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Zhiqiang Li

Ergodic Theory of Expanding Thurston Maps

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Zhiqiang Li

Ergodic Theory of Expanding Thurston Maps

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Zhiqiang Li
Institute for Mathematical Sciences
Stony Brook University
Stony Brook, NY
USA

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To the loving memory of my grandmother
Fengxian Shen

Preface

This monograph came out of my thesis work under the supervision of my Ph.D. advisor Mario Bonk during my graduate studies at the University of Michigan, Ann Arbor, and later at the University of California, Los Angeles. It focuses on the dynamics, more specifically ergodic theory, of some continuous branched covering maps on the 2-sphere, called expanding Thurston maps.

More than 15 years ago, Mario Bonk and Daniel Meyer became independently interested in some basic problems on quasisymmetric parametrization of 2-spheres, related to the dynamics of rational maps. They joined forces during their time together at the University of Michigan and started their investigation of a class of continuous (but not necessarily holomorphic) maps modeling a subclass of rational maps. These maps belong to a bigger class of continuous maps on the 2-sphere studied by William P. Thurston in his famous characterization theorem of rational maps (see [DH93]). As a result, Mario Bonk and Daniel Meyer called their maps *expanding Thurston maps*. Related studies were carried out by other researchers around the same time, notably Peter Haïssinsky and Kevin Pilgrim [HP09], and James W. Cannon, William J. Floyd, and Walter R. Parry [CFP07].

By late 2010, Mario Bonk and Daniel Meyer had summarized their findings in a reader-friendly arXiv draft [BM10] entitled *Expanding Thurston maps*, which they initially intended to publish in the AMS Mathematical Surveys and Monographs series. In order to make the material even more accessible, they decided later to expand their draft. This led to a long delay for the final published version [BM17] with almost twice the size of [BM10].

I was introduced to expanding Thurston maps by Mario Bonk soon after I joined in the graduate program at the University of Michigan. I quickly got deeply fascinated by this subject due to the connections to geometry, analysis, combinatorics, and dynamical systems.

I finished my first project on the periodic points and properties of the measures of maximal entropy of expanding Thurston maps under the supervision of Mario Bonk (later appeared in [Li13], see Chap. 4) after we moved to Los Angeles. I then decided to continue working on the ergodic theory of expanding Thurston maps, on which I eventually wrote my thesis.

This monograph covers investigations on the measures of maximal entropy, and more generally, equilibrium states of expanding Thurston maps, and their relations to the periodic points and the preimage points. In order to study the equilibrium states, the theory of thermodynamical formalism for Hölder continuous potentials is established in our context (see Chap. 5). The study of equidistribution results also leads to a close investigation on the expansion properties of our dynamical systems (see Chap. 6) and the discovery of some large deviation results (see Chap. 7).

This monograph is also intended to serve as a basic reference for the theory of thermodynamical formalism in our context. The applications to the study of the dynamical zeta functions were also kept in mind when this monograph was being prepared. As such, complex-valued function spaces are used whenever they do not introduce too much complication.

Acknowledgments I want to express my deep gratitude to Mario Bonk for introducing me to this subject, and for his encouragements, guidance, and strict standards. I would also like to thank Ilia Binder, Nhan-Phu Chung, Peter Haïssinksy, Daniel Meyer, and Kevin Pilgrim for helpful conversations. My gratitude also goes to our editors Boris Hasselblatt, Henk W. Broer, and Arjen Sevenster for their patient support and helpful comments. I want to acknowledge the partial supports from NSF grants DMS-1162471 and DMS-1344959. Last but not least, I want to thank my wife, Xuan Zhang, for her understanding, support, and love.

Stony Brook, NY, USA and Hangzhou, China — Zhiqiang Li
December 2016

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Notation

Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}}$ be the Riemann sphere. We use the convention that $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $\widehat{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$, with the order relations $<$, $\leq$, $>, \geq$ defined in the obvious way. As usual, the symbol log denotes the logarithm to the base e, and $\log_b$ the logarithm to the base b for $b > 0$.

The cardinality of a set A is denoted by card$A$. For $x \in \mathbb{R}$, we define $\lfloor X \rfloor$ as the greatest integer $\leq x$, and $\lfloor X \rfloor$ the smallest integer $\geq x$.

Let $g\colon X \to Y$ be a function between two sets X and Y. We denote the restriction of g to a subset Z of X by $g|_Z$.

Let (X, d) be a metric space. For subsets $A, B \subseteq X$, we set $d(A, B) = \inf\{d(x, y) \,|\, x \in A, y \in B\}$, and $d(A, x) = d(x, A) = d(A, \{x\})$ for $x \in X$. For each subset $Y \subseteq X$, we denote the diameter of Y by $\mathrm{diam}_d(Y) = \sup\{d(x, y) \,|\, x, y \in Y\}$, the interior of Y by $\mathrm{int}\, Y$, and the characteristic function of Y by $\mathbb{1}_Y$, which maps each $x \in Y$ to $1 \in \mathbb{R}$. We use the convention that $\mathbb{1} = \mathbb{1}_X$ when the space X is clear from the context. The identity map $\mathrm{id}_X\colon X \to X$ sends each $x \in X$ to x itself. For each $r > 0$, we define $N_d^r(A)$ to be the open r-neighborhood $\{y \in X \,|\, d(y, A) < r\}$ of A, and $\overline{N_d^r}(A)$ the closed r-neighborhood $\{y \in X \,|\, d(y, A) \leq r\}$ of A. For $x \in X$, we denote the open (resp. closed) ball of radius r centered at x by $B_d(x, r)$ (resp. $\overline{B_d}(x, r)$).

We set $C(X)$ (resp. $B(X)$) to be the space of continuous (resp. bounded Borel) functions from X to $\mathbb{R}$, by $\mathcal{M}(X)$ the set of finite signed Borel measures, and $\mathcal{P}(X)$ the set of Borel probability measures on X. We denote by $C(X, \mathbb{C})$ (resp. $B(X, \mathbb{C})$) the space of continuous (resp. bounded Borel) functions from X to $\mathbb{C}$. Obviously $C(X) \subseteq C(X, \mathbb{C})$ and $B(X) \subseteq B(X, \mathbb{C})$. We will adopt the convention that unless specifically referring to $\mathbb{C}$, we only consider real-valued functions.

For $\mu \in \mathcal{M}(X)$, we use $\|\mu\|$ to denote the total variation norm of μ, $\mathrm{supp}\mu$ the support of μ, and

$$\langle \mu, u \rangle = \int u \, \mathrm{d}\mu$$

for each $u \in C(S^2)$. If we do not specify otherwise, we equip $C(X)$ and $C(X, \mathbb{C})$ with the uniform norm $\|\cdot\|_\infty$. For a point $x \in X$, we define δ_x as the Dirac measure supported on $\{x\}$. For $g \in C(X)$ we set $\mathcal{M}(X, g)$ to be the set of g-invariant Borel probability measures on X. Unless otherwise specified, we equip $\mathcal{M}(X)$, $\mathcal{P}(X)$, and $\mathcal{M}(X, g)$ with the weak* topology.

The space of real-valued (resp. complex-valued) Hölder continuous functions with an exponent $\alpha \in (0, 1]$ on a compact metric space (X, d) is denoted as $C^{0,\alpha}(X, d)$ (resp. $C^{0,\alpha}((X, d), \mathbb{C})$). For each $\psi \in C^{0,\alpha}((X, d), \mathbb{C})$,

$$|\psi|_\alpha = \sup \left\{ \frac{|\psi(x) - \psi(y)|}{d(x, y)^\alpha} \,\middle|\, x, y \in X, \, x \neq y \right\}, \tag{0.1}$$

and the Hölder norm is defined as

$$\|\psi\|_{C^{0,\alpha}} = |\psi|_\alpha + \|\psi\|_\infty. \tag{0.2}$$

For given $f : X \to X$ and $\varphi \in C(X, \mathbb{C})$, we define

$$S_n\varphi(x) = \sum_{j=0}^{n-1} \varphi(f^j(x)) \tag{0.3}$$

and

$$W_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \tag{0.4}$$

for $x \in X$ and $n \in \mathbb{N}_0$. Note that when $n = 0$, by definition we always have $S_0\varphi = 0$, and by convention $W_0 = 0$.

Chapter 1
Introduction

Self-similar fractals have fascinated laymen and mathematicians alike due to their intrinsic beauty as well as mathematical sophistication. They appear naturally in mathematics and play important roles in the investigation of the corresponding areas of research. One particularly abundant source of self-similar fractals is the study of holomorphic dynamics, where they arise as Julia sets of rational functions and limit sets of Kleinian groups.

A powerful and fruitful point of view in the study of self-similar fractals is to look at them as metric spaces. On the other hand, due to their natural appearance in dynamics, self-similar fractals lie in the center of the interplay of dynamics and geometry. The investigation of metric and measure-theoretic properties of various self-similar fractals and their relation to dynamics and geometry has been actively carried out in different areas of mathematics.

Various tools in the study of general metric spaces become indispensable in the investigation of fractal spaces. Thanks to the new developments in the theory of quasiconformal geometry in recent years, more powerful tools become available and new perspectives become natural.

The classical theory of quasiconformal maps between Euclidean spaces dates back to the works of H. Grötzsch and L.V. Ahlfors in the early 20th century [Kü97, Ah82]. Since the groundbreaking work of O. Teichmüller on the classical moduli problem for Riemann surfaces around 1940 and later D.P. Sullivan's no-wandering-domain theorem in complex dynamics in the 1980s [Su85], nowadays the theory of planar quasiconformal maps is considered a standard tool in many areas of complex analysis such as Techmüller theory and holomorphic dynamics. Many such applications rely on an existence theorem for planar quasiconformal maps known as the Measurable Riemann Mapping Theorem. In higher dimensions, though, there is no counterpart for such an existence theorem. However, the importance of the theory of quasiconformal maps in higher dimensions became evident when G.D. Mostow used it in his celebrated rigidity theorems for rank-one symmetric spaces in the early

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1970s [Mos73]. This also inspired the generalization of the theory from $\mathbb{R}^n$ to metric spaces (see for example, [KR95, Pan89, HK98]).

We recall that in a metric space context, a homeomorphism $f: X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is *quasiconformal* if there exists a constant $H \geq 1$ such that

$$H_f(x) := \limsup_{r \to 0^+} \frac{\sup\{d_Y(f(x'), f(x)) \mid d_X(x', x) \leq r\}}{\inf\{d_Y(f(x'), f(x)) \mid d_X(x', x) \geq r\}} \leq H$$

for all $x \in X$. This definition is equivalent to the classical definition of a quasiconformal map in the context of Euclidean spaces, which we refer the reader to [Bon06]. In a context of a Euclidean space, it means, roughly speaking, that infinitesimal balls are mapped to infinitesimal ellipsoids with uniformly controlled eccentricity. In general the above definition is too weak to be useful.

A stronger and much more useful concept in the study of general metric spaces is the notion of a quasisymmetric map [TV80]. A homeomorphism $f: X \to Y$ is called *quasisymmetric* if there exists a homeomorphism $\eta: [0, +\infty) \to [0, +\infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right),$$

for all $x, y, z \in X$ with $x \neq z$. Roughly speaking, the above definition requires that balls to be mapped to "round" sets with quantitative control for their "eccentricity". This is a global version of the geometric property of a quasiconformal map. These two notions coincide in the context of Euclidean spaces.

The notion of quasisymmetry has been proved to be central in the study of various fractal metric spaces (see for example, [BM17, Bon11, BKM09]) and metric uniformization problems (see for example, [TV80, DS97, BonK02, Wi07]).

We now draw our attention back to the Riemann sphere.

Through the introduction of quasiconformal geometry techniques in his proof of the no-wandering-domain theorem in the 1980s [Su85], D.P. Sullivan revolutionized the field of complex dynamics. Originally, the theory of complex dynamics dates back to the work of G. Kœnigs, E. Schröder, and others in the 19th century. This subject, concentrating on the study of iterated rational maps on the Riemann sphere, was developed into an active and fascinating area of research, thanks to the remarkable works of S. Lattès, C. Carathéodory, P. Fatou, G. Julia, P. Koebe, L. Ahlfors, L. Bers, M. Herman, A. Douady, D.P. Sullivan, J.H. Hubbard, W.P. Thurston, J.C. Yoccoz, C. McMullen, J. Milnor, M. Lyubich, M. Shishikura, and many others. Modern research in complex dynamics centers at the study of fractals appearing in the dynamical space, namely the Julia sets, as well as ones in the parameter space such as the well-known Mandelbrot set.

In the early 1980s, D.P. Sullivan introduced a "dictionary", known as *Sullivan's dictionary* nowadays, linking the theory of complex dynamics with another classical area of conformal dynamical systems, namely, geometric group theory, mainly concerning the study of Kleinian groups acting on the Riemann sphere. Many dynamical objects in both areas can be similarly defined and results similarly proven, yet essential and important differences remain.

Sullivan's dictionary provides many connections and guiding intuitions between Kleinian groups and iterated rational maps on the Riemann sphere. A *Kleinian group* G is a discrete subgroup of the conformal automorphism group $\mathrm{Aut}\left(\widehat{\mathbb{C}}\right)$ of the Riemann sphere $\widehat{\mathbb{C}}$ and a rational map is a quotient of two polynomails on $\widehat{\mathbb{C}}$. For both subjects, there are common themes in deformation theories [MS98], and combinatorial classification theories [Mc95, Pi03]. The geometric structures of the associated fractals in both subjects are also closely related [Mc98, Mc00, SU00, SU02]. For more detailed discussions of the correspondence between these two subjects, we refer the reader to [Mc95, Mc08, HP09] and references therein.

One natural question to ask when one investigates the essential features of these two subjects is the following: "*What is special about conformal dynamical systems in a wider class of dynamical systems characterized by suitable metric-topological conditions?*"

This general question has inspired much research in both subjects (see for example, [DH93, HSS09, ZJ09, CT11, CT15, BonK02, BonK05, KK00, BM17, HP09, HP14, Th16]). Often certain combinatorial information of the dynamical systems and the metric geometry of the associated fractal spaces play an important role in such investigations.

In geometric group theory, the above question is related to a well-known conjecture by J.W. Cannon [Ca94]. Recall that a Kleinian group G extends isometrically to the hyperbolic 3-space $\mathbb{H}^3$. Cannon's Conjecture predicts that for every Gromov hyperbolic group G whose boundary at infinity $\partial_\infty G$ is homeomorphic to the 2-sphere S^2, there should exist a discrete, cocompact, and isometric action of G on the hyperbolic 3-space $\mathbb{H}^3$. Here we can consider Gromov hyperbolic groups G with 2-sphere boundary $\partial_\infty G$ as metric-topological systems generalizing the conformal dynamical systems in this context, namely, certain Kleinian groups. Recall that there are natural metrics d_{vis} on $\partial_\infty G$ called *visual metrics*. These metrics are unique up to snowflake equivalence. From the point of view of metric properties, one can formulate Cannon's Conjecture in the following way: Let G be a Gromov hyperbolic group, then $\partial_\infty G$ is homeomorphic to the 2-sphere if and only if the metric space $(\partial_\infty, d_{\mathrm{vis}})$ is quasisymmetrically equivalent to the Riemann sphere $\widehat{\mathbb{C}}$. Note that two metric spaces are *quasisymmetrically equivalent* if there exists a quasisymmetric homeomorphism between them. Considerable amount of efforts have been made to establish Cannon's Conjecture, leading to various partial results (see for example, [BonK05, BouK13, Mar13]), but the conjecture still remains open.

Cannon's Conjecture translates via Sullivan's dictionary to the celebrated characterization theorem of rational maps in complex dynamics by W.P. Thurston [DH93]. In this context, the metric-topological dynamical systems that generalize postcritially-finite rational maps on the Riemann sphere are called *Thurston maps*. These are (non-homeomorphic) branched covering maps on the 2-sphere S^2 whose finitely many critical points are all preperiodic. Thurston's combinatorial characterization of rational maps asserts that a Thurston map is essentially a rational map if and only if there does not exist so-called *Thurston obstruction*, i.e., a collection of simple closed curves on S^2 subject to certain conditions [DH93].

By imposing some additional condition of expansion, thus restricting to a subclass of Thurston maps, a characterization theorem of rational maps from a metric space point of view has been established in this context by M. Bonk and D. Meyer [BM17], and P. Haïssinsky and K. Pilgrim [HP09]. Roughly speaking, we say that a Thurston map is expanding if for each pair of points $x, y \in S^2$, their preimages under iterations of the map get closer and closer. See Definition 2.10 for a precise formulation. We also refer to [BM17, Proposition 6.3] for a list of equivalent definitions. For each expanding Thurston map, we can equip the 2-sphere S^2 with a natural class of metrics d, called *visual metrics*, that are quasisymmetrically equivalent to each other. As the name suggests, these metrics are constructed in a similar fashion as the visual metrics on the boundary $\partial_\infty G$ of a Gromov hyperbolic group G (see [BM17, Chap. 8] for details, and see [HP09] for a related construction). In the language above, the following theorem was obtained in [BM17, HP09].

Theorem 1.1 (M. Bonk and D. Meyer, P. Haïssinsky and K. Pilgrim) *An expanding Thurston map is conjugate to a rational map if and only if the sphere* (S^2, d) *equipped with a visual metric d is quasisymmetrically equivalent to the Riemann sphere* $\widehat{\mathbb{C}}$ *equipped with the spherical metric.*

The dynamics induced by iterations of expanding Thurston maps mentioned above is going to be the main subject matter of this monograph.

Various characterization theorems of rational maps correspond to Cannon's Conjecture via Sullivan's dictionary. M. Bonk and B. Kleiner proved in [BonK05] a weak form of Cannon's Conjecture by adding an additional condition on the dimensions of the visual metrics. From the same metric property point of view, P. Haïssinsky and K. Pilgrim established in [HP14] a sufficient condition for an expanding Thurston map to be essentially a rational map. Their theorem asserts that if an Ahlfors regular metric d' that is quasisymmetrically equivalent to a visual metric d of an expanding Thurston map $f : S^2 \to S^2$ realizes the Ahlfors regular conformal dimension $\operatorname{confdim}_{\mathrm{AR}}(f)$ of f, then f is conjugate to a rational map except for special cases of so-called obstructed Lattès examples. Here the Ahlfors regular conformal dimension $\operatorname{confdim}_{\mathrm{AR}}(f)$ of f is defined as the infimum of the Hausdorff dimension of all Ahlfors regular metrics that are quasisymmetrically equivalent to a visual metric of f. A metric space (X, d) is *Ahlfors regular* of dimension Q provided there is a Radon measure μ and a constant $C > 1$ such that

$$\frac{1}{C} r^Q \leq \mu\left(B_d(x, r)\right) \leq C r^Q$$

for $x \in X$ and $r \in (0, \operatorname{diam}_d(X)]$.

Due to important and fruitful applications of Thurston's theorem, many authors have worked on extending it beyond postcritically-finite rational maps using similar combinatorial obstructions. See for example, J.H. Hubbard, D. Schleicher, M. Shishikura's work on some postcritically-finite exponential maps [HSS09]; G. Cui and L. Tan's and G. Zhang and Y. Jiang's works on hyperbolic rational maps [CT11, ZJ09]; G. Zhang's work on certain rational maps with Siegel disks [Zh08]; X. Wang's work on certain rational maps with Herman rings [Wan14]; and G. Cui and L. Tan's work on some geometrically finite rational maps [CT15]. The characterization theorems of rational maps from [BM17, HP09, HP14] mentioned above provide an entirely new perspective (from properties of metric spaces) to the classical combinatorial classification theorem of rational maps by W.P. Thurston and its various generalizations.

The conditions regarding the Ahlfors regular conformal dimension in [BonK05, HP14] also reveal the relevance of dimension theory in our metric and dynamical context.

The notions of fractal dimension are widely used in many different areas of mathematics and natural sciences nowadays. C. Carathéodory, F. Hausdorff, and A.S. Besicovich laid the foundation of dimension theory in the early twentieth century. The early investigation centered around the Hausdorff dimension, which serves as an appropriate notion to measure the complexity of topological and geometric structures of subsets in metric spaces that are similar to the well-known Cantor set. Thanks to the powerful tools of computer visualization, the study of fractal objects became popular in mathematics and natural sciences. The ideas of fractal dimension were explored extensively by practitioners in sciences and applied mathematics, usually heuristically, well before rigorous mathematical theories were developed.

In the study of dynamical systems, it is strongly believed that there is a deep connection between the topology and geometry of invariant fractal sets and properties of the dynamical system acting on them. For a discussion on the relationship between various notions of fractal dimension and invariants of the dynamical systems such as Lyapunov exponents and entropy, see for example, [GOY88].

Many methods and tools developed in the study of dynamical systems have been proved to be useful in the investigation of notions of fractal dimension. The thermodynamical formalism, and more generally, ergodic theory, are such important tools. For close relationship between thermodynamical formalism and fractal dimension theory in dynamical systems, see [Pe97, Barr11, PU10].

Ergodic theory has been an important tool in the study of dynamical systems in general. The investigation of the existence and uniqueness of invariant measures and their properties has been a central part of ergodic theory. However, a dynamical system may possess a large class of invariant measures, some of which may be more interesting than others. It is therefore crucial to examine the relevant invariant measures.

The *thermodynamical formalism* is one such mechanism to produce invariant measures with some nice properties under assumptions on the regularity of their *Jacobian functions*. More precisely, for a continuous transformation on a compact metric space, we can consider the *topological pressure* as a weighted version of the *topological entropy*, with the weight induced by a real-valued continuous function, called a *potential*. The Variational Principle identifies the topological pressure with the supremum of its measure-theoretic counterpart, the *measure-theoretic pressure*, over all invariant Borel probability measures [Bow75, Wal76]. Under additional regularity assumptions on the transformation and the potential, one gets existence and uniqueness of an invariant Borel probability measure maximizing the measure-theoretic pressure, called the *equilibrium state* for the given transformation and the potential. Often the Jacobian function for the transformation with respect to the equilibrium state is prescribed by a function induced by the potential. The study of the existence and uniqueness of the equilibrium states and their various properties such as ergodic properties, equidistribution, fractal dimensions, etc., has been the main motivation for much research in the area.

This theory, as a successful approach to choosing relevant invariant measures, was inspired by statistical mechanics, and created by D. Ruelle, Ya. Sinai, and others in the early seventies [Dob68, Si72, Bow75, Wal82]. Since then, the thermodynamical formalism has been applied in many classical contexts (see for example, [Bow75, Ru89, Pr90, KH95, Zi96, MauU03, BS03, Ol03, Yu03, PU10, MayU10]). However, beyond several classical dynamical systems, even the existence of equilibrium states is largely unknown, and for those dynamical systems that do possess equilibrium states, often the uniqueness is unknown or at least requires additional conditions. The investigation of different dynamical systems from this perspective has been an active area of current research.

This monograph is intended as an introduction to the ergodic theory of expanding Thurston maps. More specifically, it focuses on the properties of important invariant measures such as the measure of maximal entropy and more generally, equilibrium states corresponding to Hölder continuous potentials, and their relationship with periodic points and preimage points.

We consent ourselves in this monograph by providing a foundation and a model case for more involved studies in this area on more general branched covering maps on S^2 such as ones investigated in [ZJ09, CT11, CT15], or between certain general topological spaces such as coarse expanding conformal maps from [HP09]; or more general and probably more useful potentials with logarithmic singularities similar to ones in [MayU10].

We develop the ergodic theory for expanding Thurston maps in three stages. In the first stage, we investigate various properties of the measure of maximal entropy (see Sect. 3.2 for definition) by direct and elementary arguments using the explicit combinatorial and geometric information of the maps. Among other things, we obtain very strong equidistribution results for preimage points, periodic points, and preperiodic points with respect to the measure of maximal entropy (see Theorems 4.2, 4.3, and Corollary 4.4). In order to establish the existence, uniqueness, and various other properties of equilibrium states for Hölder continuous potentials, one needs to apply more powerful tools, namely, the theory of thermodynamical formalism. This is what we do in the second stage. The equidistribution results with respect to equilibrium states we get from thermodynamical formalism are for preimage points only, and have less flexible choice of weight at each point compared to the corresponding results we get in the first stage (see Theorem 5.1). In order to get equidistribution results with respect to equilibrium states for periodic points, we apply in the last stage another machinery due to Y. Kifer [Ki90], which leads to some large deviations principles (see Theorem 7.1) which are stronger than equidistribution results. We are able to use this machinery for a subclass of our maps, more precisely, expanding Thurton maps without periodic critical points. For these maps, we establish the upper semi-continuity of the measure-theoretic entropy function by investigating certain weak expansion properties of our dynamical systems. However, upper semi-continuity of the measure-theoretic entropy and equidistribution of periodic points with respect to equilibrium states still remain open for expanding Thurston maps with a periodic critical point.

We now discuss our approaches in more details.

Arguably the most important measure for a dynamical system is its *measure of maximal entropy*. By definition, it is an invariant Borel probability measure that maximizes the measure-theoretic entropy. Thanks to the pioneering work of R. Bowen, D. Ruelle, P. Walters, Ya. Sinai, M. Lyubich, R. Mañé, and many others, existence and uniqueness results for the measure of maximal entropy are known for uniformly expansive continuous dynamical systems, distance expanding continuous dynamical systems, uniformly hyperbolic smooth dynamical systems, and rational maps on the Riemann sphere. In many cases, the measure of maximal entropy is also the asymptotic distribution of the period points (see [Par64, Si72, Bow75, Ly83, FLM83, Ru89, PU10]).

Expanding Thurston maps do not fall into any class of the classical dynamical systems mentioned above (see Chap. 6 for a more detailed discussion). So we have to first investigate the existence and uniqueness of such measures. As a consequence of their general results in [HP09], P. Haïssinsky and K. Pilgrim proved that for each expanding Thurston map, there exists a measure of maximal entropy and that the measure of maximal entropy is unique for an expanding Thurston map without periodic critical points. M. Bonk and D. Meyer then proved the existence and uniqueness of the measure of maximal entropy for all expanding Thurston maps using an explicit combinatorial construction [BM17]. Some equidistribution results for periodic critical points and iterated preimages with respect to the measure of maximal entropy were obtained in [HP09]. Following the philosophy of M. Bonk and D. Meyer, we establish in Chap. 4 stronger equidistribution results for (pre)periodic points and iterated preimages with respect to the measure of maximal entropy in our context. In order to do so, we carefully investigate the locations of fixed points in relation to the Markov partitions. We also establish an exact formula for the number of fixed points for an expanding Thurston map (see Theorem 4.1), which is analogous to the corresponding formula for rational maps (see for example, [Mil06, Theorem 12.1]).

After all, the measure of maximal entropy is just one important invariant measure. In order to investigate a larger class of important invariant measures, one needs to apply more powerful tools from thermodynamical formalism.

We establish the existence and uniqueness of the equilibrium state, denoted by μ_ϕ, for a Hölder continuous potential $\phi \colon S^2 \to \mathbb{R}$. Here S^2 is equipped with a visual metric. This generalizes the existence and uniqueness of the measure of maximal entropy of an expanding Thurston map in [HP09, BM17]. We also prove that the measure-preserving transformation f of the probability space (S^2, μ_ϕ) is *exact* (see Definition 5.40), and in particular, mixing and ergodic (Theorem 5.41 and