Vibro-impact Dynamics

Albert C. J. Luo | Yu Guo

VIBRO-IMPACT DYNAMICS

VIBRO-IMPACT DYNAMICS

Albert C. J. Luo
Yu Guo
Southern Illinois University, USA

滚滚红尘

海纳百川

Contents

Preface

This book is about the dynamics of vibro-impact oscillators. Vibro-impact systems extensively exist in engineering and physics. Such vibro-impact systems possess the continuous characteristics as continuous dynamical systems and discrete characteristics by impact discontinuity. Such properties require an appropriate development of discrete maps for such vibro-impact systems to investigate the corresponding complex motions. The rich dynamical behaviors in vibro-impact systems drew the authors' attention on nonlinear dynamical systems. In addition, a better understanding of such vibro-impact systems helps one study nonlinear dynamical systems with discontinuity in engineering and physics.

In 1964, Professor Weiwu Deng experimentally studied the lathe vibration reduction through impact dampers, which originated from the flutter reduction of airplane wines in Russia in the 1930s. Professor Deng found the optimal vibration reduction of the lathes is between 0.6 and 0.8 of the impact restitution coefficient with potential maximum energy dissipation. To further understand the dynamical mechanism of such impact dampers and extend applications in engineering, in 1987 Professor Deng invited the first author to work on this problem with him. After literature survey and experimental setup, it was crucial to develop an appropriate mathematical model to describe the impact dampers and to catch all possible complex motions. Since then, the first author has been working on this topic. Herein he would like to share what his group observed during the past 30 years with other scientists and engineers in vibro-impact systems.

This book mainly focussed on analytical prediction and physical mechanisms of complex motions in vibro-impact systems. After literature survey, in the next two chapters, the theory for nonlinear discrete systems is presented from the recent development of the first author primarily, including the Ying-Yang theory of discrete dynamical systems based on the positive and negative maps in discrete dynamical systems. The complete dynamics of nonlinear discrete dynamical systems is discussed and applied to one- and two-dimensional discrete systems, and a geometric method is discussed for the fractality and complexity of chaos in discrete dynamical systems. From the recent development of the first author, in Chapter 4, the theory of discontinuous dynamical systems is presented as a foundation of studying the dynamics of vibro-impact systems. In Chapter 5, bouncing ball dynamics is discussed as one of the simplest problems in vibro-impact systems to show the corresponding physical motions in this simple model. The dynamics for bouncing initiation and impacting chatter vanishing with stick motion is presented for the first time, which is significant in engineering application. After discussing the bouncing ball with the single map, a simple version of an impact damper is presented in Chapter 6 to show how to develop the complex periodic motions analytically.

The motion switching from one motion to another is discussed through the gazing phenomena. In Chapter 7, the nonlinear dynamics of the Fermi oscillator is discussed as an application in physics. The methodology presented in this book can be applied to other vibro-impact systems in general, and discontinuous dynamical systems in science and engineering.

Finally, the authors would like to thank their family's support for this work, and this book is also dedicated to Professor Weiwu Deng as a good teacher, colleague and friend. The authors hope the materials presented herein will prove durable in the field of science and engineering.

Albert C. J. Luo

Yu Guo

Edwardsville, Illinois, USA

1

Introduction

This book is about the dynamics of vibro-impact oscillators. Vibro-impact systems extensively exist in engineering. Such vibro-impact systems possess the continuous characteristics such as continuous dynamical systems and discrete characteristics by impact discontinuity. Such properties require an appropriate development of discrete maps for such vibro-impact systems to investigate the corresponding complex motions. In this book, a systematic way will be developed through a few simple vibro-oscillators in order to understand the physics of vibro-impact systems in engineering. Before discussing the nonlinear dynamical phenomena and behaviors of vibro-impact oscillators, the theory for nonlinear discrete systems will be presented, and the complete dynamics of nonlinear discrete dynamical systems will be presented, which will be applied to one- and two-dimensional discrete systems, and the a geometric method will be presented to determine the fractality and complexity of chaos in discrete dynamical systems. The theory of discontinuous dynamical systems will be presented as a base from which to study vibro-impact dynamics in engineering. Bouncing ball dynamics will be analytically discussed first and the physical motions shown in a simple model. To understand the chaotic dynamics of a bouncing ball, complex motions in a bouncing ball will be discussed. After discussing the bouncing ball with the single map, a simple version of an impact damper will be presented to show how to develop the complex periodic motions analytically. The Fermi-accelerator will be discussed in detail for application. In this chapter, a brief review about the discrete and discontinuous dynamical systems will be given first, and a brief history of Fermi-oscillators and vibro-impact oscillators will be presented. The book layout and the chapter summary will be given.

1.1 Discrete and Discontinuous Systems

A brief view of recent developments in discrete dynamical systems and discontinuous dynamical systems will be presented herein.

1.1.1 Discrete Dynamical Systems

Consider an n-dimensional discrete dynamical system defined by an implicit vector function $\mathbf{f} : D \to D$ on an open set $D \subset \mathscr{R}^n$, where the vector function is $\mathbf{f} = (f_1, f_2, \ldots, f_n)^{\mathrm{T}} \in \mathscr{R}^n$

and variable vector is $\mathbf{x}_k = (x_{k1}, x_{k2}, \ldots, x_{kn})^{\mathrm{T}} \in D$. For $\mathbf{x}_k, \mathbf{x}_{k+1} \in D$, there is a discrete relation as

$$\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = 0 \tag{1.1}$$

with a parameter vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)^{\mathrm{T}} \in \mathscr{R}^m$.

From the aforementioned discrete dynamical system, nonlinear algebraic equations are used to describe relations between two states of dynamical systems in phase space. Using such discrete relations, if one of two states is given, the rest can be determined but is not unique because the relations are given by nonlinear algebraic equations. In other words, if a final state is given, one can find multiple initial states to satisfy such nonlinear algebraic equations. On the other hand, if an initial state is given, one can find multiple final states to satisfy the nonlinear algebraic equations. For a specific set of parameters, it is very difficult to find the multiple initial or final states globally. One often uses the roughly estimated values as guessed values with linearization to resolve this puzzle. Such a computation can be done locally by the computer. For global behaviors, the discrete states in nonlinear discrete systems become more chaotic.

The nonlinear discrete systems are obtained from the nonlinear difference equations of dynamical systems. The complex dynamical behaviors in such nonlinear discrete systems are observed through the cascade of stable solutions. May (1976) used a one-dimensional discrete map to describe the dynamical processes in biological, economic and social science. The Henon map in the discrete-time dynamic system was introduced by Henon (1976) to simplify the three-dimensional Lorenz equations as a Poincare map, and chaos in such a discrete system was observed numerically by Henon. Such numerical results stimulated more attention on the Henon map. Feigenbaum (1978, 1980) discussed the universal behaviors of one-dimensional systems and qualitatively determined the universal constants for chaos. Marotto (1979) mathematically proved the existence of chaotic behaviors of the Henon map for certain parameters. Curry (1979) used Lyapunov characteristic exponent and frequency spectrum to measure the chaotic behaviors of the Henon map. Collet, Eckmann, and Koch (1981) presented a generalized theory of period-doubling bifurcations in high-dimensional dynamical systems. Cvitanovic, Gunaratne, and Procaccia (1988) investigated topologic properties and multifractality of the Henon map. Luo and Han (1992) presented a geometric approach for the period doubling bifurcation and multifractality of a general one-dimensional iterative map. Gallas (1993) numerically investigated the parameter maps for the Henon map. Zhusubaliyev *et al.* (2000) did the bifurcation analysis of the Henon map and presented a more detailed parameter map. The aforementioned investigations were based on the numerical computation. Gonchenko, Meiss, and Ovsyannikov (2006) discussed the three-dimensional Henon map generated from a homoclinic bifurcation. Hruska (2006) developed a numerical algorithm to model the dynamics of a polynomial diffeomorphism of C^2 on its chain recurrent set, and applied this algorithm to the Henon map. Gonchenko, Gonchenko, and Tatjer (2007) studied the bifurcation behaviors of periodic solutions of the generalized Henon map, and proved the existence of infinite cascades of periodic solutions in a generalized Henon map. Lorenz (2008) adopted a random searching procedure to determine the parameter maps of periodic windows embedded in chaotic solutions of Henon map. Luo (2005a) investigated the mapping dynamics of periodic motions in a non-smooth piecewise system. Luo (2010) presented the Ying-Yang theory in nonlinear discontinuous dynamics. The solutions in nonlinear discrete dynamical

systems can be divided into the "Yang", "Ying", and "Ying-Yang" states. Thus one can obtain the complete solution states for all the parameter regions. Luo and Guo (2010) discussed the complete dynamics of a discrete dynamical system with a Henon map.

Consider a one-dimensional map,

$$P : x_k \to x_{k+1} \text{ with } x_{k+1} = f(x_k, \mathbf{p}) \tag{1.2}$$

where $\mathbf{p}$ is a parameter vector. To determine the period-1 solution (fixed point) of equation (1.2), substitution of $x_{k+1} = x_k$ into equation (1.2) yields the periodic solution $x_k = x_k^*$. The stability and bifurcation of the period-1 solution is presented:

(i) Pitchfork bifurcation (period-doubling bifurcation)

$$\frac{dx_{k+1}}{dx_k} = \left. \frac{df(x_k, \mathbf{p})}{dx_k} \right|_{x_k = x_k^*} = -1. \tag{1.3}$$

(ii) Tangent (saddle-node) bifurcation.

$$\frac{dx_{k+1}}{dx_k} = \left. \frac{df(x_k, \mathbf{p})}{dx_k} \right|_{x_k = x_k^*} = 1. \tag{1.4}$$

With two conditions and fixed points $x_k = x_k^*$, the critical parameter vector $\mathbf{p}_0$ on the corresponding parameter manifolds can be determined. The two kinds of bifurcation for one-dimensional iterative maps are depicted in Figure 1.1. The pitchfork bifurcation involves an infinite cascade of period doubling bifurcations with universal scalings. An exact renormalization theory for period doubling bifurcation was developed in terms of a functional equation by Feigenbaum (1978), and Collet and Eckmann (1980). Helleman (1980a, 1980b) employed an *algebraic* renormalization procedure to determine the rescaling constants. $f(x_k, \mathbf{p})$ has a quadratic maximum at $x_k = x_k^0$. If chaotic solution ensues at $\mathbf{p}_\infty$ via the period-doubling bifurcation, the function $x_{k+1} = f(x_k, \mathbf{p}_\infty)$ is rescaled by a scale factor α and self-similar structure exists near $x_k = x_k^0$. Under the transition to chaos, the period doubling bifurcation can be discussed where two renormalization procedures are presented, that is, the renormalization
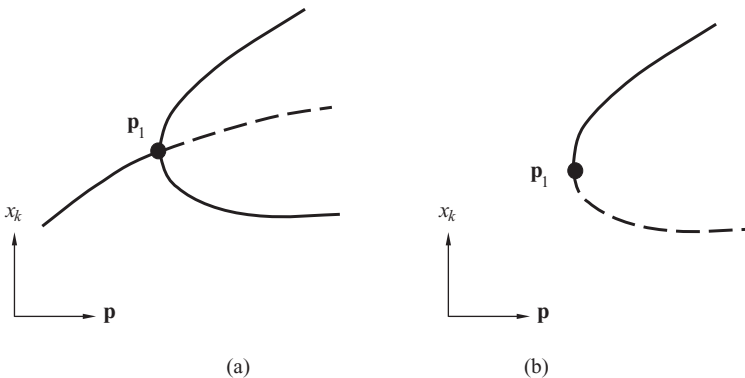

Figure 1.1   Bifurcation types: (a) period-doubling and (b) saddle-node

group approach via the functional equation method outlined by Feigenbaum (1978) and the algebraic renormalization technique described by Helleman (1980a, 1980b).

For two-dimensional invertible maps, the transition from regular motion to chaos takes place via a series of cascades of period-doubling bifurcations. Collet and Eckmann (1980) introduced an exact renormalization method for this situation. However, this exact method is not convenient to use for solving the practical problems. Therefore, Mackay (1983) and Helleman (1980a, 1980b, 1983) have developed a simple analytical approach to renormalize the period doubling bifurcation sequences of the two-dimensional iterative map. This method is similar to the algebraic renormalization technique of one-dimensional iterative map as presented before. For details, the reader can refer to the work of Eckmann and his co-workers. For a conservative system, Eckmann (1981) developed an exact renormalization procedure (see also Collet, Eckmann and Koch, 1981). Greene *et al.* (1981) carried out a more complete study of two-dimensional Hamiltonian maps.

1.1.2   *Discontinuous Dynamical Systems*

Discontinuous dynamical systems extensively exist in engineering. For instance, in mechanical engineering, there are two common and important contacts between two dynamical systems, that is, impact and friction. For example, gear transmission systems possess impact and frictions as a typical example. Such gear transmission systems are used to transmit power between parallel shafts or to change direction. During the power transmission, a pair of two gears forms a resultant dynamical system. Each gear has its own dynamical system connected with shafts and bearings. Because two subsystems are without any connection, the power transmission is completed through the impact and frictions. Because both subsystems are independent of each other except for impacting and sliding together, such two dynamical systems have a common time-varying boundary for impacts, which cause domains for the two dynamical systems to be time varying.

In the early investigation, a piecewise stiffness model was used to investigate dynamics of gear transmission systems. Such a dynamical system is discontinuous, but the corresponding domains for vector fields of the dynamical system are time-independent. For instance, den Hartog and Mikina (1932) used a piecewise linear system without damping to model gear transmission systems, and the symmetric periodic motion in such a system was investigated. For low-speed gear systems, such a linear model gave a reasonable prediction of gear-tooth vibrations. With increasing rotation speed in gear transmission systems, vibrations and noise become serious. Ozguven and Houser (1988) gave a survey on the mathematical models of gear transmission systems. The piecewise linear model and the impact model were two of the main mechanical models to investigate the origin of vibration and noise in gear transmission systems. Natsiavas (1998) investigated a piecewise linear system with a symmetric tri-linear spring, and the stability and bifurcation of periodic motions in such a system were analyzed by the variation of initial conditions. From a piecewise linear model, the dynamics of gear transmission systems were discussed in Comparin and Singh (1989), and Theodossiades and Natsiavas (2000). Pfeiffer (1984) presented an impact model of gear transmissions, and the theoretical and experimental investigations on regular and chaotic motions in the gear box were carried out in Karagiannis and Pfeiffer (1991).

To model vibrations in gear transmission systems, Luo and Chen (2005) gave an analytical prediction of the simplest, periodic motion through a piecewise linear, impacting system. In

addition, the corresponding grazing of periodic motions was observed, and chaotic motions were simulated numerically through such a piecewise linear system. From the local singularity theory in Luo (2005b), the grazing mechanism of the strange fragmentation of such a piecewise linear system was discussed by Luo and Chen (2006). Luo and Chen (2007) used the mapping structure technique to analytically predict arbitrary periodic motions of such a piecewise linear system. In this piecewise linear model, it was assumed that impact locations were fixed, and the perfectly plastic impact was considered. Separation of the two gears occurred at the same location as the gear impact. Compared with the existing models, this model can give a better prediction of periodic motions in gear transmission systems, but the related assumptions may not be realistic for practical transmission systems because all the aforementioned investigations are based on a time-independent boundary or a given motion boundary. To consider the dynamical systems with the time-varying boundary, Luo and O'Connor (2007a, 2007b) proposed a mechanical model to determine mechanism of impacting chatter and stick in gear transmission systems. The analytical conditions for such impacting chatter and stick were developed.

In mechanical engineering, the friction contact between two surfaces of two bodies is an important connection in motion transmissions (for example, clutch systems, brake systems) because two systems are independent except for friction contact. Such a problem possesses time-varying boundary and domains. For such a friction problem, den Hartog (1931) investigated the periodic motion of the forced, damped, linear oscillator contacting a surface with friction. Levitan (1960) investigated the existence of periodic motions in a friction oscillator with a periodically driven base. Filippov (1964) discussed the motion existence of a Coulomb friction oscillator, and presented a differential equation theory with discontinuous right-hand sides. The differential inclusion was introduced via the set-valued analysis for the sliding motion along the discontinuous boundary. Discontinuous differential equations with differential inclusion were summarized in Filippov (1988). However, the Filippov's theory mainly focused on the existence and uniqueness of solutions for non-smooth dynamical systems with differential inclusion. A few approximate treatments of the discontinuous dynamical systems were presented. Such a differential equation theory with discontinuity is difficult to apply to practical problems. Luo (2005b) developed a general theory to handle the local singularity of discontinuous dynamical systems. To determine the sliding and source motions in discontinuous dynamical systems, the imaginary, sink and source flows were introduced in Luo (2005c). The detailed discussions can be referred to Luo (2006, 2009a, 2011b).

On the other hand, Hundal (1979) used a periodic, continuous function to investigate the frequency-amplitude response of such a friction oscillator. Shaw (1986) used the Poincaré mapping to investigate non-stick, periodic motions of a friction oscillator. Feeny (1992) presented a mechanical model to discuss the non-smoothness of the Coulomb friction oscillator. To verify the mechanical model, Feeny and Moon (1994) investigated chaotic dynamics of a dry-friction oscillator experimentally and numerically. Feeny (1996) gave a systematical discussion of the nonlinear dynamical mechanism of stick-slip motion of friction oscillators. Hinrichs, Oestreich, and Popp (1997) discussed the nonlinear phenomena in an impact and friction oscillator under external excitations (also see, Hinrichs, Oestreich, and Popp, 1998). Natsiavas (1998) presented an algorithm to numerically determine the periodic motion and the corresponding stability of piecewise linear oscillators with viscous and dry friction damping (also see, Natsiavas and Verros, 1999). Ko *et al.* (2001) investigated the friction-induced vibrations with and without external excitations. Andreaus and Casini (2002) gave a closed

form solution of a Coulomb friction-impact model without external excitations. Thomsen and Fidlin (2003) gave an approximate estimate of response amplitude for stick-slip motion in a nonlinear friction oscillator. Kim and Perkins (2003) investigated stick-slip motions in a friction oscillator via the harmonic balance/Galerkin method. Li and Feng (2004) investigated the bifurcation and chaos in a friction-induced oscillator with a nonlinear friction model. Pilipchuk and Tan (2004) investigated the dynamical behaviors of a 2DOF mass-damper-spring system contacting on a decelerating rigid strip with friction. Awrejcewicz and Pyryev (2004) gave an investigation on frictional periodic processes by acceleration or braking of a shaft-pad system. Hetzler, Schwarzer, and Seemann (2007) considered a nonlinear friction model to analytically investigate the Hopf-bifurcation in a sliding friction oscillator with application to the low frequency disk brake noise.

In the aforesaid investigations, the conditions for motion switchability to the discontinuous boundary were not considered. Luo and Gegg (2006a) used the local singularity theory of Luo (2005b, 2006) to develop the force criteria for motion switchability on the velocity boundary in a harmonically driven linear oscillator with dry-friction (also see, Luo and Gegg, 2006b). Through such an investigation, the traditional eigenvalue analysis may not be useful for motion switching at the discontinuous boundary. Lu (2007) used the shooting method to show the existence of periodic motions in such a friction oscillator. Luo and Gegg (2007a, 2007b, 2007c) discussed the dynamics of a friction-induced oscillator contacting on time-varying belts with friction. Many researchers still considered the friction model to analyze the disk brake system (for example, Hetzler, Schwarzer, and Seemann, 2007). Luo and Thapa (2007) proposed a new model to model the brake system consisting of two oscillators, and the two oscillators are connected through a contacting surface with friction. Based on this model, the nonlinear dynamical behaviors of a brake system under a periodical excitation were investigated.

The other developments in a non-smooth dynamical system should be addressed herein. Feigin (1970) investigated the *C*-bifurcation in piecewise-continuous systems via the Floquet theory of mappings, and the motion complexity was classified by the eigenvalues of mappings, which were referred to recent publications (for example, Feigin, 1995; di Bernardo *et al.*, 1999). The C-bifurcation is also termed as the grazing bifurcation by many researchers. Nordmark (1991) used "grazing" terminology to describe the grazing phenomena in a simple impact oscillator. No strict mathematical description was given, but the grazing condition (that is, the velocity $dx/dt = 0$ for displacement x) in such an impact oscillator was obtained. From Luo (2005b, 2006, 2009a, 2011b), such a grazing condition is a necessary condition only. The grazing is the tangency between an n-D flow curve of the discontinuous dynamical systems and the boundary surface. From a differential geometry point of view, Luo (2005a) gave the strict mathematic definition of the "grazing", and the necessary and sufficient conditions of the general discontinuous boundary were presented (also see, Luo, 2006, 2009a, 2011b). Nordmark's result is a special case. Nusse and Yorke (1992) used the simple discrete mapping from Nordmark's impact oscillator and showed the bifurcation phenomena numerically. Based on the numerical observation, the sudden change bifurcation in the numerical simulation is called the so-called *border-collision bifurcation*. So, the similar discrete mappings in discontinuous dynamical system were further developed. Especially, Dankowicz and Nordmark (2000) gave a discontinuous mapping in a general way to investigate the grazing bifurcation, and the discontinuous mapping is based on the Taylor series expansion in the neighborhood of the discontinuous boundary. Following the same idea, di Bernardo, Budd, and Champneys *et al.* (2001a, 2001b), di Bernardo, Kowalczyk, and Nordmark (2002) developed a normal form

to describe the grazing bifurcation. In addition, di Bernardo *et al.* (2001c) used the normal form to obtain the discontinuous mapping and numerically observed such a border-collision bifurcation through such a discontinuous mapping. From such a discontinuous mapping and the normal form, the aforementioned bifurcation theory structure was developed for the so-called, co-dimension one dynamical system.

The discontinuous mapping and normal forms on the boundary were developed from the Taylor series expansion in the neighborhood of the boundary. However, the normal form requires the vector field with the C^r-continuity and the corresponding convergence, where the order r is the highest order of the total power numbers in each term of normal form. For piecewise linear and nonlinear systems, the C^1-continuity of the vector field cannot provide enough mathematical base to develop the normal form. The normal form also cannot be used to investigate global periodic motions in such a discontinuous system. Leine, van Campen, and van de Vrande (2000) used the Filippov theory to investigate bifurcations in nonlinear discontinuous systems. However, the discontinuous mapping techniques were employed to determine the bifurcation via the Floquet multiplier. More discussion about the traditional analysis of bifurcation in non-smooth dynamical systems can be found in Zhusubaliyev and Mosekilde (2003). From recent research, the Floquet multiplier also may not be adequate for periodic motions involved with the grazing and sliding motions in non-smooth dynamical systems. Therefore, Luo (2005b) proposed a general theory for the local singularity of non-smooth dynamical systems on connectable domains (also see, Luo, 2006, 2009a, 2011b). From recent developments in Luo (2008a, 2008b, 2008c), a generalized theory for discontinuous systems on time-varying domains was presented in Luo (2009a). Further development of discontinuous dynamical systems can be found in Luo (2011b). Such a theory will be used in vibro-impact systems.

1.2 Fermi Oscillators and Impact Problems

The Fermi acceleration oscillator was first presented by Fermi (1949), which was used to explain the very high energy of the cosmic ray. Since then, such an oscillator has been extensively investigated to interpret many physical and mechanical phenomena. Ulam (1961) pointed out the statistical properties of a particle in the Fermi oscillator. Zaslavskii and Chirikov (1964) gave a comprehensive study of the Fermi acceleration mechanism in the one-dimensional case. Lieberman and Lichtenberg (1972) discussed the stochastic and adiabatic behavior of particles accelerated by periodic forcing, and the analysis was based on the model presented by Zaslavskii and Chirikov (1964). The corresponding stability of periodic motion was discussed. Such results can be found in Lieberman and Lichtenberg (1992). Pustylnikov (1978) discussed the reducibility of the non-autonomous system in the normal form in the neighborhood of an equilibrium point, and gave a detailed description of the Fermi-acceleration mechanism (also see, Pustylnikov, 1995). Jose and Cordery (1986) studied a quantum Fermi-accelerator consisting of a particle moving between a fixed wall and a periodic oscillator. Celaschi and Zimmerman (1987) made an experimental investigation into observing the period-doubling route to chaos for a one-dimensional system with two parameters. Kowalik, Franaszek, and Pieranski (1988) made an experimental investigation into the chaotic behaviors of a ball in the bouncing ball system, and used the Zaslavski mapping to give an analysis. Luna-Acosta (1990) investigated the dynamics of the Fermi accelerator subject to a viscous

friction. Warr *et al.* (1996) determined experimentally the probability distribution function for a single-particle vibrating in one dimension. Saif *et al.* (1998) studied the dynamics of both classical and quantum Fermi acceleration oscillators and determined the dynamical localization of position and momentum for a modulation amplitude. Lopac and Dananic (1998) investigated chaotic dynamics and energy conservation in a gravitationally driven Fermi accelerator. Bouchet, Cecconi, and Vulpiani (2004) presented a simple stochastic system to generate anomalous diffusion of both position and velocity for the Fermi accelerator. Ladeira and da Silva (2008) completed scaling analysis on a Fermi-Ulam simplified accelerator, and Leonel, McClintock, and da Silva (2004) discussed the effect of a time-dependent perturbation on a Fermi accelerator model using the discrete dynamical systems. Leonel and McClintock (2006) discussed the influence of dissipation on a simplified Fermi-Ulam accelerator model. Leonel and de Carvalho (2007) presented the Fermi accelerator model with inelastic collisions through a two-dimensional nonlinear area-contracting map. Karlis *et al.* (2007) investigated the Fermi acceleration of an ensemble of non-interacted particles in two stochastic two-moving walls in the Fermi-Ulam model. Leonel, da Silva and Kampherst (2004) discussed the dynamical properties of a prototype for the Fermi acceleration through two nonlinear terms. Kamphorst, Leonel, and da Silva (2007) numerically studied the energy change of a particle bouncing in a time-varying billiard. Leonel and Livorati (2008) presented the average velocities behavior on a dissipative Fermi acceleration model with a scaling approach and the scaling characteristics of a breathing circular billiard was investigated by Ladeira and da Silva (2008). All the aforesaid investigations were based on the one motion state with impact. The motion switching was not considered.

   Similar studies of impacting systems have also been carried out because impact is an important phenomenon in mechanical engineering. For instance, Holmes (1982) investigated the dynamics of repeated impacts of a ball with a sinusoidal vibrating table. It was assumed that the mass of the ball is much smaller than the mass of the table, and the ball and table always impact at the same position. Bapat and Popplewell (1983) investigated the asymptotically stable periodic motions of a ball in an impact-pair. Shaw and Holmes (1983) studied harmonic, sub-harmonic and chaotic motions of a single-degree of freedom nonlinear oscillator and analyzed the corresponding bifurcations. Whiston (1987) studied the steady-state, vibro-impacting responses of a one-dimensional, harmonically excited, linear oscillator. Bapat (1988) used the Fourier series and perturbation method to determine the stability regions of two equi-spaced impacts and motion of an impact-pair under prescribed periodic displacement. Nordmark (1991) studied the singularities caused by grazing impact in a single degree of freedom impact oscillator. Foale and Bishop (1992) discussed a forced linear oscillator with instantaneous impacts at one or two stops. Budd and Dux (1994) investigated the chattering behavior of a periodically forced, single-degree-of-freedom impact oscillator with a restitution law for each impact. Foale (1994) tried to determine bifurcations in a sinusoidally driven impact oscillator analytically. Bapat (1995) studied the motion of an inclined impact damper with friction and collision on both sides. Luo and Han (1996) presented a reasonable model to investigate the dynamical behaviors of a bouncing ball with a sinusoidally vibrating table. Luo (2002) investigated the stability and bifurcation of unsymmetrical periodic motions in a horizontal impact oscillator with a periodic excitation. Giusepponi, Marchesoni, and Borromeo (2004) discussed the randomness of motion of the bouncing ball on a sinusoidally vibrating table. Luo (2004) discussed the stability, saddle-node and period-doubling bifurcations for the LR model motion in a horizontal impact oscillator. Luo (2005a) presented the

mapping dynamics of periodic motions in a non-smooth piecewise system. To understand the complexity in discontinuous dynamical systems, Luo (2005b, 2006) developed a theory of the non-smooth dynamical systems on connectable and accessible sub-domains. Luo and Chen (2006) applied such a theory to investigate the grazing bifurcations and periodic motions in an idealized gear transmission system with impacts. Luo and Gegg (2006a, 2006b) used such a theory to develop the force criteria of stick and non-stick motion in harmonically forced, friction-induced oscillators. Luo (2007) discussed switching bifurcations of a flow to the separation boundary. Luo and Rapp (2007) used the switching bifurcations to study the switching dynamics of flows from one domain into another adjacent domain in a periodically driven, discontinuous dynamical system. Luo and O'Connor (2009a, 2009b) discussed the dynamics mechanism of impact chatters and possible stick motions in a gear transmission system. It was observed that the moving boundaries are controlled by other dynamical systems. The dynamics mechanism of impact chatters and possible stick motions in a gear transmission system were investigated. In the gear model, the two boundaries are movable. However, the Fermi-acceleration oscillator possesses static and time-varying boundaries in phase space for impacts and motion switching. In existing investigations on the Fermi-acceleration oscillator or impact oscillators, the dynamical systems are not switched except for impacts. From the above discussion, the vibro-impact dynamics is extensively used in engineering and physics. The mechanical mechanisms and motion complexity of vibro-impact oscillators need to be understood. In this book, vibro-impact dynamics will be presented.

1.3   Book Layout

To help readers easily read this book, the main contents are summarized as follows.

In Chapter 2, basic concepts of nonlinear discrete systems will be presented. The local and global theory of stability and bifurcation for nonlinear discrete systems will be discussed. The stability switching and bifurcation on specific eigenvectors of the linearized system at fixed points under specific periods will be presented. The higher singularity and stability for nonlinear discrete systems on the specific eigenvectors will be presented.

In Chapter 3, the theory of the complete dynamics based on positive and negative discrete maps will be discussed. The basic routes of periodic solutions to chaos will be presented. The complete dynamics of a discrete dynamical system with the Henon map will be discussed briefly for a better understanding of the complete dynamics of nonlinear discrete systems. The self-similarity and multifractality of chaos generated by period-doubling bifurcation will be discussed via a geometrical approach, and a discrete system with the logistic map will be used to discuss the fractality. Finally, the complete dynamics of the logistic map will be discussed analytically to show many branches of periodic solutions to chaos via period-doubling, which is much richer than numerical simulations.

In Chapter 4, a general theory for the passability of a flow to a specific boundary in discontinuous dynamical systems will be presented from Luo (2011b). The G-functions for discontinuous dynamical systems will be introduced, and the passability of a flow from a domain to an adjacent one will be discussed. The full and half sink and source flows to the boundary will be presented with the help of real and imaginary flows. The passability of a flow to the boundary will be discussed in discontinuous dynamical systems, and the corresponding switching bifurcations between the passable and non-passable flows will be presented.

In Chapter 5, the nonlinear dynamics of a ball bouncing on a periodically oscillating table will be discussed as the simplest example of vibro-impact systems. The analytical solutions of period-1 and period-2 motions of the bouncing ball will be presented and the analytical condition of the corresponding stability and bifurcation will be presented. From mapping structures, the analytical prediction of the period-m motions will be discussed. From the theory of discontinuous dynamical systems, the analytical condition of the initialization of a ball bouncing on the vibrating table will be presented, and the impact chatter of the bouncing ball on the oscillating table will be discussed. The bouncing ball presented herein is also to show how to construct discrete maps in practical problems.

In Chapter 6, domains and boundaries for complex dynamics of impact pairs will be introduced first from impact discontinuity. The analytical periodic motions for simple impact sequences in impact pairs will be discussed, and the conditions of stability and bifurcations of such periodic motions will be developed. From generic impact mappings, the mapping structures for motions with complex impact sequences will be discussed. However, the switching complexity of motion is from grazing, and the stick motion vanishing is a key to induce impact motions in the impact pair. Thus, analytical conditions for stick and grazing motions will be discussed. The periodic motions and the corresponding stability and bifurcation in such an impact pair will be discussed. Parameter maps with different motions will be presented for a better view of motions with different parameters.

In Chapter 7, in order to understand the nonlinear dynamics of a flow from one domain to another domain, mapping dynamics of discontinuous dynamics systems will be presented, which is a generalized symbolic dynamics. Using the mapping dynamics, one can determine periodic and chaotic dynamics of discontinuous dynamical systems, and complex motions can be classified through mapping structure. The mechanism of motion switching of a particle in such a generalized Fermi oscillator will be discussed through the theory of discontinuous dynamical systems, and the corresponding analytical conditions for the motion switching will be presented. The mapping structures for periodic motions will be discussed, and such periodic motions in the Fermi oscillator will be discussed analytically. From the analytical prediction, parameter maps of regular and chaotic motions will be presented for a global view of motions in the Fermi oscillator.

2

Nonlinear Discrete Systems

In this chapter, a theory for nonlinear discrete systems will be presented. The local and global theory of stability and bifurcation for nonlinear discrete systems will be discussed. The stability switching and bifurcation on specific eigenvectors of the linearized system at fixed points under a specific period will be presented. The higher order singularity and stability for nonlinear discrete systems on the specific eigenvectors will be presented.

2.1 Definitions

Definition 2.1   For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$ with $\alpha \in \mathbb{Z}$, consider a vector function $\mathbf{f}_\alpha : \Omega_\alpha \times \Lambda \to \Omega_\alpha$ which is $C^r(r \geq 1)$-continuous, and there is a discrete (or difference) equation in a form of

$$\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) \text{ for } \mathbf{x}_k, \mathbf{x}_{k+1} \in \Omega_\alpha, k \in \mathbb{Z} \text{ and } \mathbf{p}_\alpha \in \Lambda. \tag{2.1}$$

With an initial condition of $\mathbf{x}_k = \mathbf{x}_0$, the solution of equation (2.1) is given by

$$\mathbf{x}_k = \underbrace{\mathbf{f}_\alpha(\mathbf{f}_\alpha(\cdots(\mathbf{f}_\alpha(\mathbf{x}_0, \mathbf{p}_\alpha))))}_{k}$$
$$\text{for } \mathbf{x}_k \in \Omega_\alpha, k \in \mathbb{Z} \text{ and } \mathbf{p} \in \Lambda. \tag{2.2}$$

  (i) The difference equation with the initial condition is called a *discrete dynamical system*.
 (ii) The vector function $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ is called a *discrete vector field* on Ω_α.
(iii) The solution $\mathbf{x}_k$ for each $k \in \mathbb{Z}$ is called a *flow* of discrete system.
(iv) The solution $\mathbf{x}_k$ for all $k \in \mathbb{Z}$ on domain Ω_α is called the trajectory, phase curve or orbit of the discrete dynamical system, which is defined as

$$\Gamma = \{\mathbf{x}_k \,|\, \mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) \text{ for } k \in \mathbb{Z} \text{ and } \mathbf{p}_\alpha \in \Lambda\} \subseteq \cup_\alpha \Omega_\alpha. \tag{2.3}$$

 (v) The discrete dynamical system is called *a uniform discrete system* if

$$\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \text{ for } k \in \mathbb{Z} \text{ and } \mathbf{x}_k \in \Omega_\alpha. \tag{2.4}$$
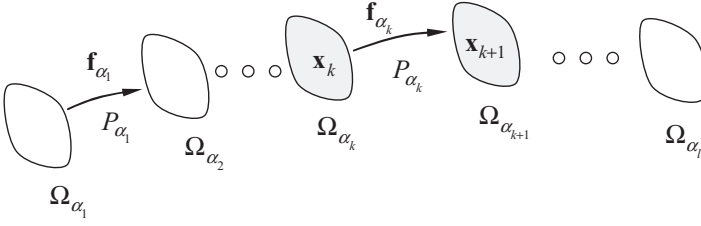
Figure 2.1   Maps and vector functions on each sub-domain for discrete dynamical system

Otherwise, this discrete dynamical system is called a *non-uniform discrete system*.

Definition 2.2   For the discrete dynamical system in equation (2.1), the relation between state $\mathbf{x}_k$ and state $\mathbf{x}_{k+1}$ ($k \in \mathbb{Z}$) is called a discrete map if

$$P_\alpha : \mathbf{x}_k \xrightarrow{\mathbf{f}_\alpha} \mathbf{x}_{k+1} \text{ and } \mathbf{x}_{k+1} = P_\alpha \mathbf{x}_k \tag{2.5}$$

with the following properties:

$$P_{(k,l)} : \mathbf{x}_k \xrightarrow{\mathbf{f}_{\alpha 1}, \mathbf{f}_{\alpha 2}, \dots, \mathbf{f}_{\alpha l}} \mathbf{x}_{k+l} \text{ and } \mathbf{x}_{k+l} = P_{\alpha_l} \circ P_{\alpha_{l-1}} \circ \cdots \circ P_{\alpha_1} \mathbf{x}_k \tag{2.6}$$

where

$$P_{(k;l)} = P_{\alpha_l} \circ P_{\alpha_{l-1}} \circ \cdots \circ P_{\alpha_1}. \tag{2.7}$$

If $P_{\alpha_l} = P_{\alpha_{l-1}} = \cdots = P_{\alpha_1} = P_\alpha$, then

$$P_{(\alpha;l)} \equiv P_\alpha^{(l)} = P_\alpha \circ P_\alpha \circ \cdots \circ P_\alpha \tag{2.8}$$

with

$$P_\alpha^{(n)} = P_\alpha \circ P_\alpha^{(n-1)} \text{ and } P_\alpha^{(0)} = \mathbf{I}. \tag{2.9}$$

The total map with l-different sub-maps is shown in Figure 2.1. The map P_{α_k} with the relation function $\mathbf{f}_{\alpha_k}(\alpha_k \in \mathbb{Z})$ is given by equation (2.5). The total map $P_{(k,l)}$ is given in equation (2.7). The domains $\Omega_{\alpha_k}(\alpha_k \in \mathbb{Z})$ can fully overlap each other or can be completely separated without any intersection.

Definition 2.3   For a vector function in $\mathbf{f}_\alpha \in \mathscr{R}^n$, $\mathbf{f}_\alpha : \mathscr{R}^n \to \mathscr{R}^n$. The operator norm of $\mathbf{f}_\alpha$ is defined by

$$||\mathbf{f}_\alpha|| = \sum_{i=1}^{n} \max_{||\mathbf{x}_k|| \leq 1, \mathbf{p}_\alpha} |f_{\alpha(i)}(\mathbf{x}_k, \mathbf{p}_\alpha)|. \tag{2.10}$$

For an $n \times n$ matrix $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) = \mathbf{A}_\alpha \mathbf{x}_k$ and $\mathbf{A}_\alpha = (a_{ij})_{n \times n}$, the corresponding norm is defined by

$$||\mathbf{A}_\alpha|| = \sum_{i,j=1}^{n} |a_{ij}|. \tag{2.11}$$

Definition 2.4 For $\Omega_\alpha \subseteq \mathcal{R}^n$ and $\Lambda \subseteq \mathcal{R}^m$ with $\alpha \in \mathbb{Z}$, the vector function $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ with $\mathbf{f}_\alpha : \Omega_\alpha \times \Lambda \to \mathcal{R}^n$ is differentiable at $\mathbf{x}_k \in \Omega_\alpha$ if

$$\left. \frac{\partial \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\partial \mathbf{x}_k} \right|_{(\mathbf{x}_k, \mathbf{p})} = \lim_{\Delta \mathbf{x}_k \to \mathbf{0}} \frac{\mathbf{f}_\alpha(\mathbf{x}_k + \Delta \mathbf{x}_k, \mathbf{p}_\alpha) - \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\Delta \mathbf{x}_k}. \tag{2.12}$$

$\partial \mathbf{f}_\alpha / \partial \mathbf{x}_k$ is called the spatial derivative of $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ at $\mathbf{x}_k$, and the derivative is given by the Jacobian matrix

$$\frac{\partial \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\partial \mathbf{x}_k} = \left[\frac{\partial f_{\alpha(i)}}{\partial x_{k(j)}} \right]_{n \times n}. \tag{2.13}$$

Definition 2.5 For $\Omega_\alpha \subseteq \mathcal{R}^n$ and $\Lambda \subseteq \mathcal{R}^m$, consider a vector function $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ with $\mathbf{f} : \Omega_\alpha \times \Lambda \to \mathcal{R}^n$ where $\mathbf{x}_k \in \Omega_\alpha$ and $\mathbf{p} \in \Lambda$ with $k \in \mathbb{Z}$. The vector function $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ satisfies the Lipschitz condition

$$||\mathbf{f}(\mathbf{y}_k, \mathbf{p}) - \mathbf{f}(\mathbf{x}_k, \mathbf{p})|| \le L ||\mathbf{y}_k - \mathbf{x}_k|| \tag{2.14}$$

with $\mathbf{x}_k, \mathbf{y}_k \in \Omega_\alpha$ and L a constant. The constant L is called the Lipschitz constant.

2.2 Fixed Points and Stability

Definition 2.6 Consider a discrete, dynamical system $\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ in equation (2.4).

(i) A point $\mathbf{x}_k^* \in \Omega_\alpha$ is called a fixed point or period-1 solution of a discrete nonlinear system $\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ under a map P_α if for $\mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}_k^*$

$$\mathbf{x}_k^* = \mathbf{f}_\alpha(\mathbf{x}_k^*, \mathbf{p}) \tag{2.15}$$

The linearized system of the nonlinear discrete system $\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ in equation (2.4) at the fixed point $\mathbf{x}_k^*$ is given by

$$\mathbf{y}_{k+1} = DP_\alpha(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k = D\mathbf{f}_\alpha(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k \tag{2.16}$$

where

$$\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^* \text{ and } \mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_{k+1}^*. \tag{2.17}$$

(ii) A set of points $\mathbf{x}_j^* \in \Omega_{\alpha_j}(\alpha_j \in \mathbb{Z})$ is called the fixed point set or period-1 point set of the total map $P_{(k;l)}$ with l-different sub-maps in nonlinear discrete system of equation (2.2) if

$$\mathbf{x}_{k+j+1}^* = \mathbf{f}_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}}) \text{ for } j \in \mathbb{Z}_+ \text{ and } j' = \text{mod}(j, l) + 1;$$
$$\mathbf{x}_{k+\text{mod}(j,l)}^* = \mathbf{x}_k^*. \tag{2.18}$$

The linearized equation of the total map $P_{(k;l)}$ gives

$$\mathbf{y}_{k+j+1} = DP_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}})\mathbf{y}_{k+j} = D\mathbf{f}_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}})\mathbf{y}_{k+j}$$
$$\text{with } \mathbf{y}_{k+j+1} = \mathbf{x}_{k+j+1} - \mathbf{x}_{k+j+1}^* \text{ and } \mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_{k+j}^* \tag{2.19}$$
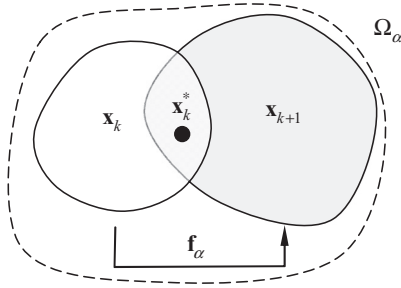$$\text{for } j \in \mathbb{Z}_+ \text{ and } j' = \text{mod}(j, l) + 1.$$

Figure 2.2   A fixed point between domains Ω_k and Ω_{k+1} for a discrete dynamical system

The resultant equation for each individual map is

$$\mathbf{y}_{k+j+1} = DP_{(k,l)}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j} \text{ for } j \in \mathbb{Z}_+ \tag{2.20}$$

where

$$DP_{(k,n)}(\mathbf{x}_k^*, \mathbf{p}) = \prod_{j=l}^{1} DP_{\alpha_j}(\mathbf{x}_{k+j-1}^*, \mathbf{p})$$

$$= DP_{\alpha_l}(\mathbf{x}_{k+l-1}^*, \mathbf{p}_{\alpha_n}) \cdot \cdots \cdot DP_{\alpha_2}(\mathbf{x}_{k+1}^*, \mathbf{p}_{\alpha_2}) \cdot DP_{\alpha_1}(\mathbf{x}_k^*, \mathbf{p}_{\alpha_1}) \tag{2.21}$$

$$= D\mathbf{f}_{(\alpha_l)}(\mathbf{x}_{k+l-1}^*, \mathbf{p}_{\alpha_n}) \cdot \cdots \cdot D\mathbf{f}_{(\alpha_2)}(\mathbf{x}_{k+1}^*, \mathbf{p}_{\alpha_2}) \cdot D\mathbf{f}_{(\alpha_1)}(\mathbf{x}_k^*, \mathbf{p}_{\alpha_1}).$$

The fixed point $\mathbf{x}_k^*$ lies in the intersected set of two domains Ω_k and Ω_{k+1}, as shown in Figure 2.2. In the vicinity of the fixed point $\mathbf{x}_k^*$, the incremental relations in the two domains Ω_k and Ω_{k+1} are different. In other words, setting $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$ and $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_{k+1}^*$, the corresponding linearization is generated as in equation (2.16). Similarly, the fixed point of the total map with n-different sub-maps requires the intersection set of two domains Ω_k and Ω_{k+n}, there is a set of equations to obtain the fixed points from equation (2.18). The other values of fixed points lie in different domains, that is, $\mathbf{x}_j^* \in \Omega_j$ $(j = k + 1, k + 2, \ldots, k + n - 1)$, as shown in Figure 2.3.

The corresponding linearized equations are given in equation (2.19). From equation (2.20), the local characteristics of the total map can be discussed as a single map. Thus, the dynamical characteristics for the fixed point of the single map will be discussed comprehensively, and the
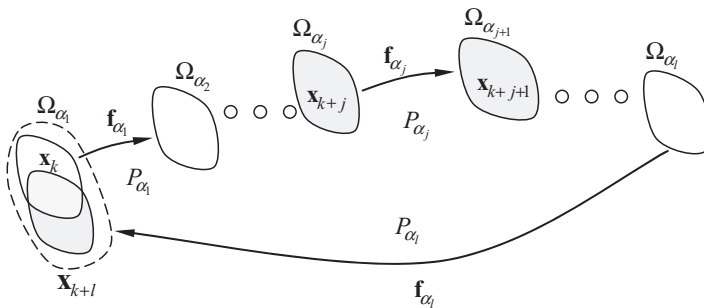

Figure 2.3   Fixed points with l-maps for discrete dynamical system

fixed points for the resultant map are applicable. The results can be extended to any period-m flows with $P^{(m)}$.

Definition 2.7 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in equation (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in equation (2.16). The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_1 real eigenvalues $|\lambda_j| < 1$ ($j \in N_1$), n_2 real eigenvalues $|\lambda_j| > 1$ ($j \in N_2$), n_3 real eigenvalues $\lambda_j = 1$ ($j \in N_3$), and n_4 real eigenvalues $\lambda_j = -1$ ($j \in N_4$). $N = \{1, 2, \ldots, n\}$ and $N_i = \{i_1, i_2, \cdots, i_{n_i}\} \cup \emptyset$ ($i = 1, 2, 3, 4$) with $i_m \in N$ ($m = 1, 2, \ldots, n_i$) and $\Sigma_{i=1}^3 n_i = n$. $N_i \subseteq N \cup \emptyset$, $\cup_{i=1}^3 N_i = N$, $N_i \cap N_p = \emptyset$ ($p \neq i$).$N_i = \emptyset$ if $n_i = 0$. The corresponding eigenvectors for contraction, expansion, invariance, and flip oscillation are $\{\mathbf{v}_j\}$ ($j \in N_i$) ($i = 1, 2, 3, 4$), respectively. The stable, unstable, invariant, and flip subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in equation (2.16) are linear subspace spanned by $\{\mathbf{v}_j\}$ ($j \in N_i$) ($i = 1, 2, 3, 4$), respectively, that is,

$$
\mathscr{E}^{\mathrm{s}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ |\lambda_j| < 1, j \in N_1 \subseteq N \cup \emptyset \end{array} \right. \right\};
$$

$$
\mathscr{E}^{\mathrm{u}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ |\lambda_j| > 1, j \in N_2 \subseteq N \cup \emptyset \end{array} \right. \right\};
$$

$$
\mathscr{E}^{\mathrm{i}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = 1, j \in N_3 \subseteq N \cup \emptyset \end{array} \right. \right\};
$$

$$
\mathscr{E}^{\mathrm{f}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = -1, j \in N_4 \subseteq N \cup \emptyset \end{array} \right. \right\}.
$$

(2.22)

where

$$
\mathscr{E}^{\mathrm{s}} = \mathscr{E}^{\mathrm{s}}_{\mathrm{m}} \cup \mathscr{E}^{\mathrm{s}}_{\mathrm{o}} \cup \mathscr{E}^{\mathrm{s}}_{\mathrm{z}} \text{with}
$$

$$
\mathscr{E}^{\mathrm{s}}_{\mathrm{m}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ 0 < \lambda_j < 1, j \in N_1^{\mathrm{m}} \subseteq N \cup \emptyset \end{array} \right. \right\};
$$

$$
\mathscr{E}^{\mathrm{s}}_{\mathrm{o}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ -1 < \lambda_j < 0, j \in N_1^{\mathrm{o}} \subseteq N \cup \emptyset \end{array} \right. \right\};
$$

$$
\mathscr{E}^{\mathrm{s}}_{\mathrm{z}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = 0, j \in N_1^{\mathrm{z}} \subseteq N \cup \emptyset \end{array} \right. \right\}
$$

(2.23)

$$
\mathscr{E}^{\mathrm{u}} = \mathscr{E}^{\mathrm{u}}_{\mathrm{m}} \cup \mathscr{E}^{\mathrm{u}}_{\mathrm{o}} \text{with}
$$

$$
\mathscr{E}^{\mathrm{u}}_{\mathrm{m}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j > 1, j \in N_2^{\mathrm{m}} \subseteq N \cup \emptyset \end{array} \right. \right\};
$$

$$
\mathscr{E}^{\mathrm{s}}_{\mathrm{o}} = span \left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ -1 < \lambda_j, j \in N_2^{\mathrm{o}} \subseteq N \cup \emptyset \end{array} \right. \right\};
$$

(2.24)

where subscripts "m" and "o" represent the monotonic and oscillatory evolutions.

Definition 2.8   Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in equation (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in equation (2.16). The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ has complex eigenvalues $\alpha_j \pm \mathbf{i}\beta_j$ with eigenvectors $\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j$ ($j \in \{1, 2, \dots, n\}$) and the base of vector is

$$\mathbf{B} = \left\{ \mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{u}_j, \mathbf{v}_j, \dots, \mathbf{u}_n, \mathbf{v}_n \right\}. \tag{2.25}$$

The stable, unstable, center subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}_k(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in equation (2.16) are linear subspaces spanned by $\{\mathbf{u}_j, \mathbf{v}_j\}$ ($j \in N_i, i = 1, 2, 3$), respectively. Set $N = \{1, 2, \cdots, n\}$ plus $N_i = \{i_1, i_2, \dots, i_{n_i}\} \cup \emptyset \subseteq N \cup \emptyset$ with $i_m \in N$ ($m = 1, 2, \dots, n_i$) and $\Sigma_{i=1}^3 n_i = n$. $\cup_{i=1}^3 N_i = N$ with $N_i \cap N_p = \emptyset (p \neq i)$. $N_i = \emptyset$ if $n_i = 0$. The stable, unstable, center subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in equation (2.16) are defined by

$$
\begin{aligned}
\mathscr{E}^{\mathrm{s}} &= span \left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} < 1, \\ (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_1 \subseteq \{1, 2, \dots, n\} \cup \emptyset \end{array} \right. \right\}; \\
\mathscr{E}^{\mathrm{u}} &= span \left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} > 1, \\ (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_2 \subseteq \{1, 2, \dots, n\} \cup \emptyset \end{array} \right. \right\}; \\
\mathscr{E}^{\mathrm{c}} &= span \left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} = 1, \\ (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_3 \subseteq \{1, 2, \dots, n\} \cup \emptyset \end{array} \right. \right\}.
\end{aligned}
\tag{2.26}
$$

Definition 2.9   Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in equation (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in equation (2.16). The fixed point or period-1 point is *hyperbolic* if no eigenvalues of $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ are on the unit circle (that is, $|\lambda_i| \neq 1$ for $i = 1, 2, \dots, n$).

Theorem 2.1   *Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in equation (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_j = \mathbf{x}_j - \mathbf{x}_k^*$ and $j = k, k+1$) in equation (2.16). The eigenspace of $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ (that is, $\mathscr{E} \subseteq \mathscr{R}^n$) in the linearized dynamical system is expressed by direct sum of three subspaces*

$$\mathscr{E} = \mathscr{E}^s \oplus \mathscr{E}^u \oplus \mathscr{E}^c. \tag{2.27}$$

where $\mathscr{E}^s$, $\mathscr{E}^u$ and $\mathscr{E}^c$ are the stable, unstable, and center subspaces, respectively.

Proof: The proof can be found in Luo (2012c).   ∎