\, d\omega + \int_D \underline{\nabla} \bullet \rho \underline{V} \otimes \underline{V}\, d\omega = \int_D \rho \underline{F}\, d\omega + \int_D \underline{\nabla} \bullet \underline{\underline{\sigma}}\, d\omega$$

$$= \int_D \left(\rho \underline{F} - \underline{\nabla p} + \underline{\nabla} \bullet \underline{\underline{\tau}}\right) d\omega\ .$$

Taking into account the expression for Kronecker's tensor, we have:

$$\int_D -\underline{\nabla} \bullet \left(p \underline{\underline{I}}\right) d\omega = \int_D -\underline{\nabla}\, p\, d\omega\ .$$

Figure 1.3.



EXAMPLE 1.2. Flowing over an inclined plain.

We consider flowing as permanent and parallel to an incompressible (water-based) liquid on an inclined plain at angle ***a*** instead of horizontal. Let us take *h* as its height. The air is immobile above the water and the pressure is equal to the atmospheric pressure.

Calculating the friction on the bottom

Let us take the reasonable assumption that there is no point knowing the nature of flow (laminar or turbulent). It will suffice to choose a domain (e.g. between the bottom, the free surface and the two distant sections *Δx)* and to apply to this domain the general theorems:

– mass conservation, we have *Q* = cte (balanced between the flow of mass entering and leaving);

– momentum in a horizontal projection. The equation is as follows:

$$\int_\Sigma \rho V_X \left(\underline{V} \bullet \underline{N}\right) d\sigma = \int_D \rho g_x \, d\omega + \int_\Sigma -pN_x \, d\sigma +$$

$$\int_\Sigma \left(\underline{\underline{\tau}} \bullet \underline{\mathrm{N}}\right)_{\mathrm{x}} \, d\sigma = \rho g \sin\alpha \left(h\Delta x\right) + \tau_p \Delta x = 0.$$

In effect, the profile for velocity only varies with *z* once the running plain has been determined (see later on the justification with the local resolution). The momentum in the flow leaving the domain is thus equal to the flow entering. Otherwise, pressure is also independent of *x,* and the forces of pressure on the entry and exit sections are balanced. We find the following result: $\boldsymbol{\tau}_p = -\rho g h \sin \alpha$.

If we consider a vertical projection, we find that $p(x\ ,0)$ = $\rho gh\cos\alpha$ + $_{\text{atm}}$ as in this case the weight balances the gap in pressure between the surface and the bottom.

1.1.2.2. *Momentum equation*

The previous equation harks back to the local form of the equation that takes into account the continuity equation.

$$\rho\left(\frac{\partial \underline{V}}{\partial t}+\underline{V}\bullet\underline{\nabla V}\right)=\rho\frac{d\underline{V}}{dt}=-\underline{\nabla p}+\underline{\nabla}\bullet\underline{\underline{\tau}}+\rho\underline{F}.$$

This is in a Cartesian marker with Einstein's formulae:

$$\rho\frac{dV_i}{dt}=\rho\left(\frac{\partial V_i}{\partial t}+V_j\frac{\partial V_i}{\partial x_j}\right)=\frac{\partial\sigma_{ij}}{\partial x_j}+\rho f_i=-\frac{\partial p}{\partial x_i}+\frac{\partial\tau_{ij}}{\partial x_j}+\rho f_i.$$

This equation demonstrates the fundamental principles of mechanics. We assert that acceleration equals the sum of external forces. The surface contact forces contribute to the balance of the divergence which is demonstrated by their diffusion within the fluid.

1.1.3. ***Theorem of kinetic energy***

1.1.3.1. *Assessment of kinetic energy*

THEOREM 1.2. *The variation in kinetic energy in a domain is balanced by the multitude of external and internal forces which act in the area*.

$$\frac{\partial}{\partial t}\int_D\left(\rho\frac{V^2}{2}\right)d\omega+\int_S\rho\frac{V^2}{2}(\underline{V}-\underline{W})\bullet\underline{N}\,d\sigma$$
$$=\int_D\rho\underline{F}\bullet\underline{V}\,d\omega+\int_\Sigma\underline{V}\bullet\underline{\underline{\sigma}}\bullet\underline{N}\ d\sigma+\int_D P_I\,d\omega\ .$$

The first member still represents the product of the larger one we are looking at. In this case, it concerns the kinetic energy in the domain.

The power of the external forces is created as a product by the velocity of each of the external forces (of both surface and volumic forces). This is therefore the power developed by each of the external forces defined previously.

The integral for power of the internal forces is at this point unknown, even if we conceive that the power is the product of the internal forces of cohesion and/or agitation.

Let us take again the example of the flow of fluid on an inclined plain:

$$\int_S \rho \frac{V^2}{2} \underline{V} \bullet \underline{N} \, d\sigma = \int_D \rho \underline{g} \bullet \underline{V} \, d\omega + \int_\Sigma -p\underline{V} \bullet \underline{N} \, d\sigma + \int_\Sigma \underline{V} \bullet \underline{\underline{\tau}} \bullet \underline{N} \, d\sigma + P_D .$$

The kinetic energy flux entering and leaving is zero. The power of the forces of pressure is balanced and in this instance friction is zero (zero friction at the surface and velocity is zero at the bottom). We deduce that the work of the weight is dissipated by the internal friction. From this we have $P_D = -pgQh \sin\alpha \Delta x$.

1.1.3.2. *Generalized Bernoulli theorem*

In an incompressible and constant system, the total variation of mechanical energy (kinetic + potential + pressure energy) in a domain is equal to the loss of charge in the domain.

We can already affirm in going back to the equation for the assessment of kinetic energy that in the case of incompressible fluids:

$$\int_S \rho \left(\frac{V^2}{2} + \frac{p}{\rho} + gz \right) \underline{V} \bullet \underline{N} \, d\sigma$$

$$= \int_D P_I \, d\omega - \int_D \frac{\partial}{\partial t} \left(\rho \frac{V^2}{2} \right) d\omega - \int_D gz \frac{\partial \rho}{\partial t} d\omega + \int_\Sigma \underline{V} \bullet \underline{\underline{\tau}} \bullet \underline{N} \, d\sigma .$$

As the velocity is zero on the inner surfaces, $\int_\Sigma \underline{V} \bullet \underline{\underline{\tau}} \bullet \underline{N} \, d\sigma = 0$, we deduce that there is a permanent movement between the two sections of flow:

$$\Delta \left(\int_\Sigma \left(\rho \frac{V^2}{2} + p + \rho gz \right) d\sigma \right) = \frac{\int_D P_I \partial\omega}{Q} = \frac{P_{ID}}{Q} = \Delta P_T .$$