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Characterizations of Univariate Continuous Distributions

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To my grand children, Zakir, Samil, Amil and Julian.

Preface

Characterization of distributions plays an important role in statistical science. Using the basic properties of data, characterizations provide the type of distributions of that data set. Significant findings in this area have been published over the last several decades, and this book serves to be an extensive compilation of many important characterizations of univariate continuous distributions. Chapter 1 presents basic properties common to all univariate continuous distributions, while Chap. 2 discusses the properties of some select important distributions. Chapter 3 discusses ways to use independent copies of random variables to characterize distributions. Chapters 4–6 characterize distributions using order statistics, record values, and generalized order statistics, respectively.

I would like to thank Prof. Chris Tsokos for his encouragement to publish a book on characterization of distributions and Zeger Karssen of Atlantis Press for his support of this publication. I would also like to thank my wife Masuda for all her support. Finally, I would like to thank Rider University for a summer grant and a sabbatical leave that provided resources for me to complete this book.

Lawrenceville, NJ, USA
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Mohammad Ahsanullah

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Chapter 1
Introduction

In this chapter some basic materials will be presented which will be used in the book. We will restrict ourselves to continuous univariate probability distributions.

1.1 Distribution of Univariate Continuous Distribution

Let X be an absolutely continuous random variable with cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x). We define

$F(x) = P(X \le x)$ for all x, $-\infty < x < \infty$ and $f(x) = \frac{d}{dx}F(x)$. F(x) has the following properties

(i) $0 \le F(x) \le 1$

$$\lim_{x \to -\infty} F(x) = 0 \text{ and } \lim_{x \to \infty} F(x) = 1$$

(ii) F(x) is non decreasing
(iii) F(x) is right continuous, $F(x) = F(x + 0)$ for all x.

1.2 Moment Generating and Characteristic Functions

The moment generating function $M_X(t)$ of the random variable X with pdf $f_X(x)$ is defined as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, \quad -\infty < t < \infty$$

provided the integral converge absolutely. $M_X(0)$ always exists and equal to 1.

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The characteristic function $\varphi_X(t)$ of a random variable with pdf f(x) always exits and it is given by

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)dx, \quad -\infty < t < \infty.$$

The characteristic function has the following properties:

(i) A characteristic function is uniformly continuous on the entire real line,
(ii) It is non vanishing around zero and $\varphi_X(0) = 1$,
(iii) It is bounded, $|\varphi_X(t)| \le 1$,
(iv) It is Hermitian,

$$\varphi_X(-t) = \overline{\varphi}_X(t),$$

(v) If a random variable has kth moment, then $\varphi_X(t)$ is k times differentiable on the entire real line,
(vi) If the characteristic function $\varphi_X(t)$ of a random variable X has k-th derivative at t = 0, then the random variable X has all moments up to k if k is even and k − 1 if k is odd.

A necessary and sufficient condition for two random variables X_1 and X_2 to have identical cdf is that their characteristic functions be identical.

There is a one to one correspondence between the cumulative distribution function and characteristic function.

Theorem 1.2.1 *If characteristic function $\varphi_X(t)$ is integrable, then $F_X(x)$ is absolutely continuous, and X has the probability density function $f_X(x)$ that is given by*

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_X(t)dt$$

1.3 Some Reliability Properties

Hazard Rate

The hazard rate (r(t)) of a positive random variable random variable with F(0) = 0 is defined as follows.

$r(t) = \frac{f(t)}{\overline{F}(t)}$, $\overline{F}(t) = 1 - F(t)$, provided $\overline{F}(x)$ is not zero.

By integrating both sides of the above equation, we obtain

$$\overline{F}(x) = \exp(-\int_0^x r(t)dt).$$

An alternative representation is

$$1 - \mathrm{F}(\mathrm{x}) = e^{-R(x)}.\ \mathrm{R}(\mathrm{x}) = -\ln(1 - \mathrm{F}(\mathrm{x})).$$

We will say that the random variable X belongs to class C_1 if the hazard rate is monotonically increasing or decreasing.

New Better (Worse) Than Used (NBU(NWU))
A cumulative distribution function F(x) is NBU(NWU) if

$$\overline{F}(x+y) \leq (\geq) \overline{F}(x)\overline{F}(y), \text{ for } \mathrm{x} \geq 0, \mathrm{y} \geq 0.$$

We will say the random variable X whose cdf F(x) belongs to the class C_2 if it is NBU or NWU.

Memoryless Property
Suppose the random variable X has the property

$P(X > t + s|X > t) = P(X > s)$ for all s, t $\geq$ 0, then we say that X has memory less property.

The exponential distribution with $F(x) = 1 - e^{-(x-\mu)/\sigma}$ for $\sigma > 0, -\infty < x < \mu < \infty$. is the only continuous distribution that has this memoryless property.

1.4 Cauchy Functional Equations

We will consider the following three Cauchy functional equations for a non zero continuous function g(x).

(i) $g(x+y) = g(x) + g(y),\ x \geq 0,\ y \geq 0$

(ii) $g(xy) = g(x) + g(y), x \geq 0,\ y \geq 0$

(iii) $g(xy) = g(x)g(y), x \geq 0,\ y \geq 0$

We will take the solutions as of the functional equations as $g(x) = e^{cx}$, $g(x) = c\ln(x)$ and $g(x) = x^c$, where c is a constant respectively. For details about the solutions see Aczel (1966).

1.5 Order Statistics

Let X_1, X_2,…,X_n be independent and identically distributed (i.i.d.) absolutely continuous random variables. Suppose that F(x) be their cumulative distribution function (cdf) and f(x) be the their probability density function (pdf). Let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the corresponding order statistics. We denote $F_{k,n}(x)$ and $f_{k,n}(x)$ as the cdf and pdf respectively of $X_{k,n}$, k = 1,2,…,n. We can write

$$f_{k,n}(x) = \frac{n!}{(k-1)!(n-k)!}\,(F(x))^{k-1}(1-F(x))^{n-k}f(x),$$

The joint probability density function of order statistics $X_{1,n}$, $X_{2,n}$,…,$X_{n,n}$ has the form

$$f_{1,2,\ldots,n,n}(x_1, x_2, \ldots, x_n) = n!\prod_{k=1}^{n} f(x_k), \quad -\infty < x_1 < x_2 < \cdots < x_n < \infty$$

and

$$= 0, \textit{otherwise}$$

There are some simple formulae for pdf's of the maximum ($X_{n,n}$) and the minimum ($X_{1,n}$) of the n random variables._

The pdfs of the smallest and largest order statistics are given respectively as

$$f_{1,n}(x) = n(1-F(x))^{n-1}f(x)$$

and

$$f_{n,n}(x) = n(F(x))^{n-1}f(x)$$

The joint pdf $f_{1,n,n}$ (x, y) of $X_{1,n}$ and $X_{n.n}$ is given by

$$f_{1,n}(x,y) = n(n-1)(F(y)-F(x))^{n-2}\,f(x)f(y),$$

$$-\infty < x < y < \infty.$$

Example 1.5.1. Exponential distribution.

Suppose that X_1, X_2,…,X_n are n i.i.d. random variables with cdf F(x) as

$$F(x) = 1 - e^{-x}, \ x \ge 0$$

The pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}$ (x) are respectively