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Bing-Yuan Cao
Pei-Zhuang Wang
Zeng-Liang Liu
Yu-Bin Zhong *Editors*

International Conference on Oriental Thinking and Fuzzy Logic

Celebration of the 50th Anniversary in the era of Complex Systems and Big Data

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Preface

Oriental Thinking and Fuzzy Logic in Dalian, China, an international conference, was held during August 17–20, 2015, to celebrate the 50th anniversary of Fuzzy Sets. The honorary chair for this conference was Prof. L.A. Zadeh, the founder of fuzzy sets theory, who has guided an information revolution, and constructed a great bridge between qualitative and quantitative.

The conference focused on six main topics as follows: fuzzy information processing; fuzzy engineering; Internet and big data applications; factor space and factorial neural networks; information granulation and granular computing; extension and innovation methods. Here, topic three, the theory of factor space was initiated by Prof. Pei-Zhuang Wang with the oriental thinking. And extension topic six, is a new field of the disciplinary initiated by Prof. Wen Cai, who achieved innovation facing a problem where impossible cases seem to be possible.

There were 15 plenary talks in the conference including Wen Cai, fuzzy logic and extenics; Y.X. Chen, inter-definability and application of fuzzy logic operators; I. Dzitac, fuzzy logic and artificial intelligence; J.L. Feng, theory of meta-synthetic wisdom based on fusion of qualitative, quantitative and imagery operations; J.F. Gu, system science and Chinese medicine; Ouyang He, a mathematical foundation for factor spaces; Qing He, uncertainty learning; C.F. Huang, an approach checking whether an intelligent internet can be improved into intelligence; D.Y. Li, cognitive physics; Z.L. Liu, factorial neural networks; W. Pedrycz, new frontiers of computing and reasoning with qualitative information: a perspective of granular computing; Germano Resconi, from inconsistent topology to consistent in big data; Yong Shi and Y.J. Tian, uncertainty and big databases; P.Z. Wang, fuzzy sets and factor space; Z.S. Xu, complex information decision making. As a special guest, Mr. H.R. Lin, with his 18-year teaching practice in Shanghai Middle School, introduced his book "Preliminary of Fuzzy Mathematics" for pupils in his schools.

Apart from the organized speeches, we much appreciated the articles from individuals with natural interest and deep friendship toward Prof. L. Zadeh. They developed fuzzy theory along probability representation, rough sets, intuitionistic fuzzy sets, nonlinear Particle Swarm Optimization, ranking method to structure elements and they apply fuzzy theory into recommendation, feature extraction, qualitative mapping, etc. Among all papers presented at the conference, we carefully selected over 60 papers to form this book as assorted appetizers to commemorate the 50th anniversary of fuzzy sets from the Dalian conference.

Finally, we thank the publisher, Springer, for publishing the proceedings as Advance in Intelligent and Soft Computing.

December 2015

Bing-Yuan Cao
Pei-Zhuang Wang
Zeng-Liang Liu
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Editorial Committee

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Part I
Fuzzy Information Processing

$(\in, \in \vee q_{(\lambda,\mu)})$-Fuzzy Weak Ideal of Complemented Semirings

Zuhua Liao, Chan Zhu, Xiaotang Luo, Xiaoying Zhu, Wangui Yuan and Juan Tong

Abstract In this paper,the notions of generalized fuzzy weak ideal of complemented semiring, $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal of complemented semiring are introduced. By discussing, the two new concepts are found to be equivalent. Furthermore, some fundamental properties of their intersection, union, level sets, homomorphic image and homomorphic preimage are investigated.

Keywords $(\in, \in \vee q_{(\lambda,\mu)})$-Fuzzy weak ideal · Generalized fuzzy weak ideal · Homomorphic image · Homomorphic preimage

1 Introduction

Rosenfeld in 1971 introduced fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group [1]. Since then, many researches have extended the concepts of abstract algebra to a fuzzy framework. In 2006, Liao etc. generalized "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy set of Liu to "generalized quasi-coincident with" relation $(q_{(\lambda,\mu)})$ between a fuzzy point and a fuzzy set, and extended Rosenfeld's $(\in, \in)$-fuzzy algebra, Bhakat and Das's $(\in, \in \vee q)$-fuzzy algebra and $(\overline{\in}, \overline{\in} \vee \overline{q}_{(\lambda,\mu)})$-fuzzy algebra to $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy algebra [2] with more abundant hierarchy [3].

Vandiver [4] in 1939 put forward the concept of semiring. The applications of semirings to areas such as optimization theory, graph theory, theory of discrete event dynamical systems, generalized fuzzy computation, automata theory, formal language theory, coding theory and analysis of computer programs have been extensively studied in the literature [5, 6]. Liu [7] introduced fuzzy ideals in a ring.

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Following this definition, Mukherjee and Sen [8, 9]obtained many interesting results in the theory of rings.

On the study of fuzzy semiring, there has been a large number of researches at home and abroad. Feng and Zhan [10] proposed complemented semiring. They put the Boolean algebra as its proper class, and studied the algebraic structure of it.

This paper is the continuation of the above work.

Section 2 of this paper list some necessary preliminaries that support our results. Section 3 is the kernel of the whole paper, which display main results obtained by the authors, including the relationships among generalized fuzzy weak ideal, $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal and level subsets of a fuzzy set and relative properties about the intersection, union, homomorphic image and homomorphic preimage of such generalized fuzzy weak ideal. In section, we make a conclusion and prospect the further study of generalized fuzzy weak ideal.

2 Preliminaries

In this section we recall some basic notions and results which will be needed in the sequel.

Throughout the paper we always consider S as a semigroup.

Definition 1 [11] A semiring S is a structure consisting of a nonempty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that

(1) S together with addition is a semigroup. o is additive identity element;
(2) S together with multiplication is a semigroup.1 is multiplicative identity element;
(3) $a(b + c) = ab + ac, (a + b)c = ac + bc, \forall a, b, c \in S$;
(4) $o \cdot a = a \cdot o = o$.

Definition 2 [10] Assume a is an element of semiring S, if there exists a complement $\overline{a}$ which makes $a\overline{a} = o, a + \overline{a} = 1$, a is called complemented. S is said to be a complemented semiring if every element of S has a complement.

Definition 3 [12] Let S be a semiring. A nonempty subset A of S is said to be a complemented subsemiring of S if A is closed under three binary operations on S:

(1) If $a, b \in A$, then $a + b \in A$;
(2) If $a, b \in A$, then $ab \in A$;
(3) If $a \in A$, then $\overline{a} \in A$.

From now on, we write S and H for complemented semirings.

Definition 4 [13] Let $\alpha, \lambda, \mu \in [0, 1]$ and $\lambda < \mu$, if $A(x) \geq \alpha$, then a fuzzy point x_α is said to belongs to a fuzzy subset A written $x_\alpha \in A$; if $\lambda < \alpha$ and $A(x) + \alpha > 2\mu$, then a fuzzy point x_α is called to be generalized quasi-coincident with a fuzzy subset A, denoted by $x_\alpha q_{(\lambda,\mu)} A$. If $x_\alpha \in A$ or $x_\alpha q_{(\lambda,\mu)} A$, then $x_\alpha \in \vee q_{(\lambda,\mu)} A$.

Definition 5 [12] Let $\alpha, \lambda, \mu \in [0, 1]$ and $\lambda < \mu$, A fuzzy subset A of S is called an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy complemented subsemiring of S, if $\forall t, r \in (\lambda, 1]$, $a, b \in S$ satisfy:

(1) If $a_t, b_r \in A$, we have $(a + b)_{t\wedge r} \in \vee q_{(\lambda,\mu)} A$;
(2) If $a_t, b_r \in A$, there exist $(ab)_{t\wedge r} \in \vee q_{(\lambda,\mu)} A$;
(3) If $a_t \in A$, $a_t \in \vee q_{(\lambda,\mu)} A$ holds.

Definition 6 [12] Let $\alpha, \lambda, \mu \in [0, 1]$ and $\lambda < \mu$, A is a fuzzy set of S. We call A a generalized fuzzy complemented subsemiring of S if $\forall a, b \in S$ satisfy:

(1) $A(a + b) \vee \lambda \geq A(a) \wedge A(b) \wedge \mu$;
(2) $A(ab) \vee \lambda \geq A(a) \wedge A(b) \wedge \mu$;
(3) $A(\overline{a}) \vee \lambda \geq A(a) \wedge \mu$.

Definition 7 [14] Let $S_i (1 \leq i \leq n)$ be complemented semirings and direct product: $\prod_{1\leq i\leq n} S_i = \{(a_1, a_2, \dots a_n) | a_i \in S_i\}$. Then $\prod_{1\leq i\leq n} S_i$ is a complemented semiring under the operations as following:

$(a_1, a_2, \dots a_n) + (b_1, b_2, \dots b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$;
$(a_1, a_2, \dots a_n)(b_1, b_2, \dots b_n) = (a_1 b_1, a_2 b_2, \dots a_n b_n)$;
$\overline{(a_1, a_2, \dots a_n)} = (\overline{a_1}, \overline{a_2}, \dots \overline{a_n})$.

Definition 8 [14] Let $A_i (1 \leq i \leq n)$ be fuzzy subsets of S_i, then a fuzzy set $\prod_{1\leq i\leq n} A_i$ defined as $(\prod_{1\leq i\leq n} A_i)(x_1, x_2, \dots x_n) = \inf_{1\leq i\leq n} A_i(x_i)$ is called fuzzy direct product.

Theorem 1 [12] *Let A be a fuzzy subset of S, then A is a generalized fuzzy complemented subsemiring of S if and only if A is an* $(\in, \in \vee q_{(\lambda,\mu)})$*-fuzzy complemented subsemiring of S.*

Theorem 2 [12] *Let A be a fuzzy subset of S, then A is a generalized fuzzy complemented subsemiring of S if and only if* $\forall \alpha \in (\lambda, \mu]$*, nonempty* A_α *is a subsemiring of S.*

Theorem 3 [12] *Let A and B be generalized fuzzy complemented subsemirings of S, then* $A \cap B$ *is a generalized fuzzy complemented subsemiring of S.*

Theorem 4 [12] *A subset A of S is a complemented subsemiring of S if and only if* χ_A *is a generalized fuzzy complemented subsemiring of S.*

Theorem 5 [12] *Let* $f : S \to H$ *be a full homomorphism. If A is a generalized fuzzy complemented subsemiring of S, then* $f(A)$ *is a generalized fuzzy complemented subsemiring of H.*

Theorem 6 [12] *Let* $f : S \to H$ *be a homomorphism. If B is a generalized fuzzy complemented subsemiring of H, then* $f^{-1}(B)$ *is a generalized fuzzy complemented subsemiring of S.*

3 $(\in, \in \vee q_{(\lambda,\mu)})$-Fuzzy Completely Prime Ideals

The following if no special instructions, we suppose $\lambda, \mu \in [0, 1]$ and $\lambda < \mu$. Firstly, the definitions of weak idea of complemented semiring generalized fuzzy weak ideal and $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal are given.

Definition 9 Let A be a complemented subsemiring of S, then A is said to be a weak idea of complemented semiring S if for all $x \in A, y \in S, xy + yx \in A$.

Definition 10 Let A be a generalized fuzzy complemented subsemiring of S, then A is called a generalized fuzzy weak ideal of S if $A(xy + yx) \vee \lambda \geq A(x) \wedge \mu$. $\forall x, y \in S$.

Definition 11 Let A be an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy complemented subsemiring of S. Then A is called an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal of S, if $\alpha \in (\lambda, 1], y \in S, x_\alpha \in A$ imply $(xy + yx)_\alpha \in \vee q_{(\lambda,\mu)} A$.

By the research on the relationships among generalized fuzzy complemented subsemiring, $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal, and level subsets of a fuzzy set, we obtain the following result.

Theorem 7 *Let A be a fuzzy subset of S, then the following conditions are equivalent:*

(1) A is an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal of S;
(2) A is a generalized fuzzy weak ideal of S;
(3) $\forall \alpha \in (\lambda, \mu]$, nonempty set A_α is a weak ideal of S.

Proof (1) $\Rightarrow$ (2):

From Theorem 1, we know A is an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy complemented subsemiring of S thus A is a generalized fuzzy complemented subsemiring of S.

Next we prove A is a generalized fuzzy weak ideal of A. Assume that there exist $x_0, y_0 \in S$ such that $A(x_0y_0 + y_0x_0) \vee \lambda < A(x_0) \wedge \mu$. choose α such that $A(x_0y_0 + y_0x_0) \vee \lambda < \alpha < A(x_0) \wedge \mu$, then $A(x_0y_0 + y_0x_0) < \alpha, A(x_0) > \alpha$ and $\lambda < \alpha < \mu$, so $(x_0)_\alpha \in A$. Since A is an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal of S, thus $(x_0y_0 + y_0x_0)_\alpha \in \vee q_{(\lambda,\mu)} A$. But $A(x_0y_0 + y_0x_0) + \alpha < \alpha + \alpha < 2\mu$, a contradiction.

So A is a generalized fuzzy weak ideal of S.

(2) $\Rightarrow$ (1)

It is easy to prove that A is an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy complemented subsemiring of S. $\forall x, y \in S, \alpha \in (\lambda, 1]$, if $x_\alpha \in A$, then $A(x) \geq \alpha$. Since A is a generalized fuzzy weak ideal of S, then $A(xy + yx) \vee \lambda \geq A(x) \wedge \mu \geq \alpha \wedge \mu$.

Case 1: If $\alpha > \mu$, then $A(xy + yx) \vee \lambda \geq \mu$. By $\lambda \leq \mu$, so $A(xy + yx) \geq \mu$. Then $A(xy + yx) + \alpha \geq \mu + \alpha > 2\mu$, i.e.$(xy + yx)_\alpha q_{(\lambda,\mu)} A$.

Case 2: If $\alpha \leq \mu$, then we can obtain that $A(xy + yx) \geq \alpha$, i.e.$(xy + yx)_\alpha \in A$. So $(xy + xy)_\alpha \in \vee q_{(\lambda,\mu)} A$. Therefore A is an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal of S.

(2) $\Rightarrow$ (3)

We know that A_α is a subsemiring of S based on Theorem 2. $\forall x \in A_\alpha, \alpha \in (\lambda, \mu]$ and $y \in S$, then $A(x) \geq \alpha$. Since A is a generalized fuzzy weak ideal of S, then $A(xy + yx) \vee \lambda \geq A(x) \wedge \mu \geq \alpha \wedge \mu = \alpha$, by $\lambda < \alpha$, so $A(xy + yx) \geq \alpha$, i.e., $xy + yx \in A_\alpha$.

Therefore A_α is a weak ideal of S, $\forall \alpha \in (\lambda, \mu]$.

(3) $\Rightarrow$ (2)

We obtain that A is a generalized fuzzy complemented subsemiring of S based on Theorem 2. Assume that there exist $x_0, y_0 \in S$ such that $A(x_0y_0 + y_0x_0) \vee \lambda < A(x_0) \wedge \mu$. Choose α such that $A(x_0y_0 + y_0x_0) \vee \lambda < \alpha < A(x_0) \wedge \mu$, then $A(x_0y_0 + y_0x_0) < \alpha$, $A(x_0) > \alpha$ and $\lambda < \alpha < \mu$. So $x_0 \in A_\alpha$. Since A_α is a weak ideal of S, then $A(x_0y_0 + y_0x_0) \geq \alpha$, a contradiction. Therefore A is a generalized fuzzy weak ideal of S.

The above theorem shows that generalized fuzzy weak ideal and $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal are equivalent. Thus we can prove a normal fuzzy set be an $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal by proving it be a generalized fuzzy weak ideal, which is easier than the former. Meanwhile, Theorem 7 establishes a kind of link between generalized fuzzy weak ideal and ordinary weak ideal.

Theorem 8 *Let A and B be generalized fuzzy weak ideal of S, then $A \cap B$ is a generalized fuzzy weak ideal of S.*

Proof We obtain that $A \cap B$ is a generalized fuzzy complemented subsemiring of S based on Theorem 3. For all $x, y \in S$, we have $(A \cap B)(xy + yx) \vee \lambda = (A(xy + yx) \wedge B(xy+yx)) \vee \lambda = (A(xy + yx) \vee \lambda) \wedge (B(xy + yx) \vee \lambda) \geq (A(x) \wedge \mu) \wedge (B(x) \wedge \mu) = (A \cap B)(x) \wedge \mu$.

Therefore $A \cap B$ is a generalized fuzzy weak ideal of S.

Corollary 1 *Let $A_i (i \in I)$ be generalized fuzzy weak ideals of S, then $\cap_{i\in I} A_i$ is a generalized fuzzy weak ideal of S.*

Theorem 9 *Let $A_i (i \in I)$ be generalized fuzzy weak ideals of S, and $\forall i, j \in I$, $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Then $\cup_{i\in I} A_i$ is a generalized fuzzy weak ideal of S.*

Proof Firstly, we prove that $\vee_{i\in I}(A_i(x) \wedge A_i(y) \wedge \mu) = (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$. Obviously, $\vee_{i\in I}(A_i(x) \wedge A_i(y) \wedge \mu) \leq (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$. Assume that $\vee_{i\in I}(A_i(x) \wedge A_i(y) \wedge \mu) \neq (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$, then there exists r such that $\vee_{i\in I}(A_i(x) \wedge A_i(y) \wedge \mu) < r < (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$. Since $\forall i, j \in I$, $A_i \subseteq A_j$ or $A_j \subseteq A_i$, then $\exists k \in I$, such that $r < A_k(x) \wedge A_k(y) \wedge \mu$. But $A_i(x) \wedge A_i(y) \wedge \mu < r$, $\forall i \in I$, a contradiction. Thus $\{\vee_{i\in I}(A_i(x) \wedge A_i(y) \wedge \mu)\} = (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$. Next, $\forall x, y \in S$, we have $(\cup_{i\in I} A_i)(x + y) \vee \lambda = \vee_{i\in I} A_i(x + y) \vee \lambda = \vee_{i\in I}(A_i(x + y) \vee \lambda) \geq \vee_{i\in I}(A_i(x) \wedge A_i(y) \wedge \mu) = (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$. Similarly, we can prove that $(\cup_{i\in I} A_i)(xy) \vee \lambda \geq (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$. $\forall i \in I$, since A_i is a generalized fuzzy complemented subsemiring of S, so $A_i(\overline{a}) \vee \lambda \geq A_i(a) \wedge \mu$. Then $(\vee_{i\in I} A_i(\overline{a})) \vee \lambda \geq A_i(\overline{a}) \vee \lambda \geq A_i(a) \wedge \mu$. We have $(\vee_{i\in I} A_i(\overline{a})) \vee \lambda \geq (\vee_{i\in I} A_i(a)) \wedge \mu$. That is $(\cup_{i\in I} A_i(\overline{a})) \vee \lambda \geq (\cup_{i\in I} A_i(a)) \wedge \mu$.

Thus, $\cup_{i\in I} A_i$ is a generalized fuzzy complemented subsemiring of S.

Finally, $\forall x, y \in S$, $\cup_{i\in I}(xy + yx) \vee \lambda = \vee_{i\in I} A_i(xy + yx) \vee \lambda = \vee_{i\in I}(A_i(xy + yx) \vee \lambda) \geq \vee_{i\in I}(A_i(x) \wedge \mu) = \vee_{i\in I} A_i(x) \wedge \mu = (\cup_{i\in I} A_i)(x) \wedge \mu$.

Therefore $\cup_{i\in I} A_i$ is a generalized fuzzy weak ideal of S.

Theorem 10 *Let A_1 and A_2 be generalized fuzzy weak ideals of S_1 and S_2 respectively, then $A_1 \times A_2$ is a generalized fuzzy weak ideal of $S_1 \times S_2$.*

Proof Firstly, we prove that $A_1 \times A_2$ is a generalized fuzzy subsemiring of $S_1 \times S_2$. For all $x, y \in S_1 \times S_2$, where $x = (x_1, x_2), y = (y_1, y_2)$, since A_1 and A_2 are generalized fuzzy subsemirings of S_1 and S_2 respectively, then $(A_1 \times A_2)(x + y) \vee \lambda = (A_1 \times A_2)(x_1 + y_1, x_2 + y_2) \vee \lambda = (A_1(x_1 + y_1) \wedge A_2(x_2 + y_2)) \vee \lambda = (A_1(x_1 + y_1) \vee \lambda) \wedge (A_2 (x_2 + y_2) \vee \lambda) \geq (A_1(x_1) \wedge A_1(y_1) \wedge \mu) \wedge (A_2(x_2) \wedge A_2(y_2) \wedge \mu) = (A_1 \times A_2)((x_1, x_2)) \wedge (A_1 \times A_2)((y_1 , y_2)) \wedge \mu = (A_1 \times A_2)(x) \wedge (A_1 \times A_2)(y) \wedge \mu$.

Similarly, we can prove that $(A_1 \times A_2)(\overline{x}) \vee \lambda \geq (A_1 \times A_2)(x) \wedge \mu$ and $(A_1 \times A_2)(xy) \vee \lambda \geq (A_1 \times A_2)(x) \wedge (A_1 \times A_2)(y) \wedge \mu$. So $A_1 \times A_2$ is a generalized fuzzy complemented subsemiring of $S_1 \times S_2$.

Next, for all $x, y \in S_1 \times S_2$, where $x = (x_1, x_2), y =, (y_1, y_2)$, $(A_1 \times A_2)(xy + yx) \vee \lambda = (A_1 \times A_2)(x_1y_1 + y_1x_1, x_2y_2 + y_2x_2) \vee \lambda = (A_1(x_1y_1 + y_1x_1) \wedge A_2(x_2y_2 + y_2x_2)) \vee \lambda = (A_1(x_1y_1 + y_1x_1) \vee \lambda) \wedge (A_2(x_2y_2 + y_2x_2) \vee \lambda) \geq (A_1(x_1) \wedge \mu) \wedge (A_2(x_2) \wedge \mu) = (A_1 \times A_2)(x_1, x_2) \wedge \mu = (A_1 \times A_2)(x) \wedge \mu$.

Therefore $A_1 \times A_2$ is a generalized fuzzy weak ideal of $S_1 \times S_2$.

Theorem 11 *Let A_i be generalized fuzzy weak ideals of S, then $\prod_{1 \leq i \leq n} A_i$ is a generalized fuzzy weak ideal of $\prod_{1 \leq i \leq n} S_i$.*

Proof Firstly, we prove that $\prod_{1 \leq i \leq n} A_i$ is a generalized fuzzy subsemiring of $\prod_{1 \leq i \leq n} S_i$. For all $x, y \in \prod_{1 \leq i \leq n} S_i$, where $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$, then $(\prod_{1 \leq i \leq n} A_i)(x + y) \vee \lambda = \inf A_i(x_i + y_i) \vee \lambda = \inf(A_i(x_i + y_i) \vee \lambda) \geq \inf(A_i(x_i) \wedge A_i(y_i) \wedge \mu) = \inf A_i(x_i) \wedge \inf A_i(y_i) \wedge \mu = (\prod_{1 \leq i \leq n} A_i)(x) \wedge (\prod_{1 \leq i \leq n} A_i)(y) \wedge \mu$. Similarly, we can prove $(\prod_{1 \leq i \leq n} A_i)(xy) \vee \lambda \geq (\prod_{1 \leq i \leq n} A_i)(x) \wedge (\prod_{1 \leq i \leq n} A_i)(y) \wedge \mu$. Next in addition, $(\prod_{1 \leq i \leq n} A_i)(\overline{x}) \vee \lambda = \inf A_i(\overline{x_i}) \vee \lambda = \inf(A_i(\overline{x_i}) \vee \lambda) \geq \inf(A_i(x_i) \wedge \mu) = \inf A_i(x_i) \wedge \mu = (\prod_{1 \leq i \leq n} A_i)(x) \wedge \mu$.

Therefore $\prod_{1 \leq i \leq n} A_i$ is a generalized fuzzy complemented subsimiring of $\prod_{1 \leq i \leq n} S_i$.

Finally, for all $x, y \in \prod_{1 \leq i \leq n} S_i$, where $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$, then $(\prod_{1 \leq i \leq n} A_i)(xy + yx) \vee \lambda = \inf A_i(x_iy_i + y_ix_i) \vee \lambda = \inf(A_i(x_iy_i + y_ix_i) \vee \lambda) \geq \inf(A_i(x_i) \wedge \mu) = \inf A_i(x_i) \wedge \mu = (\prod_{1 \leq i \leq n} A_i)(x) \wedge \mu$. Thus $\prod_{1 \leq i \leq n} A_i$ is a generalized fuzzy weak ideal of $\prod_{1 \leq i \leq n} S_i$.

Theorem 12 *Let A be a subset of S, then χ_A is a generalized fuzzy weak ideal of S if and only if A is a weak ideal of S.*

Proof We know that A is a complemented subsemiring of S based on Theorem 4. For all $x \in A$ and $y \in S$, since χ_A is a generalized fuzzy weak ideal of S, then $\chi_A(xy + yx) \vee \lambda \geq \chi_A(x) \wedge \mu = 1 \wedge \mu = \mu$. By $\lambda < \mu$, so $\chi_A(xy + yx) \geq \mu > 0$ and $\chi_A(xy + yx) = 1$. Then $xy + yx \in A$. Therefore A is a weak ideal of S.

Conversely, we can obtain that χ_A is a generalized fuzzy complemented subsemiring of S based on Theorem 4.

Assume that there exist $x_0, y_0 \in S$ such that $\chi_A(x_0y_0 + y_0x_0) \vee \lambda < \chi_A(x_0) \wedge \mu$. Choose α such that $\chi_A(x_0y_0 + y_0x_0) < \alpha < \chi_A(x_0) \wedge \mu < \alpha < \chi_A(x_0) \wedge \mu$, then $\chi_A(x_0y_0 + y_0x_0) < \alpha, \chi_A(x_0) > \alpha$ and $\lambda < \alpha < \mu$. i.e. $x_0 \in A$. Since A is a weak ideal of S then $x_0y_0 + y_0x_0 \in A$. So $\chi_A(x_0y_0 + y_0x_0) = 1 > \alpha$, a contradiction.

Thus χ_A is a generalized fuzzy weak ideal of S.

Theorem 13 *Let A be a generalized fuzzy weak ideals of S, then $A_{\underline{\lambda}} = \{x|A(x) > \lambda\}$ is a generalized weak ideal of S.*

Proof Firstly, we prove that $A_{\underline{\lambda}}$ is a complemented subsemiring of S. For all $x, y \in A_{\underline{\lambda}}$, since A is a generalized fuzzy weak ideals of S, then $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu > \lambda$, so $A(x+y) > \lambda$, i.e., $x+y \in A_{\underline{\lambda}}$. Similarly, we can prove that $xy, \overline{x} \in A_{\underline{\lambda}}$.

Therefore $A_{\underline{\lambda}}$ is a complemented submiring of S. Next, for all $x \in A_{\underline{\lambda}}$ and $y \in S$, then $A(x) > \lambda$. Since A is a generalized fuzzy weak ideals of S, then $A(xy+yx) \vee \lambda \geq A(x) \wedge \mu > \lambda$, so $A(xy+yx) > \lambda$, i.e., $xy+yx \in A_{\underline{\lambda}}$.

Therefore $A_{\underline{\lambda}}$ is a weak ideal of S.

Theorem 14 *Let $f : S \to H$ be a full homomorphism, if A is a generalized fuzzy weak ideal of S, then $f(A)$ is a generalized fuzzy weak ideal of H.*

Proof Based on Theorem 5, we know that $f(A)$ is a generalized fuzzy complemented subsemiring of H. For all $z_1, z_2 \in H$, there exist $x_1, x_2 \in S$, such that $f(x_1) = z_1, f(x_2) = z_2$, then $f(x_1x_2 + x_2x_1) = f(x_1x_2) + f(x_2x_1) = f(x_1)f(x_2) + f(x_2)f(x_1) = z_1z_2 + z_2z_1$, So $f(A)(z_1z_2 + z_2z_1) \vee \lambda = \sup\{A(x)|f(x) = z_1z_2 + z_2z_1\} \vee \lambda = \sup\{A(x) \vee \lambda|f(x) = z_1z_2 + z_2z_1\} \geq \sup\{A(x_1x_2 + x_2x_1) \vee \lambda|f(x_1) = z_1, f(x_2) = z_2\} \geq \sup\{A(x_1) \wedge \mu|f(x_1) = z_1\} = \sup\{A(x_1) \wedge \mu|f(x_1) = z_1\} == f(A)(z_1) \wedge \mu$.

Therefore $f(A)$ is a generalized fuzzy ideal of H.

Theorem 15 *Let $f : S \to H$ be a homomorphism, if B is a generalized fuzzy weak ideal of H, then $f^{-1}(B)$ is a generalized fuzzy weak ideal of S.*

Proof Based on Theorem 6, we can obtain that $f^{-1}(B)$ is a generalized fuzzy complemented subsemiring of S. For all $x, y \in S$, then $f(x), f(y) \in H$. Since B is a generalized fuzzy weak ideal of H, so $f^{-1}(B)(xy+yx) \vee \lambda = B(f(xy+yx)) \vee \lambda = B(f(x)f(y) + f(y)f(x)) \vee \lambda \geq B(f(x)) \wedge \mu = f^{-1}(B)(x) \wedge \mu$.

Therefore $f^{-1}(B)$ is a generalized fuzzy weak ideal of S.

4 Conclusion

In this present investigation, the concepts of generalized fuzzy weak ideal and $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy weak ideal are proposed. Moreover, the relevant properties are studied. Further we will do some relevant properties on chain condition of fuzzy weak ideal.

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Generalized Fuzzy Sets and Fuzzy Relations

Yan-cai Zhao, Zu-hua Liao, Teng Lu and Juan Tong

Abstract In classical fuzzy set theory, a fuzzy set is a membership function which associates with each element a real number in $[0,1]$, a fuzzy relation is a function which associates with each pair of elements a real number in $[0,1]$. In the present paper, we generalize the above two concepts by associating with each set a real number in $[0,1]$, and associating with each pair of sets a real number in $[0,1]$, respectively. We then give a series of properties for these two types of generalized concepts. We also show that a generalized fuzzy relation can be induced by a classical fuzzy relation, which shows the communication of our generalized fuzzy concepts with the classical fuzzy theory.

Keywords Fuzzy set · Fuzzy relation · Generalized fuzzy set · Generalized fuzzy relation · Power set

1 Introduction

The concept of a fuzzy set was introduced by Zadeh [10]. A *fuzzy set* in a referential (universe of discourse) X is characterized by a membership function A which associates with each element $x \in X$ a real number $A(x) \in [0,1]$, having the interpretation that $A(x)$ is the membership degree of x in the fuzzy set A. For convenience, we also call the set X in above definition the *base set* of a the fuzzy set A.

Let X, Y be two sets. A mapping $R : X \times Y \to [0,1]$ is called a *fuzzy relation* [10]. The number $R(x,y) \in [0,1]$ can be interpreted as the degree of relationship between x and y.

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Since fuzzy set was introduced by Zadeh in 1965 [10], many extensions have been developed, such as intutionistic fuzzy set [1], type-2 fuzzy set [2, 4], type-n fuzzy set [2], fuzzy multiset [5, 9] and hesitant fuzzy set [6–8, 11]. So far as we know, all types of fuzzy set assign a value or a set of values to an element of a classical set. However, there are many situations in our real lives in which one has to make decision on a whole set. For example, a patient usually has several symptoms. His/Her doctor has to make a decision by whole of his/her symptoms, which means that the doctor assigns a value to the set of all the patient's symptoms. We will give the details of this example later.

In this paper, we first generalize the classical fuzzy set on a base set X to one on a collection of sets. Then, we concentrate on the study of the generalized fuzzy sets on base set $\mathcal{P}(X)$, the power set of X, and obtain a series of properties; We also generalize the classical fuzzy relation between two elements to several types of relations between two sets, and obtain a series of properties for these generalized fuzzy relations. Among them, we construct a equivalence fuzzy relation at the end of the paper.

2 Generalized Fuzzy Sets and Their Operations

In our real lives, there exist many situations in which we should make decisions on a collection of sets.

Example 1 A doctor usually judge that if a patient has caught a cold by the following symptoms of this patient: Fever (F for short), Headache (H for short) and Cough (C for short). The following table give the corresponding numbers to different sets of symptoms. Each number means a degree of a patient catch a cold.

In the following table, the collection D of decisions is a fuzzy set on the collection of the set of different symptoms of a patient (Fig. 1).

$$D = \frac{0}{\emptyset} + \frac{0.2}{\{F\}} + \frac{0.1}{\{H\}} + \frac{0.2}{\{C\}} + \frac{0.3}{\{F,H\}} + \frac{0.5}{\{F,C\}} + \frac{0.4}{\{H,C\}} + \frac{0.9}{\{F,H,C\}}.$$

A classical fuzzy set assigns a value to an element. We now give a generalized type of fuzzy set which assigns a value to a set.

Definition 1 Let $\mathcal{S} = \{S_i | \ i \in I\}$ be a collection of sets. Then a mapping A such that $A : \mathcal{S} \to [0, 1], S \mapsto A(S), \forall S \in \mathcal{S}$ is called a fuzzy set on $\mathcal{S}$. $A(S)$ is called the *membership degree* of S in $\mathcal{S}$. Let the collection of all the fuzzy sets on $\mathcal{S}$ be $\mathcal{F}(\mathcal{S})$.

Fig. 1 Fuzzy decisions on the power set of a set

symptoms	$\emptyset$	$\{F\}$	$\{H\}$	$\{C\}$	$\{F,H\}$	$\{F,C\}$	$\{H,C\}$	$\{F,H,C\}$
decision	0	0.2	0.1	0.2	0.3	0.5	0.4	0.9

As an special collection of sets, the power set of a set is interesting. So we will concentrate our studies on the fuzzy sets on a power set.

Definition 2 Let X be a set. Then a mapping $A_{\mathcal{P}}$ such that

$$A_{\mathcal{P}} : \mathcal{P}(X) \to [0,1], P \mapsto A_{\mathcal{P}}(P), \forall P \in \mathcal{P}(X)$$

is called a fuzzy set on $\mathcal{P}(X)$. $A_{\mathcal{P}}(P)$ is called the *membership degree* of P in $\mathcal{P}(X)$. Let the collection of all the fuzzy sets on $\mathcal{P}(X)$ be $\mathcal{F}(\mathcal{P}(X))$.

Give a fuzzy set on X, we can deduce a fuzzy set on $\mathcal{P}(X)$ as follows.

Definition 3 Given a fuzzy set A on X, the mapping such that

$$A_{max} : \mathcal{P}(X) \to [0,1], P \mapsto A_{max}(P) = \vee_{x \in P} A(x)$$

is called the *max-type induced fuzzy set* by A. Denote $\mathcal{F}_{max}(\mathcal{P}(X)) = \{A_{max} \mid A \in \mathcal{F}(X)\}$.

Definition 4 Given a fuzzy set A on X, the mapping such that

$$A_{min} : \mathcal{P}(X) \to [0,1], P \mapsto A_{min}(P) = \vee_{x \in P} A(x)$$

is called the *min-type induced fuzzy set* by A. Denote $\mathcal{F}_{min}(\mathcal{P}(X)) = \{A_{min} \mid A \in \mathcal{F}(X)\}$.

Given any fuzzy set A on X, then A induces a relation $\sim$ on X by defining that $x \sim y \Leftrightarrow A(x) = A(y)$. It is easy to see that $\sim$ is an equivalence relation on X. Therefore, there exists the quotient set $X/\sim$. Similarly, A_{max} or A_{min} induces a relation $\sim'$ on $\mathcal{P}(X)$ such that $P_1 \sim' P_2 \Leftrightarrow A_{\mathcal{P}}(x) = A_{\mathcal{P}}(y)$ for any $P_1, P_2 \in \mathcal{P}(X)$. $\sim'$ is an equivalence relation on $\mathcal{P}(X)$ and thus there exists the quotient set $\mathcal{P}(X)/\sim'$.

The order relation of elements in a set is usually defined as $x \leq y \Leftrightarrow A(x) \leq A(y)$. Now we define the order relation of two sets as follows.

Definition 5 Let A be a fuzzy set on X. For any $P_1, P_2 \in \mathcal{P}(X), P_1 \leq P_2 \Leftrightarrow A_{\mathcal{P}}(P_1) \leq A_{\mathcal{P}}(P_2)$.

We further give a more general order relation on $\mathcal{P}(X)$ as follows.

Definition 6 Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}(\mathcal{P}(X))$. If $\forall P \in \mathcal{P}(X), \widetilde{A}(P) \leq \widetilde{B}(P)$, then we say $\widetilde{A} \subseteq \widetilde{B}$. If $\forall P \in \mathcal{P}(X), \widetilde{A}(P) = \widetilde{B}(P)$, then we say that $\widetilde{A} = \widetilde{B}$.

Theorem 1 *Let $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{F}(\mathcal{P}(X))$, then we have the follows.*

(1) Self-reflexivity. $\widetilde{A} \subseteq \widetilde{A}$.
(2) Anti-symmetry. $\widetilde{A} \subseteq \widetilde{B}, \widetilde{B} \subseteq \widetilde{A} \Rightarrow \widetilde{A} = \widetilde{B}$.
(3) Transitivity. $\widetilde{A} \subseteq \widetilde{B}, \widetilde{B} \subseteq \widetilde{C} \Rightarrow \widetilde{A} \subseteq \widetilde{C}$.

From Theorem 1 we know that $(\mathcal{F}(\mathcal{P}(X)), \subseteq)$ is a partially ordered set.

3 Generalized Fuzzy Relations

The classical fuzzy set theory defined the relations between two elements as follows. Let X, Y be two classical sets. A mapping $R : X \times Y \to [0, 1]$ is called a *fuzzy relation* [10]. The number $R(x, y) \in [0, 1]$ can be interpreted as the degree of relationship between x and y.

Now we define the fuzzy relation between two sets as follows.

Definition 7 Let X, Y be two classical sets. A mapping $R : \mathcal{P}(X) \times \mathcal{P}(Y) \to [0, 1]$ is called a *fuzzy relation*. The number $R(A, B) \in [0, 1]$ can be interpreted as the degree of relationship between A and B.

Further, we give two types of induced relations between two sets as follows.

Definition 8 Let $R : X \times X \to [0, 1]$ be a fuzzy relation on X. The mapping $\widetilde{R} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, 1]$ such that $\widetilde{R}(A, B) = \bigvee_{a\in A, b\in B} R(a, b)$ for $A, B \in \mathcal{P}(X)$ is called *the max-type induced fuzzy relation between A and B.*

Definition 9 Let $R : X \times X \to [0, 1]$ be a fuzzy relation on X. The mapping $\widetilde{R} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, 1]$ such that $\widetilde{R}(A, B) = \bigwedge_{a\in A, b\in B} R(a, b)$ for $A, B \in \mathcal{P}(X)$ is called *the min-type induced fuzzy relation between A and B.*

Theorem 2 *If R is a fuzzy relation on X, then the max-type induced fuzzy relation $\widetilde{R}$ on $\mathcal{P}(X)$ has the following properties.*

(1) $A \subseteq C, B \subseteq D \Rightarrow \widetilde{R}(A, B) \le \widetilde{R}(C, D)$*;*
(2) If R is self-reflexive and $A \cap B \ne \emptyset$, then $\widetilde{R}(A, B) = 1$*;*
(3) $\widetilde{R}(A, B \cup C) = \widetilde{R}(A, B) \vee \widetilde{R}(A, C)$*;*
(4) $\widetilde{R}(A, B \cap C) \le \widetilde{R}(A, B) \wedge \widetilde{R}(A, C)$*.*

Proof (1). By Definition 8, it is easy to see.
(2). Since R is self-reflexive, $R(x, x) = 1$ for any $x \in X$. Choose an element $Y \in A \cap B$. Then by (1) $\widetilde{R}(A, B) \ge \widetilde{R}(\{x\}, \{x\}) = R(x, x) = 1$. So $\widetilde{R}(A, B) = 1$.
(3).

$$\begin{aligned}
\widetilde{R}(A, B \cup C) &= \bigvee_{\substack{a\in A\\ u\in B\cup C}} R(a, u) \\
&= \bigvee_{\substack{a\in A\\ u\in B}\ or\ \substack{a\in A\\ u\in C}} R(a, u) \\
&= \left\{ \bigvee_{\substack{a\in A\\ u\in B}} R(a, u) \right\} \vee \left\{ \bigvee_{\substack{a\in A\\ u\in C}} R(a, u) \right\} \\
&= \widetilde{R}(A, B) \vee \widetilde{R}(A, C).
\end{aligned}$$

(4). It is easy to be deduced from (1).

Theorem 3 *If R is a fuzzy relation on X, then the min-type induced fuzzy relation $\widetilde{R}$ on $\mathcal{P}(X)$ has the following properties.*

(1) $A \subseteq C, B \subseteq D \Rightarrow \widetilde{R}(A, B) \geq \widetilde{R}(C, D)$;
(2) If R is self-reflexive and $A - B \neq \emptyset$ (Res. $B - A \neq \emptyset$), then $\widetilde{R}(A, B) = \widetilde{R}(A - B, B)$ (Res. $\widetilde{R}(A, B) = \widetilde{R}(A, B - A)$);
(3) $\widetilde{R}(A, B \cup C) = \widetilde{R}(A, B) \wedge \widetilde{R}(A, C)$;
(4) $\widetilde{R}(A, B \cap C) \geq \widetilde{R}(A, B) \vee \widetilde{R}(A, C)$.

Proof The proofs are similar to those in Theorem 2, and thus we omit.

Now we give two types of definitions of transitivity of fuzzy relations on $\mathcal{P}(X)$, and will provide an equivalence fuzzy relation on $\mathcal{P}(X)$.

Definition 10 A fuzzy relation R on $\mathcal{P}(X)$ is called *I-type transitive*, if $R^2 = R \circ R \subseteq R$, that is, $\forall (U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, $\bigvee_{W \in \mathcal{P}(X)} R(U, W) \bigwedge R(W, V) \leq R(U, V)$.

Definition 11 A fuzzy relation R on $\mathcal{P}(X)$ is called *II-type transitive*, if $R^2 = R \circ R \subseteq R$, that is, $\forall (U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, $\bigvee_{\{x\} \in \mathcal{P}(X)} R(U, \{x\}) \bigwedge R(\{x\}, V) \leq R(U, V)$.

Definition 12 Let $R : X \times X \to [0, 1]$ be a fuzzy relation on X. For any two different elements $x, y \in X$, a path from x to y, denoted $P(x, y)$, is a set of continuous pairs $(x, a_1), (a_1, a_2), (a_2, a_3), \ldots, (a_n, y)$, where, each pair is called an *edge* of the path. Let $E(P(x, y))$ be the set of all edges in a path $P(x, y)$. The *degree* of a path $P(x, y)$ is $S(P(x, y)) = \bigwedge_{e \in E(P(x,y))} R(e)$. Suppose that there are l paths $P_1, P_2, \ldots, P_l$ from x to y. Then the *connective degree* between x and y is

$$S(x, y) == \begin{cases} \bigvee_{i=1}^{l} S(P_i), & \text{if } x \neq y; \\ 1, & \text{if } x = y. \end{cases}$$

The set of paths from A to B is $P(A, B) = \{P(a, b) | a \in A, b \in B\}$. The *connective degree* between A and B is $S(A, B) = \bigvee_{a \in A, b \in B} S(a, b)$.

The induced fuzzy relation S on $\mathcal{P}(X)$ has the following properties.

Theorem 4 *If R is a fuzzy relation on X, then the fuzzy relation S on $\mathcal{P}(X)$ has the following properties.*

(1) $A \subseteq C, B \subseteq D \Rightarrow S(A, B) \leq S(C, D)$;
(2) If $A \cap B \neq \emptyset$, *then* $S(A, B) = 1$;
(3) $S(A, B \cup C) = S(A, B) \vee S(A, C)$;
(4) $S(A, B \cap C) \leq S(A, B) \wedge S(A, C)$.

Proof The proof is similar to that of Theorem 2, and thus we omit.

Theorem 5 *If R is a symmetric fuzzy relation on X, then S is an equivalence relation on $\mathcal{P}(X)$, under the meaning of the II-type transitivity.*