EAA Series

Michael Radtke
Klaus D. Schmidt
Anja Schnaus *Editors*

Handbook on Loss Reserving

european actuarial academy

Springer

EAA Series

EAA series is successor of the EAA Lecture Notes and supported by the European Actuarial Academy (EAA GmbH), founded on the 29 August, 2005 in Cologne (Germany) by the Actuarial Associations of Austria, Germany, the Netherlands and Switzerland. EAA offers actuarial education including examination, permanent education for certified actuaries and consulting on actuarial education.

actuarial-academy.com

More information about this series at http://www.springer.com/series/7879

Michael Radtke · Klaus D. Schmidt
Anja Schnaus
Editors

Handbook on Loss Reserving

Springer

Editors
Michael Radtke
Lohmar
Germany

Anja Schnaus
Kerpen
Germany

Klaus D. Schmidt
Freital
Germany

Based on a translation from the German language edition: *Handbuch zur Schadenreservierung* by Michael Radtke and Klaus D. Schmidt Copyright © Verlag Versicherungswirtschaft GmbH 2012. All Rights Reserved.

ISSN 1869-6929 ISSN 1869-6937 (electronic)
EAA Series
ISBN 978-3-319-30054-2 ISBN 978-3-319-30056-6 (eBook)
DOI 10.1007/978-3-319-30056-6

Library of Congress Control Number: 2016942503

Mathematics Subject Classification (2010): 62-00

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Preface

In property and casualty insurance the provisions for payment obligations from losses that have occurred but have not yet been settled usually constitute the largest item on the liabilities side of an insurer's balance sheet. For this reason, the determination and evaluation of these technical provisions, which are also called *loss reserves*, is of considerable economic importance for every property and casualty insurer. Therefore, the application of actuarial methods of loss reserving is indispensable.

This *Handbook on Loss Reserving* presents the basic aspects of actuarial loss reserving. Besides the traditional methods it also includes a description of more recent ones and a discussion of certain problems occurring in actuarial practice, like inflation, scarce data, large claims, slow loss development, the use of market statistics, the need for simulation techniques, and last but not least, the task of calculating best estimates and ranges of future losses.

The actuarial methods of loss reserving form a substantial part of this book. These methods are presented in separate articles which are to a large extent self-contained. In the articles on traditional methods, the description of the method is accompanied by two numerical examples; these examples are the same for all methods and illustrate their sensitivity with respect to a small change in the data. While the traditional methods are univariate in the sense that they aim at prediction for a single portfolio of risks, the new multivariate methods, developed about ten years ago, aim at simultaneous prediction for several portfolios and take dependencies between these portfolios into account. Such methods are presented as well.

Almost all of the traditional methods are related to the *Bornhuetter–Ferguson principle*, which consists of an analytical part and a synthetical part. The analytical part provides a unified form of the predictors of most traditional methods such that the differences between these methods can be explained by the use of different estimators of parameters related to accident years or development years, and hence also by the use of different kinds of information, and the synthetical part consists of the construction of new methods by using new combinations of such estimators.

The methods of loss reserving and their properties can only be understood on the basis of stochastic models, which describe the generation of the run-off data and express the assumptions on the development (run-off) behaviour. For this reason, the articles on methods also discuss stochastic models that justify the respective method. By contrast, some other articles emphasize a stochastic model and then use the model together with a classical principle of mathematical statistics to construct a method of loss reserving.

There are basically two types of stochastic models that can be used to justify a method of loss reserving:

- *Development patterns* formalize the idea that, up to random fluctuations, the development of losses is identical for all different accident years, and they involve only assumptions on the expectations of the incremental or cumulative losses.
- *Linear models* and *credibility models* involve assumptions not only on the expectations but also on the variances and covariances of the incremental or cumulative losses. They thus enable the determination of the expected squared prediction error and its estimation.

While the traditional univariate methods result from heuristic considerations and were justified by a stochastic model later, the new multivariate methods result from generalizations of such models.

This book addresses actuarial students and academics as well as practicing actuaries. It is not intended as a complete presentation of all aspects of loss reserving, but rather as an invitation to gain an overview of the most important actuarial methods, to understand their underlying stochastic models and to get an idea of how to solve certain problems which may occur in practice. To proceed further and to become acquainted with other models and methods of loss reserving which are outside the scope of this book, the advanced reader may consult the survey articles by England & Verrall (2002) and by Schmidt (2012) and the monographs by Taylor (1986, 2000), and by Wüthrich & Merz (2008). We also refer to *A Bibliography on Loss Reserving*

http://www.math.tu-dresden.de/sto/schmidt/dsvm/reserve.pdf

which will be completed from time to time.

This *Handbook on Loss Reserving* is a free translation of the second edition of the *Handbuch zur Schadenreservierung*, published in 2012 as an update and extension of its first edition which appeared in 2004. A few articles of the German editions have been excluded since they are either outdated or specific to the German market.

The articles of this book are arranged in alphabetical order. They allow for a quick access to the different subjects, and the following guide *How to Read This Book* contains several hints on connections between certain articles and on possible starting points for reading this book.

The editors are most grateful to the authors who contributed to this book and whose expertise enabled a concise and consistent presentation of diverse theoretical and practical aspects of loss reserving, to Helga Mettke and Christiane Weber for their very delicious work in producing the graphics of this book, and to Christiane Weber for thoughtful proofreading.

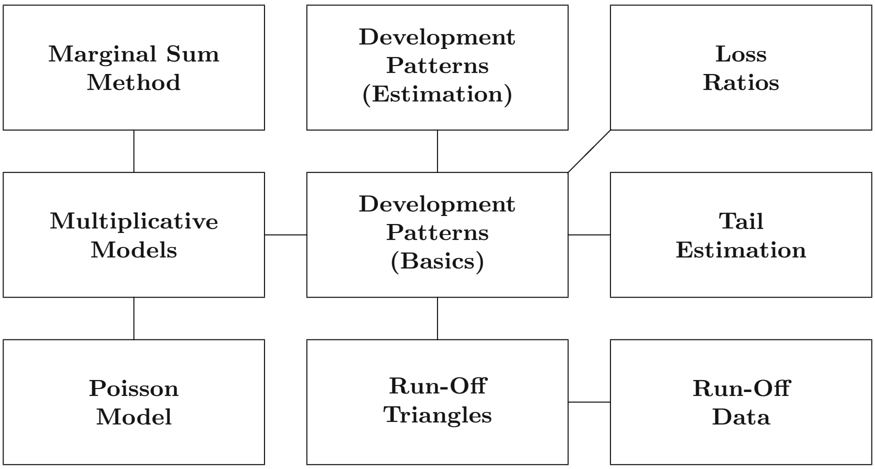
Lohmar, Germany Michael Radtke
Freital, Germany Klaus D. Schmidt
Kerpen, Germany Anja Schnaus
December 2015

How to Read This Book

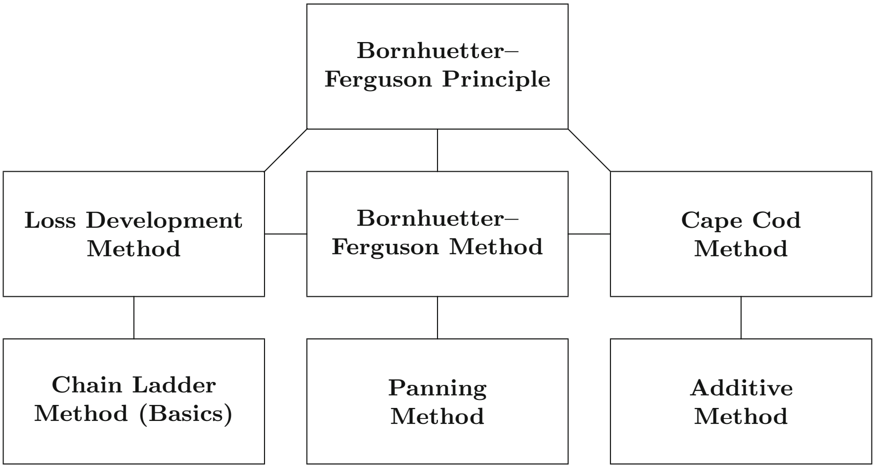
The articles of this book are to a large extent self-contained. Nonetheless, there are many interconnections between the different articles, and these are accessible via the keywords given at the end of an article.

With regard to actuarial practice, many articles focus on methods. In spite of this there is a close relationship between methods, stochastic models, and general principles of statistics. For example, many methods can be justified by the assumption of an underlying development pattern, which presents an elementary stochastic model, and for some methods it is even possible to show that the predictors of future losses are optimal in a certain sense since they turn out to be the Gauss–Markov predictors in a suitable linear model.

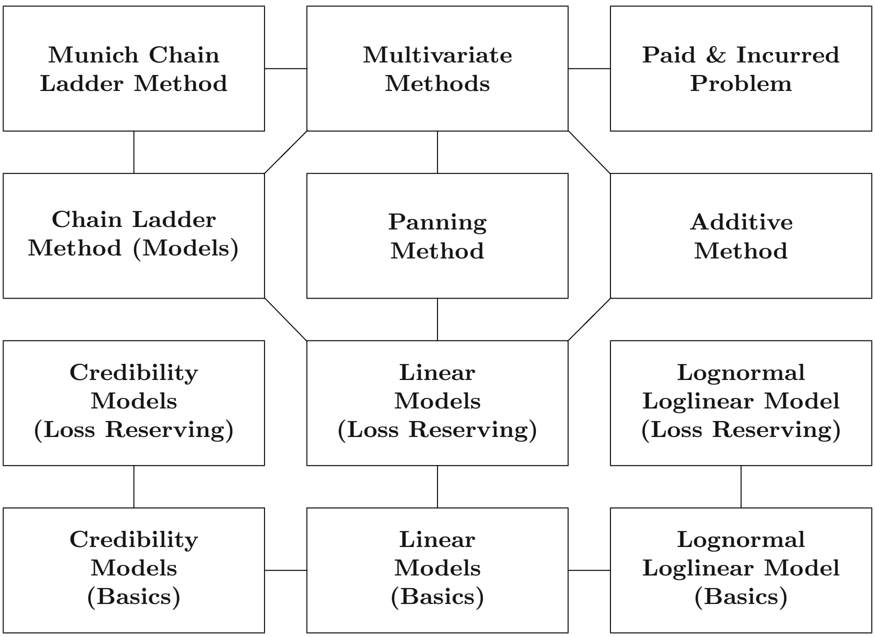
For a general background and a systematic approach to methods and models of loss reserving, and to become familiar with the notation, it is useful to start with the articles *Run-Off Data* and *Run-Off Triangles* as well as *Development Patterns* (*Basics*), *Development Patterns* (*Estimation*) and *Loss Ratios*:



A central group of articles presents the basic methods of loss reserving and the *Bornhuetter–Ferguson principle*, which provides a general framework for a unified presentation and possible extensions of these methods:



Some of these methods can also be justified by a linear model or can be extended to the multivariate case involving different lines of business or different kinds of data of the same line of business:



Of course, there are many other possible paths through the variety of methods and models of loss reserving.

The notions and also the notation used in loss reserving are far from being uniform in the literature. In this book we have tried to use uniform notions and notation as far as possible in order to simplify the recognition of interrelations between the different topics. The subject index should be helpful in this regard.

We have also tried not to burden the book too much with technicalities. For example, in an identity like

$$a = b/c$$

it is tacitly assumed that $c \neq 0$. This remark is far from being trivial since many methods of loss reserving involve divisions and are thus not applicable when the data lead to a division by 0.

Contents

Additive Method

Klaus D. Schmidt and Mathias Zocher

Consider the run-off square of incremental losses:

Accident year	**Development year** 0	1	…	k	…	$n-i$	…	$n-1$	n
0	$Z_{0,0}$	$Z_{0,1}$	…	$Z_{0,k}$	…	$Z_{0,n-i}$	…	$Z_{0,n-1}$	$Z_{0,n}$
1	$Z_{1,0}$	$Z_{1,1}$	…	$Z_{1,k}$	…	$Z_{1,n-i}$	…	$Z_{1,n-1}$	$Z_{1,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
i	$Z_{i,0}$	$Z_{i,1}$	…	$Z_{i,k}$	…	$Z_{i,n-i}$	…	$Z_{i,n-1}$	$Z_{i,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
$n-k$	$Z_{n-k,0}$	$Z_{n-k,1}$	…	$Z_{n-k,k}$	…	$Z_{n-k,n-i}$	…	$Z_{n-k,n-1}$	$Z_{n-k,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
$n-1$	$Z_{n-1,0}$	$Z_{n-1,1}$	…	$Z_{n-1,k}$	…	$Z_{n-1,n-i}$	…	$Z_{n-1,n-1}$	$Z_{n-1,n}$
n	$Z_{n,0}$	$Z_{n,1}$	…	$Z_{n,k}$	…	$Z_{n,n-i}$	…	$Z_{n,n-1}$	$Z_{n,n}$

We assume that the incremental losses $Z_{i,k}$ are observable for $i + k \le n$ and that they are non-observable for $i + k \ge n + 1$. For $i, k \in \{0, 1, \dots, n\}$ we denote by

$$S_{i,k} := \sum_{l=0}^{k} Z_{i,l}$$

the cumulative loss from accident year i in development year k.

K.D. Schmidt (✉) · M. Zocher
Technische Universität Dresden, Dresden, Germany
e-mail: klaus.d.schmidt@tu-dresden.de

M. Radtke et al. (eds.), *Handbook on Loss Reserving*,
EAA Series, DOI 10.1007/978-3-319-30056-6_1

The additive method, which is also called the *incremental loss ratio method*, involves known *volume measures* $v_0, v_1, \ldots, v_n$ of the accident years and is based on the development pattern for incremental loss ratios:

Development Pattern for Incremental Loss Ratios: *There exist parameters* $\zeta_0, \zeta_1, \ldots, \zeta_n$ *such that the identity*

$$E\left[\frac{Z_{i,k}}{v_i}\right] = \frac{E[Z_{i,k}]}{v_i} = \zeta_k$$

holds for all $k \in \{0, 1, \ldots, n\}$ *and for all* $i \in \{0, 1, \ldots, n\}$.

In this article, we assume that there exists a development pattern for incremental loss ratios. Then the parameters $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ given by

$$\vartheta_k := \frac{\zeta_k}{\sum_{l=0}^{n} \zeta_l}$$

form a *development pattern for incremental quotas* and the parameters $\gamma_0, \gamma_1, \ldots, \gamma_n$ given by

$$\gamma_k := \frac{\sum_{l=0}^{k} \zeta_l}{\sum_{l=0}^{n} \zeta_l}$$

form a *development pattern for quotas*. Moreover, the identity

$$E[Z_{i,k}] = v_i \zeta_k$$

yields the existence of a *multiplicative model*, and the identity

$$E\left[\frac{S_{i,n}}{v_i}\right] = \sum_{l=0}^{n} E\left[\frac{Z_{i,l}}{v_i}\right] = \sum_{l=0}^{n} \zeta_l$$

shows that the expected ultimate loss ratios of all accident years are identical.

The additive method consists of two steps:

- For every development year $k \in \{0, 1, \ldots, n\}$, the expected incremental loss ratio ζ_k is estimated by the *additive incremental loss ratio*

$$\zeta_k^{\text{AD}} := \frac{\sum_{j=0}^{n-k} Z_{j,k}}{\sum_{j=0}^{n-k} v_j}$$

Since

$$\zeta_k^{\text{AD}} = \sum_{j=0}^{n-k} \frac{v_j}{\sum_{h=0}^{n-k} v_h} \frac{Z_{j,k}}{v_j}$$

the additive incremental loss ratio ζ_k^{AD} is a weighted mean of the observable *individual incremental loss ratios* $Z_{j,k}/v_j$ of development year k, with weights proportional to the volume measures of the accident years.

- For every accident year i and every development year k such that $i + k \geq n + 1$, the future incremental loss $Z_{i,k}$ is predicted by the *additive predictor*

$$Z_{i,k}^{\mathrm{AD}} := v_i \zeta_k^{\mathrm{AD}}$$

The definition of the additive predictors of the incremental losses reflects the identity

$$E[Z_{i,k}] = v_i \zeta_k$$

which results from the development pattern for incremental loss ratios.

Using the additive predictors of the future incremental losses, we define the *additive predictors*

$$S_{i,k}^{\mathrm{AD}} := S_{i,n-i} + \sum_{l=n-i+1}^{k} Z_{i,l}^{\mathrm{AD}} = S_{i,n-i} + v_i \sum_{l=n-i+1}^{k} \zeta_l^{\mathrm{AD}}$$

of the future cumulative losses $S_{i,k}$ and the *additive predictors*

$$R_i^{\mathrm{AD}} := \sum_{l=n-i+1}^{n} Z_{i,l}^{\mathrm{AD}}$$

$$R_{(c)}^{\mathrm{AD}} := \sum_{l=c-n}^{n} Z_{c-l,l}^{\mathrm{AD}}$$

$$R^{\mathrm{AD}} := \sum_{l=1}^{n} \sum_{j=n-l+1}^{n} Z_{j,l}^{\mathrm{AD}}$$

of the accident year reserves R_i with $i \in \{1, \ldots, n\}$, the calender year reserves $R_{(c)}$ with $c \in \{n+1, \ldots, 2n\}$ and the aggregate loss reserve R. The additive predictors of reserves are also called *additive reserves*. Moreover, the *additive ultimate loss ratio*

$$\kappa^{\mathrm{AD}} := \sum_{l=0}^{n} \zeta_l^{\mathrm{AD}}$$

is an estimator of the expected ultimate loss ratio

$$\kappa := E\left[\frac{S_{i,n}}{v_i}\right] = \sum_{l=0}^{n} E\left[\frac{Z_{i,l}}{v_i}\right] = \sum_{l=0}^{n} \zeta_l$$

which is identical for all accident years.

Example A. Calculation of the additive predictors of incremental losses:

Accident year i	Development year k 0	1	2	3	4	5	Volume v_i	Sum
0	1001	854	568	565	347	148	4025	
1	1113	990	671	648	422	*164*	4456	
2	1265	1168	800	744	*482*	*195*	5315	
3	1490	1383	1007	*849*	*543*	*220*	5986	
4	1725	1536	*1068*	*984*	*629*	*255*	6939	
5	1889	*1811*	*1256*	*1157*	*740*	*300*	8158	
ζ_k^{AD}	*0.24*	*0.22*	*0.15*	*0.14*	*0.09*	*0.04*		*0.89*
ϑ_k^{AD}	*0.27*	*0.25*	*0.17*	*0.16*	*0.10*	*0.04*		*1*
γ_k^{AD}	*0.27*	*0.52*	*0.70*	*0.86*	*0.96*	*1*		

Reserves:

Accident year i	Reserve R_i^{AD}
1	*164*
2	*677*
3	*1612*
4	*2937*
5	*5264*
total	*10654*

Calendar year c	Reserve $R_{(c)}^{\text{AD}}$
6	*4374*
7	*2979*
8	*2007*
9	*995*
10	*300*
total	*10654*

The estimators of the development pattern for incremental quotas and quotas are not needed for the additive method and are given only for the sake of comparison with other methods.

Example B. In this example the incremental loss $Z_{4,1}$ is increased by 1000:

Accident year i	Development year k 0	1	2	3	4	5	Volume v_i	Sum
0	1001	854	568	565	347	148	4025	
1	1113	990	671	648	422	*164*	4456	
2	1265	1168	800	744	*482*	*195*	5315	
3	1490	1383	1007	*849*	*543*	*220*	5986	
4	1725	2536	*1068*	*984*	*629*	*255*	6939	
5	1889	*2116*	*1256*	*1157*	*740*	*300*	8158	
ζ_k^{AD}	*0.24*	*0.26*	*0.15*	*0.14*	*0.09*	*0.04*		*0.93*
ϑ_k^{AD}	*0.26*	*0.28*	*0.17*	*0.15*	*0.10*	*0.04*		*1*
γ_k^{AD}	*0.26*	*0.54*	*0.71*	*0.86*	*0.96*	*1*		

Reserves:

Accident year i	**Reserve** R_i^{AD}
1	*164*
2	*677*
3	*1612*
4	*2937*
5	*5569*
total	*10959*

Calendar year c	**Reserve** $R_{(c)}^{\mathrm{AD}}$
6	*4679*
7	*2979*
8	*2007*
9	*995*
10	*300*
total	*10959*

The outlier $Z_{4,1}$ affects the estimator of the parameter ζ_1 and hence the predictors of the incremental loss $Z_{5,1}$, the cumulative losses $S_{5,k}$ with $k \in \{1, \ldots, 5\}$, the accident year reserve R_5 and the calendar year reserve $R_{(6)}$.

Bornhuetter–Ferguson Principle

Define now

$$\gamma_k^{\mathrm{AD}} := \frac{\sum_{l=0}^{k} \zeta_l^{\mathrm{AD}}}{\sum_{l=0}^{n} \zeta_l^{\mathrm{AD}}} \qquad \text{and} \qquad \alpha_i^{\mathrm{AD}} := v_i \sum_{l=0}^{n} \zeta_l^{\mathrm{AD}}$$

Then the additive predictors of the future cumulative losses satisfy

$$S_{i,k}^{\mathrm{AD}} = S_{i,n-i} + \left(\gamma_k^{\mathrm{AD}} - \gamma_{n-i}^{\mathrm{AD}}\right) \alpha_i^{\mathrm{AD}}$$

Therefore, the additive method is subject to the *Bornhuetter–Ferguson principle*.
Because of the definition of κ^{AD}, we also have $\alpha_i^{\mathrm{AD}} = v_i \kappa^{\mathrm{AD}}$ and hence

$$S_{i,k}^{\mathrm{AD}} = S_{i,n-i} + \left(\gamma_k^{\mathrm{AD}} - \gamma_{n-i}^{\mathrm{AD}}\right) v_i \kappa^{\mathrm{AD}}$$

Moreover, if the Cape Cod ultimate loss ratio κ^{CC} is computed by using the *additive quotas* γ_k^{AD}, then it satisfies

$$\kappa^{\mathrm{CC}} = \kappa^{\mathrm{AD}}$$

This means that the additive method is a special case of the *Cape Cod method*. Furthermore, since the development pattern for incremental loss ratios yields a development pattern $\gamma_0, \gamma_1, \ldots, \gamma_n$ for quotas, we have

$$E\left[\frac{S_{i,k}}{v_i \gamma_k}\right] = E\left[\frac{S_{i,n}}{v_i}\right] = \kappa$$

for all $i, k \in \{0, 1, \ldots, n\}$. This is an assumption of the Cape Cod model.

Linear Model

The development pattern for incremental loss ratios concerns the structure of the expectations of the incremental losses. This elementary model can be refined by adding an assumption on the structure of their covariances. Such an assumption is part of the *additive model*:

Additive Model: *There exist known volume measures $v_0, v_1, \ldots, v_n$ of the accident years as well as unknown parameters $\zeta_0, \zeta_1, \ldots, \zeta_n$ and parameters $\sigma_0^2, \sigma_1^2, \ldots, \sigma_n^2$ such that the identities*

$$E\left[\frac{Z_{i,k}}{v_i}\right] = \zeta_k$$
$$\operatorname{cov}\left[\frac{Z_{i,k}}{v_i}, \frac{Z_{j,l}}{v_j}\right] = \frac{1}{v_i} \sigma_k^2 \, \delta_{i,j} \, \delta_{k,l}$$

hold for all $i, j, k, l \in \{0, 1, \ldots, n\}$.

The conditions of the additive model can be also represented in the form

$$E[Z_{i,k}] = v_i \, \zeta_k$$
$$\operatorname{cov}[Z_{i,k}, Z_{j,l}] = v_i \, \sigma_k^2 \, \delta_{i,j} \, \delta_{k,l}$$

Therefore, the additive model is a *linear model* and it is obvious that all additive predictors are linear in the observable incremental losses. Further properties of the additive predictors result from the theory of linear models:

Theorem. *In the additive model, the additive predictor of the future incremental loss $Z_{i,k}$ is unbiased, it is optimal in the sense that it minimizes the expected squared prediction error*

$$E\left[\left(\widehat{Z}_{i,k} - Z_{i,k}\right)^2\right]$$

over all unbiased linear predictors $\widehat{Z}_{i,k}$ of $Z_{i,k}$, and it is the only predictor having this property. These properties also hold for the additive predictors of cumulative losses and reserves.

Under the assumptions of the additive model, the theorem asserts that the additive predictors are precisely the *Gauss–Markov predictors*. In particular, it is possible to determine the expected squared prediction errors of the additive reserves and one obtains

$$E\left[\left(R_i^{\mathrm{AD}} - R_i\right)^2\right] = v_i^2 \sum_{l=n-i+1}^{n} \left(\frac{1}{\sum_{h=0}^{n-l} v_h} + \frac{1}{v_i} \right) \sigma_l^2$$

$$E\left[\left(R_{(c)}^{\mathrm{AD}} - R_{(c)}\right)^2\right] = \sum_{l=c-n}^{n} v_{c-l}^2 \left(\frac{1}{\sum_{h=0}^{n-l} v_h} + \frac{1}{v_{c-l}} \right) \sigma_l^2$$

$$E\left[\left(R^{\mathrm{AD}} - R\right)^2\right] = \sum_{l=1}^{n} \left(\sum_{j=n-l+1}^{n} v_j \right)^2 \left(\frac{1}{\sum_{h=0}^{n-l} v_h} + \frac{1}{\sum_{h=n-l+1}^{n} v_h} \right) \sigma_l^2$$

To estimate the prediction errors, one has to replace the variance parameters $\sigma_1^2, \dots, \sigma_n^2$ occurring in these formulae by appropriate estimators. Usually, the unbiased estimators

$$\widehat{\sigma}_k^2 := \frac{1}{n-k} \sum_{j=0}^{n-k} v_j \left(\frac{Z_{j,k}}{v_j} - \zeta_k^{\mathrm{AD}} \right)^2$$

are chosen for $k \in \{1, \dots, n-1\}$, and an estimator $\widehat{\sigma}_n^2$ is determined by extrapolation.

Remarks

The structure of the additive method is very similar to that of the *chain ladder method* and that of the *Panning method*. Correspondingly, the additive model is quite similar to the *chain ladder model of Schnaus* and the *Panning model*.

The additive method can be modified by changing the weights in the additive incremental loss ratios

$$\zeta_k^{\mathrm{AD}} = \sum_{j=0}^{n-k} \frac{v_j}{\sum_{h=0}^{n-k} v_h} \frac{Z_{j,k}}{v_j}$$

and such a change of the weights can be captured by an appropriate change of the accident year factors $1/v_i$ in the covariance condition

$$\operatorname{cov}\left[\frac{Z_{i,k}}{v_i}, \frac{Z_{j,l}}{v_j} \right] = \frac{1}{v_i} \sigma_k^2 \, \delta_{i,j} \, \delta_{k,l}$$

of the additive model.

It is interesting to note that there is also a micro model leading to the additive model:

- Assume that the volume measures are positive integers.

- Assume further that for every cell (i, k) with $i, k \in \{0, 1, \ldots, n\}$ there exists a family of random variables $\{X_{i,k,l}\}_{l \in \{1,\ldots,v_i\}}$ with $E[X_{i,k,l}] = \zeta_k$ and $\mathrm{var}[X_{i,k,l}] = \sigma_k^2$ as well as

$$Z_{i,k} = \sum_{l=1}^{v_i} X_{i,k,l}$$

- Assume also that any two of the random variables $X_{i,k,l}$ are uncorrelated.

Then the family $\{Z_{i,k}\}_{i,k \in \{0,1,\ldots,n\}}$ satisfies the assumptions of the additive model. The quantities of this micro model may be interpreted as follows: In accident year i there are v_i contracts, and for contract $l \in \{1, \ldots, v_i\}$ from accident year i the incremental loss in development year k is given by $X_{i,k,l}$.

Notes

Keywords: Aggregation, Bornhuetter–Ferguson Method, Bornhuetter–Ferguson Principle, Cape Cod Method, Chain Ladder Method (Basics), Development Pattern (Basics), Development Pattern (Estimation), Linear Models (Loss Reserving), Loss Ratios, Multiplicative Models, Multivariate Methods, Paid & Incurred Problem, Panning Method, Run-Off Triangles, Volume Measures.

References: Ludwig, Schmeißer & Thänert [2009], Mack [2002], Schmidt [2009, 2012], Schmidt & Zocher [2008].

Aggregation

Sebastian Fuchs, Heinz J. Klemmt and Klaus D. Schmidt

For small or highly volatile portfolios the standard methods of loss reserving tend to produce highly volatile predictors of future losses and hence of reserves. One might be tempted to combine a small or highly volatile portfolio with a large and stable one and to apply the corresponding methods to the resulting total portfolio. A typical example of such a situation is the combination of bodily injury claims with pure property damage claims in *motor third party liability insurance* insurance.

However, aggregation of sub-portfolios to a total portfolio turns out to be problematic since it can lead to a systematic distortion of the predictors. This is, in particular, the case when the sub-portfolios show different development patterns and develop differently also over the accident years. Moreover, if the standard methods of loss reserving are interpreted not just as algorithms but rather as statistical methods based on a stochastic model, then the problem arises that a model which is acceptable for each of the sub-portfolios will not necessarily be appropriate for the total portfolio. We discuss these aspects of aggregation for the chain ladder method and the additive method.

S. Fuchs · H.J. Klemmt · K.D. Schmidt (✉)
Technische Universität Dresden, Dresden, Germany
e-mail: klaus.d.schmidt@tu-dresden.de

M. Radtke et al. (eds.), *Handbook on Loss Reserving*,
EAA Series, DOI 10.1007/978-3-319-30056-6_2

Consider the run-off square of incremental losses:

Accident year	**Development year** 0	1	…	k	…	$n-i$	…	$n-1$	n
0	$Z_{0,0}$	$Z_{0,1}$	…	$Z_{0,k}$	…	$Z_{0,n-i}$	…	$Z_{0,n-1}$	$Z_{0,n}$
1	$Z_{1,0}$	$Z_{1,1}$	…	$Z_{1,k}$	…	$Z_{1,n-i}$	…	$Z_{1,n-1}$	$Z_{1,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
i	$Z_{i,0}$	$Z_{i,1}$	…	$Z_{i,k}$	…	$Z_{i,n-i}$	…	$Z_{i,n-1}$	$Z_{i,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
$n-k$	$Z_{n-k,0}$	$Z_{n-k,1}$	…	$Z_{n-k,k}$	…	$Z_{n-k,n-i}$	…	$Z_{n-k,n-1}$	$Z_{n-k,n}$
⋮	⋮	⋮		⋮		⋮		⋮	⋮
$n-1$	$Z_{n-1,0}$	$Z_{n-1,1}$	…	$Z_{n-1,k}$	…	$Z_{n-1,n-i}$	…	$Z_{n-1,n-1}$	$Z_{n-1,n}$
n	$Z_{n,0}$	$Z_{n,1}$	…	$Z_{n,k}$	…	$Z_{n,n-i}$	…	$Z_{n,n-1}$	$Z_{n,n}$

We assume that the incremental losses $Z_{i,k}$ are observable for $i + k \leq n$ and that they are non-observable for $i + k \geq n + 1$. For $i, k \in \{0, 1, \dots, n\}$, let

$$S_{i,k} := \sum_{l=0}^{k} Z_{i,l}$$

denote the cumulative loss from accident year i in development year k.

Chain Ladder Method

The chain ladder method is usually described by means of the cumulative losses. It is based on the *chain ladder factors*

$$\varphi_k^{\mathrm{CL}} := \frac{\sum_{j=0}^{n-k} S_{j,k}}{\sum_{j=0}^{n-k} S_{j,k-1}}$$

with $k \in \{1, \dots, n\}$ and it consists primarily in the prediction of the future cumulative losses $S_{i,k}$ with $i + k \geq n + 1$ by the *chain ladder predictors*

$$S_{i,k}^{\mathrm{CL}} := S_{i,n-i} \prod_{l=n-i+1}^{k} \varphi_l^{\mathrm{CL}}$$

For the prediction of the future incremental losses $Z_{i,k}$ with $i + k \geq n + 1$ one uses the *chain ladder predictors*

$$Z_{i,k}^{\mathrm{CL}} := S_{i,n-i} \, (\varphi_k^{\mathrm{CL}} - 1) \prod_{l=n-i+1}^{k-1} \varphi_l^{\mathrm{CL}}$$

(with $Z^{\text{CL}}_{i,n-i+1} = S_{i,n-i}\,(\varphi^{\text{CL}}_{n-i+1} - 1)$) from which the *chain ladder predictors* of the reserves result by summation.

By analogy with the chain ladder factors, we define for $i \in \{1, \ldots, n\}$ the *dual chain ladder factors*

$$\psi^{\text{CL}}_i := \frac{\sum_{j=0}^{i} S_{j,n-i}}{\sum_{j=0}^{i-1} S_{j,n-i}}$$

Here the analogy and the notion of duality result from the identities

$$\varphi^{\text{CL}}_k = \frac{\sum_{j=0}^{n-k} \sum_{l=0}^{k} Z_{j,l}}{\sum_{j=0}^{n-k} \sum_{l=0}^{k-1} Z_{j,l}} \qquad \text{and} \qquad \psi^{\text{CL}}_i = \frac{\sum_{l=0}^{n-i} \sum_{j=0}^{i} Z_{j,l}}{\sum_{l=0}^{n-i} \sum_{j=0}^{i-1} Z_{j,l}}$$

The dual chain ladder factors are exactly the chain ladder factor in the *reflected run-off triangle* of incremental losses, in which the roles of accident years and of development years are interchanged. Therefore they describe the development over accident years instead of development years.

We consider now two sub-portfolios with the respective incremental losses $\bar{Z}_{i,k} > 0$ and $\tilde{Z}_{i,k} > 0$ as well as the total portfolio with the incremental losses $Z_{i,k} := \bar{Z}_{i,k} + \tilde{Z}_{i,k}$. We also denote all other quantities of the sub-portfolios in the same way as the incremental losses.

Theorem.

(1) *If $\bar{\varphi}^{\text{CL}}_k > \tilde{\varphi}^{\text{CL}}_k$ and $\bar{\psi}^{\text{CL}}_i > \tilde{\psi}^{\text{CL}}_i$ holds for all $i, k \in \{1, \ldots, n\}$, then the inequality*

$$\bar{Z}^{\text{CL}}_{i,k} + \tilde{Z}^{\text{CL}}_{i,k} > Z^{\text{CL}}_{i,k}$$

holds for all $i, k \in \{1, \ldots, n\}$ such that $i + k \geq n + 1$.

(2) If $\bar{\varphi}^{\text{CL}}_k = \tilde{\varphi}^{\text{CL}}_k$ *holds for all $i, k \in \{1, \ldots, n\}$, then the identity*

$$\bar{Z}^{\text{CL}}_{i,k} + \tilde{Z}^{\text{CL}}_{i,k} = Z^{\text{CL}}_{i,k}$$

holds for all $i, k \in \{1, \ldots, n\}$ such that $i + k \geq n + 1$.

(3) If $\bar{\varphi}^{\text{CL}}_k < \tilde{\varphi}^{\text{CL}}_k$ *and $\bar{\psi}^{\text{CL}}_i > \tilde{\psi}^{\text{CL}}_i$ holds for all $i, k \in \{1, \ldots, n\}$, then the inequality*

$$\bar{Z}^{\text{CL}}_{i,k} + \tilde{Z}^{\text{CL}}_{i,k} < Z^{\text{CL}}_{i,k}$$

holds for all $i, k \in \{1, \ldots, n\}$ such that $i + k \geq n + 1$.

By summation, the results of the theorem for the chain ladder predictors of incremental losses yield corresponding results for the chain ladder predictors of cumulative losses and for the chain ladder reserves.

Example. Sub-portfolio I: Incremental losses and predictors of incremental losses:

Accident year i	**Development year** k 0	1	2	3	$\bar{\psi}_i^{\text{CL}}$
0	230	110	60	20	
1	240	120	80	*22*	*2.10*
2	230	120	*70*	*21*	*1.50*
3	280	*140*	*84*	*25*	*1.40*
$\bar{\varphi}_k^{\text{CL}}$		*1.50*	*1.20*	*1.05*	

Sub-portfolio II: Incremental losses and predictors of incremental losses:

Accident year i	**Development year** k 0	1	2	3	$\tilde{\psi}_i^{\text{CL}}$
0	780	140	80	10	
1	760	120	100	*10*	*1.98*
2	410	130	*54*	*6*	*1.30*
3	390	*78*	*47*	*5*	*1.20*
$\tilde{\varphi}_k^{\text{CL}}$		*1.20*	*1.10*	*1.01*	

Sums of the predictors of the two sub-portfolios:

Accident year i	**Development year** k 0	1	2	3	
0					
1				*32*	
2			*124*	*27*	
3		*218*	*131*	*30*	

Total portfolio: Incremental losses and predictors of incremental losses:

Accident year i	**Development year** k 0	1	2	3	
0	1010	250	140	30	
1	1000	240	180	*30*	
2	640	250	*114*	*21*	
3	670	*187*	*110*	*21*	
φ_k^{CL}		*1.28*	*1.13*	*1.02*	

The results confirm assertion (1) of the theorem.

The chain ladder method is based on the assumption of the existence of a development pattern for factors.

- If the existence of a development pattern for factors is assumed for each of the sub-portfolios, then there exist parameters $\bar{\varphi}_k$ and $\tilde{\varphi}_k$ such that

$$E[\bar{S}_{i,k}] = E[\bar{S}_{i,k-1}]\,\bar{\varphi}_k$$
$$E[\tilde{S}_{i,k}] = E[\tilde{S}_{i,k-1}]\,\tilde{\varphi}_k$$

  holds for all $k \in \{1, \ldots, n\}$ and $i \in \{0, 1, \ldots, n\}$.
- If the existence of a development pattern for factors is assumed for the total portfolio, then there exist parameters φ_k such that

$$E[S_{i,k}] = E[S_{i,k-1}]\,\varphi_k$$

  holds for all $k \in \{1, \ldots, n\}$ and $i \in \{0, 1, \ldots, n\}$.

It thus follows that a development pattern for factors exists for each of the sub-portfolios and also for the total portfolio if and only if there exists, for every $k \in \{1, \ldots, n\}$, some c_{k-1} such that the identity

$$\frac{E[\bar{S}_{i,k-1}]}{E[\tilde{S}_{i,k-1}]} = c_{k-1}$$

holds for all $i \in \{0, 1, \ldots, n\}$. As this proportionality condition is not plausible in general, this raises the problem of a consistent modelling of the sub-portfolios and the total portfolio.

One possibility of a consistent modelling of the sub-portfolios and the total portfolio is provided by the *multivariate chain ladder model*, which provides a justification of the *multivariate chain ladder method*. The multivariate chain ladder model describes not only the individual sub-portfolios, but also the correlations between the sub-portfolios.

In actuarial practice, the application of the multivariate chain ladder method may cause problems, but the method represents a *benchmark* and in many cases the multivariate chain ladder predictors are approximated quite well by the univariate chain ladder predictors for the individual sub-portfolios.

Additive Method

The additive method uses known *volume measures* $v_0, v_1, \ldots, v_n$ of the accident years. It is based on the *additive incremental loss ratios*

$$\zeta_k^{\mathrm{AD}} := \frac{\sum_{j=0}^{n-k} Z_{j,k}}{\sum_{j=0}^{n-k} v_j}$$

with $k \in \{0, 1, \ldots, n\}$ and it consists primarily in the prediction of the future incremental losses $Z_{i,k}$ with $i + k \geq n + 1$ by the *additive predictors*

$$Z_{i,k}^{\mathrm{AD}} := v_i \zeta_k^{\mathrm{AD}}$$

from which the *additive predictors* of the future cumulative losses and of the reserves result by summation.

We consider now two sub-portfolios with the respective incremental losses $\bar{Z}_{i,k} > 0$ and $\tilde{Z}_{i,k} > 0$ and the respective volume measures $\bar{v}_i > 0$ and $\tilde{v}_i > 0$ as well as the total portfolio with the incremental losses $Z_{i,k} := \bar{Z}_{i,k} + \tilde{Z}_{i,k}$ and the volume measures $v_i := \bar{v}_i + \tilde{v}_i$. We also denote all other quantities of the sub-portfolios in the same way as the incremental losses and the volume measures.

Lemma. *For all $i, k \in \{1, \ldots, n\}$ such that $i + k \geq n + 1$ there exists a constant $v_{i,k} > 0$ determined by the volume measures such that*

$$\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} - Z_{i,k}^{\mathrm{AD}} = v_{i,k} \left(\frac{\bar{v}_i}{\sum_{j=0}^{n-k} \bar{v}_j} - \frac{\tilde{v}_i}{\sum_{j=0}^{n-k} \tilde{v}_j} \right) \left(\bar{\zeta}_k^{\mathrm{AD}} - \tilde{\zeta}_k^{\mathrm{AD}} \right)$$

This lemma provides a complete solution to the problem of additivity for the additive method (and even for the additive predictors of the individual future incremental losses). In particular, the additive method is always additive if there exists some c such that the identity

$$\bar{v}_i / \tilde{v}_i = c$$

holds for all $i \in \{0, 1, \ldots, n\}$.

An analogon to the theorem on the additivity in the chain ladder method results immediately from the lemma:

Theorem.

(1) *If $\bar{\zeta}_k^{\mathrm{AD}} > \tilde{\zeta}_k^{\mathrm{AD}}$ and $\bar{v}_i / \sum_{j=0}^{n-k} \bar{v}_j > \tilde{v}_i / \sum_{j=0}^{n-k} \tilde{v}_j$* holds for all *$i, k \in \{1, \ldots, n\}$ such that $i + k \geq n + 1$, then the inequality*

$$\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} > Z_{i,k}^{\mathrm{AD}}$$

holds for all $i, k \in \{1, \ldots, n\}$ such that $i + k \geq n + 1$.

(2) *If $\bar{\zeta}_k^{\mathrm{AD}} = \tilde{\zeta}_k^{\mathrm{AD}}$* or *$\bar{v}_i / \sum_{j=0}^{n-k} \bar{v}_j = \tilde{v}_i / \sum_{j=0}^{n-k} \tilde{v}_j$ holds for all $i, k \in \{1, \ldots, n\}$* such that *$i + k \geq n + 1$, then the identity*

$$\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} = Z_{i,k}^{\mathrm{AD}}$$

holds for all $i, k \in \{1, \ldots, n\}$ such that *$i + k \geq n + 1$.*

(3) *If* $\bar{\zeta}_k^{\text{AD}} < \tilde{\zeta}_k^{\text{AD}}$ *and* $\bar{v}_i / \sum_{j=0}^{n-k} \bar{v}_j > \tilde{v}_i / \sum_{j=0}^{n-k} \tilde{v}_j$ *holds for all* $i, k \in \{1, \dots, n\}$ *such that* $i + k \geq n + 1$, *then the inequality*

$$\bar{Z}_{i,k}^{\text{AD}} + \tilde{Z}_{i,k}^{\text{AD}} < Z_{i,k}^{\text{AD}}$$

holds for all $i, k \in \{1, \dots, n\}$ *such that* $i + k \geq n + 1$.

By summation, the results of the theorem for the additive predictors of incremental losses yield corresponding results for the additive predictors of cumulative losses and for the additive reserves.

The additive method is based on the assumption of the existence of a development pattern for incremental loss ratios.

- If the existence of a development pattern for incremental loss ratios is assumed for each of the sub-portfolios, then there exist parameters $\bar{\zeta}_k$ and $\tilde{\zeta}_k$ such that

$$E[\bar{Z}_{i,k}] = \bar{v}_i \bar{\zeta}_k$$
$$E[\tilde{Z}_{i,k}] = \tilde{v}_i \tilde{\zeta}_k$$

  holds for all $k \in \{0, 1, \dots, n\}$ and $i \in \{0, 1, \dots, n\}$.
- If the existence of a development pattern for incremental loss ratios is assumed for the total portfolio, then there exist parameters ζ_k such that

$$E[Z_{i,k}] = v_i \zeta_k$$

  holds for all $k \in \{0, 1, \dots, n\}$ and $i \in \{0, 1, \dots, n\}$.

It thus follows that development patterns for incremental loss ratios exist for each of the sub-portfolios and also for the total portfolio if and only if there exists some c such that the identity

$$\bar{v}_i / \tilde{v}_i = c$$

holds for all $i \in \{0, 1, \dots, n\}$.

Example. Depending on the choice of the volume measure, different effects arise from the application of the additive method in *motor third party liability insurance*: If the number of contracts is chosen as the volume measure, the separation of bodily injury claims and pure property damage claims can be omitted, as the same volume measure is used for both types of losses and since the additive method is additive in this case.[1] By contrast, if the corresponding expected number of claims is chosen as the volume measure, then the volume measures for bodily injury claims and for pure property damage claims are usually not proportional and in this case the additive method is not additive in general.

---

[1] Let w_i denote the number of contracts in accident year i. Then one has $\bar{v}_i = w_i$ and $\tilde{v}_i = w_i$, and hence $v_i = 2w_i$. The additive method applied to either v_i and w_i produces the same results since scaling of the volume measures does not affect the predictors.

One possibility of a consistent modelling of the sub-portfolios and the total portfolio is provided by the *multivariate additive model*, which provides a justification of the *multivariate additive method*.

The remarks made on the multivariate chain ladder method also apply to the multivariate additive method.

Remarks

Assertion (1) of both theorems essentially states that, for every future incremental loss, the sum of the predictors from the sub-portfolios is always greater than the predictor from the total portfolio when one of the two sub-portfolios has at the same time a lower *development speed* and a higher *expansion speed* than the other. Similar interpretations can be given for assertions (2) and (3) of these theorems. The expansion over accident years is sometimes called *accident year inflation*.

The theoretical results of this article provide sufficient conditions for underestimation or overestimation of the reserves caused by the aggregation of sub-portfolios. Presumably, in actuarial practice these conditions will only be checked once the predictors have been computed and compared. If, however, it then turns out that the appropriate sufficient condition is fulfilled, then this check provides some useful information on the sub-portfolios.

Notes

Keywords: Additive Method, Chain Ladder Method (Basics), Development Patterns (Basics), Multivariate Methods, Volume Measures.

References: Ajne [1994], Barnett, Zehnwirth & Dubossarski [2005], Fuchs [2014], Klemmt [2005], Schmidt [2006b, 2012].