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Wolfgang Arendt
Charles J.K. Batty
Matthias Hieber
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Wolfgang Arendt
Angewandte Analysis
Universität Ulm
89069 Ulm
Germany
wolfgang.arendt@uni-ulm.de

Charles J.K. Batty
St. John's College
Oxford OX1 3JP
UK
charles.batty@sjc.ox.ac.uk

Matthias Hieber
Fachbereich Mathematik
TU Darmstadt
Schlossgartenstr. 7
64289 Darmstadt
Germany
hieber@mathematik.tu-darmstadt.de

Frank Neubrander
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803
USA
neubrand@math.lsu.edu

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Prefaces

Preface to the First Edition

Linear evolution equations in Banach spaces have seen important developments in the last two decades. This is due to the many different applications in the theory of partial differential equations, probability theory, mathematical physics, and other areas, and also to the development of new techniques. One important technique is given by the Laplace transform. It played an important role in the early development of semigroup theory, as can be seen in the pioneering monograph by Hille and Phillips [HP57]. But many new results and concepts have come from Laplace transform techniques in the last 15 years. In contrast to the classical theory, one particular feature of this method is that functions with values in a Banach space have to be considered.

The aim of this book is to present the theory of linear evolution equations in a systematic way by using the methods of vector-valued Laplace transforms.

It is simple to describe the basic idea relating these two subjects. Let A be a closed linear operator on a Banach space X. The *Cauchy problem* defined by A is the initial value problem

$$(CP) \quad \begin{cases} u'(t) = & Au(t) \quad (t \geq 0), \\ u(0) = & x, \end{cases}$$

where $x \in X$ is a given initial value. If u is an exponentially bounded, continuous function, then we may consider the Laplace transform

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\, dt$$

of u for large real λ. It turns out that u is a (mild) solution of (CP) if and only if

$$(\lambda - A)\hat{u}(\lambda) = x \qquad (\lambda \text{ large}). \tag{1}$$

Thus, if λ is in the resolvent set of A, then

$$\hat{u}(\lambda) = (\lambda - A)^{-1} x. \tag{2}$$

Now it is a typical feature of concrete evolution equations that no explicit information on the solution is known and only in exceptional cases can the solution be given by a formula. On the other hand, in many cases much can be said about the resolvent of the given operator. The fact that the Laplace transform allows us to reduce the Cauchy problem (CP) to the characteristic equation (1) explains its usefulness. The Laplace transform is the link between solutions and resolvents, between Cauchy problems and spectral properties of operators.

There are two important themes in the theory of Laplace transforms. The first concerns representation theorems; i.e., results which give criteria to decide whether a given function is a Laplace transform. Clearly, in view of (2), such results, applied to the resolvent of an operator, give information on the solvability of the Cauchy problem.

The other important subject is asymptotic behaviour, where the most challenging and delicate results are *Tauberian theorems* which allow one to deduce asymptotic properties of a function from properties of its transform. Since in the case of solutions of (CP) the transform is given by the resolvent, such results may allow one to deduce results of asymptotic behaviour from spectral properties of A.

These two themes describe the essence of this book, which is divided into three parts. In the first, representation theorems for Laplace transforms are given, and corresponding to this, well-posedness of the Cauchy problem is studied. The second is a systematic study of asymptotic behaviour of Laplace transforms first of arbitrary functions, and then of solutions of (CP). The last part contains applications and illustrative examples. Each part is preceded by a detailed introduction where we describe the interplay between the diverse subjects and explain how the sections are related.

We have assumed that the reader is already familiar with the basic topics of functional analysis and the theory of bounded linear operators, Lebesgue integration and functions of a complex variable. We require some standard facts from Fourier analysis and slightly more advanced areas of functional analysis for which we give references in the text. There are also four appendices (A, B, C and E) which collect together background material on other standard topics for use in various places in the book, while Appendix D gives a proof of a technical result in the geometry of Banach spaces which is needed in Section 4.6.

Finally, a few words should be said about the realization of the book. The collaboration of the authors is based on two research activities: the common work of W. Arendt, M. Hieber and F. Neubrander on integrated semigroups and the work of W. Arendt and C. Batty on asymptotic behaviour of semigroups over many years. Laplace transform methods are common to both.

The actual contributions are as follows.

Part I: All four authors wrote this part.

Part II was written by W. Arendt and C. Batty.

Part III was written by W. Arendt (Chapters 6 and 7) and M. Hieber (Chapter 8).

C. Batty undertook the coordination needed to make the material into a consistent book.

The authors are grateful to many colleagues and friends with whom they had a fruitful cooperation, frequently over many years, which allowed them to discuss the material presented in the book. We would especially like to acknowledge among them H. Amann, B. Bäumer, Ph. Bénilan, J. van Casteren, R. Chill, O. El-Mennaoui, J. Goldstein, H. Kellermann, V. Keyantuo, R. deLaubenfels, G. Lumer, R. Nagel, J. van Neerven, J. Prüss, F. Räbiger, A. Rhandi, W. Ruess, Q.P. Vũ, and L. Weis

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Ulm, Oxford, Darmstadt, Baton Rouge
August, 2000

Wolfgang Arendt
Charles Batty
Matthias Hieber
Frank Neubrander

Preface to the Second Edition

Ten years after the publication of the first edition of this monograph, it is clear that vector-valued Laplace transform methods continue to play an important role in the analysis of partial differential equations and other disciplines of analysis. Among the most notable new achievements of this period are the characterization of generators of cosine functions on Hilbert space due to Crouzeix, and quantitative Tauberian theorems for Laplace transforms with applications to energy estimates for wave equations.

In this second edition, the new developments have been taken into account by updating the Notes on each Chapter and the Bibliography. For example, the characterization of generators of cosine functions on Hilbert space by a purely geometric condition on the numerical range is precisely stated in Theorem 3.17.5. The main text has not been substantially changed, except in Section 4.4 where some results are now presented in quantitative forms. Their applications in the study of damped wave equations are explained in detail in the Notes of the section.

A few minor mathematical gaps and typographical errors have been corrected, and we are grateful to M. Haase, J. van Neerven, R. Schumann and D. Seifert for alerting us to some of them.

September 2010 The Authors

Part I

Laplace Transforms and Well-Posedness of Cauchy Problems

As a guide-line for Part I, as well as for the entire book, we have in mind the formula

$$\hat{u}(\lambda) = R(\lambda, A)x \tag{3}$$

saying that a mild solution of the Cauchy problem, $u'(t) = Au(t)$ with initial value x, is given by the resolvent of the underlying operator A evaluated at x. Thus, if we want to find solutions, we first have to characterize those functions which are Laplace transforms; i.e., we study representation theorems. Correspondingly, on the side of evolution equations, we investigate existence and uniqueness of solutions of the Cauchy problem. Other subjects treated here include regularity and positivity.

Part I contains three chapters as follows:

1. The Laplace Integral

2. The Laplace Transform

3. Cauchy Problems

We start with an introduction to the vector-valued Lebesgue integral; i.e., the Bochner integral. For our purposes it suffices to consider functions defined on the real line. Then we introduce the Laplace integral and investigate its analytic properties, giving special attention to its diverse abscissas. This will play an important role when solutions of the Cauchy problem are considered, as the abscissas give information about the asymptotic behaviour for large time. Operational properties of the Laplace integral are also discussed. Finally, we introduce functions of (semi) bounded variation defined on the half-line and the Laplace-Stieltjes transform. They will be needed when we study resolvent positive operators (Section 3.11) and Hille-Yosida operators (Section 3.5).

The vector-valued Fourier transform on the line is introduced in Section 1.6 and we prove the Paley-Wiener theorem for functions with values in a Hilbert space. This is the first of several representation theorems for Laplace transforms which we present in this book.

In Chapter 2, real representation theorems are the central subject. We prove a vector-valued version of Widder's classical theorem which describes those functions which are Laplace transforms of bounded measurable functions. The vector-valued version (Section 2.2) will lead directly to generation theorems in Chapter 3 for semigroups and integrated semigroups (Section 3.3) and for cosine functions (Section 3.15). A particularly simple representation theorem is valid for holomorphic functions (Section 2.6). The Laplace transform is an isomorphism between certain classes of holomorphic functions defined on sectors in the complex plane. This will lead directly to the generation theorem for holomorphic semigroups in Section 3.7. The third representation theorem is a vector-valued version of Bernstein's theorem describing Laplace-Stieljes transforms of monotonic functions (Section 2.7). It has its counterpart for Cauchy problems in Section 3.11 where resolvent positive operators are considered.

The uniqueness theorem for Laplace transforms is easy to prove (Section 1.7), but it has striking consequences. It gives directly an approximation result for sequences of Laplace transforms. In Chapter 3 we find its counterpart for Cauchy problems in the form of the Trotter-Kato theorem (Section 3.6).

For Cauchy problems, the most satisfying situation is when there corresponds exactly one (mild) solution to each initial value. This notion of well-posedness is equivalent to existence of a C_0-semigroup (Section 3.1). We also consider weaker forms of well-posedness which are characterized by the existence of integrated semigroups. In applications, they allow one to describe precise regularity for certain partial differential equations in $L^p(\mathbb{R}^n)$, and Chapter 8 is devoted to this. Here in Part I, there are three situations where integrated semigroups occur in a natural way. Operators satisfying the Hille-Yosida condition generate locally Lipschitz continuous integrated semigroups. Using convolution properties established in Section 1.3, we prove a beautiful existence and uniqueness theorem due to Da Prato and Sinestrari for the inhomogeneous Cauchy problem defined by such operators. The second interesting class of examples are resolvent positive operators which always generate twice integrated semigroups. This will be proved in Section 3.11. In Chapter 6 a resolvent positive operator will provide an elegant transition from elliptic to parabolic problems. Finally, in Section 3.14 we show that the second order Cauchy problem is well-posed on a space X if and only if the associated canonical system generates an integrated semigroup on the product space $X \times X$.

In Section 3.10 we show that integrated semigroups and semigroups are equivalent, up to the choice of the underlying Banach space. This choice is particularly interesting in the context of the second order Cauchy problem. In Section 3.14 we will show the remarkable result that the space of well-posedness is unique, and we find the phase space associated to the second order problem. In the applications to the wave equation given in Chapter 7 we will see how this space is well adapted to perturbation theory, allowing us to prove well-posedness of the wave equation defined by very general second order elliptic operators.

Special attention is given to C_0-groups; i.e., to Cauchy problems allowing unique mild solutions on the line. In Section 3.9 we study when a holomorphic semigroup of angle $\pi/2$ has a boundary group. This problem will occur again in Section 3.16 where we investigate which cosine functions allow a square root reduction. A striking theorem due to Fattorini shows that on UMD-spaces a square root reduction is always possible; i.e., each generator A of a cosine function is of the form $A = B^2 - \omega$ where B generates a C_0-group and $\omega \geq 0$. This beautiful result concludes the three sections on the second order Cauchy problem, applications of which will be given in Chapters 7 and 8.

Chapter 1

The Laplace Integral

The first three sections of this chapter are of a preliminary nature. There, we collect properties of the Bochner integral of functions of a real variable with values in a Banach space X. We then concentrate on the basic properties of the Laplace integral

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt := \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} f(t)\, dt$$

for locally Bochner integrable functions $f : \mathbb{R}_+ \to X$. In Section 1.4 we describe the set of complex numbers λ for which the Laplace integral converges. It will be shown that the domain of convergence is non-empty if and only if the antiderivative of f is of exponential growth. In Section 1.5 we discuss the holomorphy of $\lambda \mapsto \hat{f}(\lambda)$ and in Section 1.7 we show that f is uniquely determined by the Laplace integrals $\hat{f}(\lambda)$ (uniqueness and inversion). In Section 1.6 we prove the operational properties of the Laplace integral which are essential in applications to differential and integral equations. In particular, we show that the Laplace integral of the convolution $k * f : t \mapsto \int_0^t k(t-s) f(s)\, ds$ of a scalar-valued function k with a vector-valued function f is given by

$$\widehat{(k * f)}(\lambda) = \hat{k}(\lambda)\hat{f}(\lambda)$$

if $\hat{f}(\lambda)$ exists and $\hat{k}(\lambda)$ exists as an absolutely convergent integral. In Section 1.8 we consider vector-valued Fourier transforms and we show that Plancherel's theorem and the Paley-Wiener theorem extend to functions with values in a Hilbert space. Finally, after introducing the basic properties of the Riemann-Stieltjes integral in Section 1.9, we extend in Section 1.10 the basic properties of Laplace integrals to Laplace-Stieltjes integrals

$$\widehat{dF}(\lambda) := \int_0^\infty e^{-\lambda t}\, dF(t) := \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t}\, dF(t)$$

of functions F of bounded semivariation.

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If f is Bochner integrable, then the normalized antiderivative $t \mapsto F(t) := \int_0^t f(s)\,ds$ is of bounded variation. We will see that $\hat{f}(\lambda)$ exists if and only if $\widehat{dF}(\lambda)$ exists, and in this case $\hat{f}(\lambda) = \widehat{dF}(\lambda)$. Thus, the Laplace-Stieltjes integral is a natural extension of the Laplace integral. This extension is crucial for our discussion of the Laplace transform in Chapter 2 since there are many functions $r : (\omega, \infty) \to X$ which can be represented as a Laplace-Stieltjes integral, but not as a Laplace integral of a Bochner integrable function. Examples are, among others, Dirichlet series $r(\lambda) = \sum_{n=1}^{\infty} a_n e^{-\lambda n} = \widehat{dF}(\lambda)$, where F is the step function $\sum_{n=1}^{\infty} a_n \chi_{(n,\infty)}$, or any function $r(\lambda) = \widehat{dF}(\lambda)$, where F is of bounded semivariation, but not the antiderivative of a Bochner integrable function.

1.1 The Bochner Integral

This section contains some properties of the Bochner integral of vector-valued functions. We shall consider only those properties which are used in later sections, and we shall assume that the reader is familiar with the basic facts about measure and integration of scalar-valued functions.

Let X be a complex Banach space, and let I be an interval (bounded or unbounded) in $\mathbb{R}$, or a rectangle in $\mathbb{R}^2$. A function $f : I \to X$ is *simple* if it is of the form $f(t) = \sum_{r=1}^{n} x_r \chi_{\Omega_r}(t)$ for some $n \in \mathbb{N} := \{1, 2, \ldots\}, x_r \in X$ and Lebesgue measurable sets $\Omega_r \subset I$ with finite Lebesgue measure $m(\Omega_r)$; f is a *step function* when each Ω_r can be chosen to be an interval, or a rectangle in $\mathbb{R}^2$. Here χ_Ω denotes the characteristic (indicator) function of Ω. In the representation of a simple function, the sets Ω_r may always be arranged to be disjoint, and then

$$f(t) = \begin{cases} x_r & (t \in \Omega_r; r = 1, 2, \ldots, n) \\ 0 & \text{otherwise.} \end{cases}$$

A function $f : I \to X$ is *measurable* if there is a sequence of simple functions g_n such that $f(t) = \lim_{n\to\infty} g_n(t)$ for almost all $t \in I$. Since any χ_Ω for Ω measurable is a pointwise almost everywhere (a.e.) limit of a sequence of step functions, it is not difficult to see that the functions g_n can be chosen to be step functions. When $X = \mathbb{C}$, this definition agrees with the usual definition of (Lebesgue) measurable functions. It is easy to see that if $f : I \to X$, $g : I \to X$ and $h : I \to \mathbb{C}$ are measurable, then $f + g$ and $h \cdot f$ are measurable. Moreover, if $k : X \to Y$ is continuous (where Y is any Banach space), then $k \circ f$ is measurable whenever f is measurable. In particular, $\|f\|$ is measurable. If X is a closed subspace of Y, and f is measurable as a Y-valued function, then f is also measurable as an X-valued function.

To verify measurability of a function we often use the characterization given by Pettis's theorem below. We say that $f : I \to X$ is *countably valued* if there is a countable partition $\{\Omega_n : n \in \mathbb{N}\}$ of I into subsets Ω_n such that f is constant on each Ω_n; it is easy to see that f is measurable if each Ω_n is measurable (and conversely, $\{t \in I : f(t) = x\}$ is measurable whenever f is measurable and $x \in X$). Also, $f : I \to X$ is called *almost separably valued* if there is a null set Ω_0 in I such that $f(I \setminus \Omega_0) := \{f(t) : t \in I \setminus \Omega_0\}$ is separable (equivalently, $f(I \setminus \Omega_0)$ is contained in a separable closed subspace of X); f is *weakly measurable* if $x^* \circ f : t \mapsto \langle f(t), x^* \rangle$ is Lebesgue measurable for each x^* in the dual space X^* of X.

Here and subsequently, $\langle \cdot, \cdot \rangle$ denotes the duality between X and X^*. For a subset D of X, we shall denote the closure of D in X by $\overline{D}$. For $x \in X$ and $\varepsilon > 0$, we shall let $B_X(x, \varepsilon) := B(x, \varepsilon) := \{y \in X : \|y - x\| < \varepsilon\}$ and $\overline{B}(x, \varepsilon) := \{y \in X : \|y - x\| \le \varepsilon\}$. We shall also use this notation when $X = \mathbb{R}^n$ or $X = \mathbb{C}$, when it will be implicit that the norm is the Euclidean norm.

Theorem 1.1.1 (Pettis). *A function $f : I \to X$ is measurable if and only if it is weakly measurable and almost separably valued.*

Proof. If f is measurable, then there exist a null set Ω_0 and simple functions g_n such that $g_n \to f$ pointwise on $I \setminus \Omega_0$. The simple functions $x^* \circ g_n$ converge to $x^* \circ f$ on $I \setminus \Omega_0$ for all $x^* \in X^*$. Therefore, f is weakly measurable. The values taken by the functions g_n form a countable set D and $f(I \setminus \Omega_0) \subset \overline{D}$. Thus, f is almost separably valued.

To prove the converse statement one may replace X by the smallest closed subspace which contains $f(I \setminus \Omega_0)$ and then choose a countable dense set $\{x_n : n \in \mathbb{N}\}$. By the Hahn-Banach theorem, there are unit vectors $x_n^* \in X^*$ with $|\langle x_n, x_n^* \rangle| = \|x_n\|$. For any $\varepsilon > 0$ and $x \in X$ there exists x_k such that $\|x - x_k\| < \varepsilon$. Hence,

$$\begin{aligned} \sup_n |\langle x, x_n^* \rangle| &\le \|x\| \le \|x_k\| + \varepsilon = |\langle x_k, x_k^* \rangle| + \varepsilon \\ &\le |\langle x - x_k, x_k^* \rangle| + |\langle x, x_k^* \rangle| + \varepsilon \\ &\le \sup_n |\langle x, x_n^* \rangle| + 2\varepsilon. \end{aligned}$$

So

$$\|x\| = \sup_n |\langle x, x_n^* \rangle| \quad \text{for all } x \in X. \tag{1.1}$$

This implies that $t \mapsto \|f(t) - x\| = \sup_n |\langle f(t) - x, x_n^* \rangle|$ is measurable for all $x \in X$. Let

$$\Delta := \{t \in I \setminus \Omega_0 : \|f(t)\| > 0\} \text{ and } \Delta_{n,\varepsilon} := \{t \in \Delta : \|f(t) - x_n\| < \varepsilon\}$$

for $\varepsilon > 0$ and $n \in \mathbb{N}$. The sets $\Delta_{n,\varepsilon}$ are measurable and $\bigcup_n \Delta_{n,\varepsilon} = \Delta$. For fixed $\varepsilon > 0$, the sets $\Omega_{1,\varepsilon} := \Delta_{1,\varepsilon}$ and $\Omega_{n,\varepsilon} := \Delta_{n,\varepsilon} \setminus \bigcup_{k<n} \Delta_{k,\varepsilon}$ $(n \ge 2)$ form a measurable partition of Δ. Define a measurable, countably valued function g_ε on I by $g_\varepsilon := \sum_{i=1}^\infty x_i \chi_{\Omega_{i,\varepsilon}}$. Let $t \in I \setminus \Omega_0$. If $t \notin \Delta$, then $f(t) = 0 = g_\varepsilon(t)$. If $t \in \Delta$, then there exists $n \in \mathbb{N}$ such that $t \in \Omega_{n,\varepsilon}$. Hence,

$$\|f(t) - g_\varepsilon(t)\| < \varepsilon \quad \text{for all } t \in I \setminus \Omega_0.$$

This shows that f is the uniform limit almost everywhere of a sequence of measurable, countably valued functions.

Let (I_n) be an increasing sequence of bounded subintervals of I with $I = \bigcup_n I_n$. For each $n \in \mathbb{N}$, define a simple function $h_n := g_{2^{-n}}\chi_{H_n}$, where $H_n := I_n \cap \bigcup_{i=1}^{k_n} \Omega_{i,2^{-n}}$ and k_n is chosen such that the Lebesgue measure $m(I_n \setminus H_n) < 2^{-n}$. If $t \in \bigcap_{n=k}^{\infty} H_n$ for some $k \geq 1$, then

$$\|f(t) - h_n(t)\| = \|f(t) - g_{2^{-n}}(t)\| < 2^{-n}$$

for all $n \geq k$. Thus, $\lim_{n\to\infty} h_n(t) = f(t)$ for all $t \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} H_n$. For $k \geq j$,

$$m\left(I_j \setminus \bigcap_{n=k}^{\infty} H_n\right) \leq \sum_{n=k}^{\infty} m(I_n \setminus H_n) < 2^{-k+1}.$$

Hence, $I_j \setminus \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} H_n$ is null, for each j. This shows that $\lim_{n\to\infty} h_n(t) = f(t)$ for almost all $t \in I$. □

Corollary 1.1.2. *Let $f : I \to X$. Then the following statements hold:*

a) *The function f is measurable if and only if it is the uniform limit almost everywhere of a sequence of measurable, countably valued functions.*

b) *If X is separable, then f is measurable if and only if it is weakly measurable.*

c) *If f is continuous, then it is measurable.*

d) *If $f_n : I \to X$ are measurable functions and $f_n \to f$ pointwise a.e., then f is measurable.*

Proof. The statement b) is an immediate consequence of Pettis's Theorem 1.1.1. For d), observe first that f is weakly measurable. Define $\Omega_0 := \cup_n \Omega_n$ where Ω_n is a null set such that $f_n(I \setminus \Omega_n)$ is separable. Then $m(\Omega_0) = 0$ and $\Delta := \bigcup_n f_n(I \setminus \Omega_0)$ is separable. Since the least closed subspace containing Δ is separable and contains $f(I \setminus \Omega_0)$ it follows that f is almost separably valued. Thus, f is measurable. If f is continuous, then f is weakly measurable and the countable set $\{f(t) : t \in \mathbb{Q}\}$ is dense in the range of f. Again by Pettis's theorem, f is measurable. One implication of a) was established in the proof of Pettis's theorem and the converse follows from d). □

Pettis's theorem can be improved considerably in the following way. A subset W of X^* is called *separating* if for all $x \in X \setminus \{0\}$ there exists $x^* \in W$ such that $\langle x, x^* \rangle \neq 0$.

Corollary 1.1.3. *Let $f : I \to X$ be an almost separably valued function. Assume that $x^* \circ f$ is measurable for all x^* in a separating subset W of X^*. Then f is measurable.*

Proof. Changing f on a set of measure 0 and replacing X by a subspace, we can assume that X is separable. Let

$$Y := \{x^* \in X^* : \ x^* \circ f \ \text{ is measurable}\}.$$

Then Y is a subspace of X^* which contains W. By the Hahn-Banach theorem, Y is weak* dense in X^*. Let $Y_1 = Y \cap \overline{B}_{X^*}(0,1)$. We show that Y_1 is weak* closed. Let x^* be in the weak* closure of Y_1. Since X is separable, the weak* topology on $\overline{B}_{X^*}(0,1)$ is metrizable (see [Meg98, Theorem 2.6.23], for example). Thus, there exists a sequence $(x_n^*)_{n\in\mathbb{N}}$ in Y_1 converging to x^*. Hence, $x_n^* \circ f \to x^* \circ f$ as $n \to \infty$ pointwise on I. Thus, $x^* \circ f$ is measurable; i.e., $x^* \in Y_1$. This proves the claim. It follows from the Krein-Smulyan theorem (Theorem A.6) that Y is weak* closed. Since Y is weak* dense, we have $Y = X^*$; i.e., f is weakly measurable. Now the result follows from Theorem 1.1.1. □

For a simple function $g : I \to X,\ g = \sum_{i=1}^n x_i \chi_{\Omega_i}$, we define

$$\int_I g(t)\, dt := \sum_{i=1}^n x_i m(\Omega_i)$$

where $m(\Omega)$ is the Lebesgue measure of Ω. It is routine to verify that the definition is independent of the representation $g = \sum_{i=1}^n x_i \chi_{\Omega_i}$, and the integral so defined is linear.

A function $f : I \to X$ is called *Bochner integrable* if there exist simple functions g_n such that $g_n \to f$ pointwise a.e., and $\lim_{n\to\infty} \int_I \|f(t) - g_n(t)\|\, dt = 0$. If $f : I \to X$ is Bochner integrable, then the *Bochner integral of f on I* is

$$\int_I f(t)\, dt := \lim_{n\to\infty} \int_I g_n(t)\, dt.$$

It is easy to see that this limit exists and is independent of the choice of the sequence (g_n). If Ω is measurable with finite measure, then χ_Ω can be approximated in L^1-norm by step functions, and it follows that the functions g_n can always be chosen to be step functions. The integral $\int_I f(t)\, dt$ lies in the closed linear span of $\{f(t) : \ t \in I\}$. The set of all Bochner integrable functions from I to X is a linear space and the Bochner integral is a linear mapping. When $X = \mathbb{C}$, the definitions of Bochner integrability and integrals agree with those of Lebesgue integration theory.

When I is a rectangle, we may denote a Bochner integral by $\int_I f(s,t)\, d(s,t)$.

It is one of the great virtues of the Bochner integral that the class of Bochner integrable functions is easily characterized.

Theorem 1.1.4 (Bochner). *A function $f : I \to X$ is Bochner integrable if and only if f is measurable and $\|f\|$ is integrable. If f is Bochner integrable, then*

$$\left\| \int_I f(t)\, dt \right\| \le \int_I \|f(t)\|\, dt. \tag{1.2}$$

Proof. If f is Bochner integrable, then there exists an approximating sequence of simple functions g_n. Thus f and $\|f\|$ are measurable. The integrability of $\|f\|$ follows from

$$\int_I \|f(t)\|\,dt \le \int_I \|g_n(t)\|\,dt + \int_I \|f(t)-g_n(t)\|\,dt.$$

Moreover,

$$\begin{aligned}\left\|\int_I f(t)\,dt\right\| = \lim_{n\to\infty}\left\|\int_I g_n(t)\,dt\right\| &\le \lim_{n\to\infty}\int_I \|g_n(t)\|\,dt \\ &= \int_I \|f(t)\|\,dt.\end{aligned}$$

To prove the converse statement, let (h_n) be a sequence of simple functions approximating f pointwise on $I \setminus \Omega_0$, where $m(\Omega_0) = 0$. Define simple functions by

$$g_n(t) := \begin{cases} h_n(t) & \text{if } \|h_n(t)\| \le \|f(t)\|(1+n^{-1}), \\ 0 & \text{otherwise.}\end{cases}$$

Then $\|g_n(t)\| \le \|f(t)\|(1+n^{-1})$ and $\lim_{n\to\infty}\|g_n(t)-f(t)\| = 0$ for all $t \in I \setminus \Omega_0$. Because the functions $\|f\|$ and $\|g_n - f\|$ are integrable and $\|g_n(t) - f(t)\| \le 3\|f(t)\|$, we can apply the scalar dominated convergence theorem and obtain that $\lim_{n\to\infty}\int_I \|g_n(t)-f(t)\|\,dt = 0$. □

Example 1.1.5. a) Let X be the Lebesgue space $L^\infty(0,1)$ of all (equivalence classes of) bounded measurable functions from $(0,1)$ to $\mathbb{C}$. Let $f : (0,1) \to L^\infty(0,1)$ be given by $f(t) := \chi_{(0,t)}$. Then f is not almost separably valued since $\|f(t)-f(s)\| = 1$ for $t \ne s$. Thus, f is not measurable and therefore not Bochner integrable.

b) Let X be the Banach space c_0 of all complex sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = 0$, with $\|x\| = \sup_n |x_n|$. Identify X^* with the space ℓ^1 of all complex sequences $a = (a_n)_{n\in\mathbb{N}}$ such that $\|a\| := \sum_{n=1}^\infty |a_n| < \infty$. Let $f : [0,1] \to c_0$ be given by $f(t) := (f_n(t))_{n\in\mathbb{N}}$ where $f_n(t) := n\chi_{[0,\frac{1}{n}]}(t)$. Let $x^* = (a_n)_{n\in\mathbb{N}} \in \ell^1$. Then $t \mapsto \langle f(t), x^*\rangle = \sum_{n=1}^\infty n a_n \chi_{[0,\frac{1}{n}]}(t)$ is measurable on $[0,1]$. Since c_0 is separable, it follows from Pettis's theorem that f is measurable. Moreover,

$$\int_0^1 |\langle f(t), x^*\rangle|\,dt \le \sum_{n=1}^\infty |a_n| = \|x^*\| < \infty.$$

However, $\|f(t)\| = n$ for $t \in (\frac{1}{n+1}, \frac{1}{n}]$, so $t \mapsto \|f(t)\|$ is not integrable on $[0,1]$. Thus, f is not Bochner integrable on $[0,1]$. □

Now we will consider the behaviour of the Bochner integral under linear operators. The following result is a straightforward consequence of the definition of the Bochner integral, and we shall use it frequently without comment, especially in the case of a linear functional ($Y = \mathbb{C}$).

Proposition 1.1.6. *Let $T : X \to Y$ be a bounded linear operator between Banach spaces X and Y, and let $f : I \to X$ be Bochner integrable. Then $T \circ f : t \mapsto T(f(t))$ is Bochner integrable and $T \int_I f(t)\,dt = \int_I T(f(t))\,dt$.*

We shall also need a version of Proposition 1.1.6 for a closed operator A on X (see Appendix B for notation and terminology).

Proposition 1.1.7. *Let A be a closed linear operator on X. Let $f : I \to X$ be Bochner integrable. Suppose that $f(t) \in D(A)$ for all $t \in I$ and $A \circ f : I \to X$ is Bochner integrable. Then $\int_I f(t)\,dt \in D(A)$ and*

$$A \int_I f(t)\,dt = \int_I A(f(t))\,dt.$$

Proof. Consider $X \times X$ as a Banach space in the norm $\|(x, y)\| = \|x\| + \|y\|$. The graph $G(A)$ of A is a closed subspace of $X \times X$. Define $g : I \to G(A) \subset X \times X$ by $g(t) = (f(t), A(f(t)))$. It is easy to see that g is measurable and

$$\int_I \|g(t)\|\,dt = \int_I \|f(t)\|\,dt + \int_I \|A(f(t))\|\,dt < \infty.$$

By Theorem 1.1.4, g is Bochner integrable. Moreover, $\int_I g(t)\,dt \in G(A)$. Applying Proposition 1.1.6 to the two projection maps of $X \times X$ onto X shows that

$$\int_I g(t)\,dt = \left(\int_I f(t)\,dt\ ,\ \int_I A(f(t))\,dt \right).$$

This gives the result. □

Now we give vector-valued versions of two classical theorems of integration theory.

Theorem 1.1.8 (Dominated Convergence). *Let $f_n : I \to X$ $(n \in \mathbb{N})$ be Bochner integrable functions. Assume that $f(t) := \lim_{n\to\infty} f_n(t)$ exists a.e. and there exists an integrable function $g : I \to \mathbb{R}$ such that $\|f_n(t)\| \le g(t)$ a.e. for all $n \in \mathbb{N}$. Then f is Bochner integrable and $\int_I f(t)\,dt = \lim_{n\to\infty} \int_I f_n(t)\,dt$. Furthermore, $\int_I \|f(t) - f_n(t)\|\,dt \to 0$ as $n \to \infty$.*

Proof. The function f is Bochner integrable since it is measurable (by Corollary 1.1.2) and since $\|f\|$ is integrable (because $\|f(t)\| \le g(t)$ a.e.). Define $h_n(t) := \|f(t) - f_n(t)\|$ for $t \in I$. Since $|h_n(t)| \le 2g(t)$ and $h_n(t) \to 0$ a.e., the scalar dominated convergence theorem implies that $\int_I \|f(t) - f_n(t)\|\,dt \to 0$ as $n \to \infty$. By (1.2),

$$\left\| \int_I f(t)\,dt - \int_I f_n(t)\,dt \right\| \to 0.$$

□

Theorem 1.1.9 (Fubini's Theorem). *Let $I = I_1 \times I_2$ be a rectangle in $\mathbb{R}^2$, let $f : I \to X$ be measurable, and suppose that*

$$\int_{I_1} \int_{I_2} \|f(s,t)\| \, dt \, ds < \infty.$$

Then f is Bochner integrable and the repeated integrals

$$\int_{I_1} \int_{I_2} f(s,t) \, dt \, ds, \quad \int_{I_2} \int_{I_1} f(s,t) \, ds \, dt$$

exist and are equal, and they coincide with the double integral $\int_I f(s,t) \, d(s,t)$.

Proof. Since any measurable function is almost separably valued, we may assume that X is separable.

The scalar-valued case of Fubini's theorem implies that $\|f\|$ is integrable on I, $\int_{I_2} \|f(s,t)\| \, dt$ exists for almost all $s \in I_1$, and for each $x^* \in X^*$ the repeated integrals

$$\int_{I_1} \int_{I_2} \langle f(s,t), x^* \rangle \, dt \, ds, \quad \int_{I_2} \int_{I_1} \langle f(s,t), x^* \rangle \, ds \, dt$$

exist and are equal. It follows from Theorem 1.1.4 that $f : I \to X$ is Bochner integrable and $\int_{I_2} f(s,t) \, dt$ exists for almost all $s \in I_1$, and from Theorem 1.1.1 that $s \mapsto \int_{I_2} f(s,t) \, dt$ is measurable. Moreover,

$$\int_{I_1} \left\| \int_{I_2} f(s,t) \, dt \right\| \, ds \le \int_{I_1} \int_{I_2} \|f(s,t)\| \, dt \, ds < \infty,$$

so Theorem 1.1.4 shows that $\int_{I_1} \left(\int_{I_2} f(s,t) \, dt \right) \, ds$ exists. Since

$$\int_{I_2} \int_{I_1} \|f(s,t)\| \, ds \, dt = \int_{I_1} \int_{I_2} \|f(s,t)\| \, dt \, ds,$$

it follows similarly that $\int_{I_2} \left(\int_{I_1} f(s,t) \, ds \right) \, dt$ exists. For any $x^* \in X^*$,

$$\begin{aligned}
\left\langle \int_{I_1} \left(\int_{I_2} f(s,t) \, dt \right) \, ds, x^* \right\rangle &= \int_{I_1} \int_{I_2} \langle f(s,t), x^* \rangle \, dt \, ds \\
&= \int_I \langle f(s,t), x^* \rangle \, d(s,t) \\
&= \left\langle \int_I f(s,t) \, d(s,t), x^* \right\rangle \\
&= \int_{I_2} \int_{I_1} \langle f(s,t), x^* \rangle \, ds \, dt \\
&= \left\langle \int_{I_2} \left(\int_{I_1} f(s,t) \, ds \right) \, dt, x^* \right\rangle.
\end{aligned}$$

The Hahn-Banach theorem implies that

$$\int_{I_1} \left(\int_{I_2} f(s,t)\, dt \right) ds = \int_I f(s,t)\, d(s,t) = \int_{I_2} \left(\int_{I_1} f(s,t)\, ds \right) dt. \qquad \square$$

Let $L^1(I,X)$ denote the space of all Bochner integrable functions $f : I \to X$, and let

$$\|f\|_1 := \int_I \|f(t)\|\, dt.$$

In the usual way, we shall identify functions which differ only on sets of measure zero. Then $\|\cdot\|_1$ is a norm on $L^1(I,X)$.

Theorem 1.1.10. *The space $L^1(I,X)$ is a Banach space.*

Proof. Let (f_n) be a sequence in $L^1(I,X)$ with $\sum \|f_n\|_1 < \infty$. By the monotone convergence theorem for series of positive scalar-valued functions, $\sum \|f_n(t)\| < \infty$ a.e., $\sum_{n=1}^{\infty} \|f_n(\cdot)\|$ is integrable, and

$$\int_I \sum_{n=1}^{\infty} \|f_n(t)\|\, dt = \sum_{n=1}^{\infty} \int_I \|f_n(t)\|\, dt.$$

Hence, $\sum_{n=1}^{\infty} f_n(t)$ converges a.e. to a sum $g(t)$ in the Banach space X. By Corollary 1.1.2, g is measurable. Moreover, $\|g(t)\| \le \sum_{n=1}^{\infty} \|f_n(t)\|$, so $\|g\|$ is integrable. By Theorem 1.1.4, g is integrable. Finally,

$$\left\| g - \sum_{n=1}^{k} f_n \right\|_1 \le \int_I \|g(t) - \sum_{n=1}^{k} f_n(t)\|\, dt \le \int_I \sum_{n=k+1}^{\infty} \|f_n(t)\|\, dt \to 0$$

as $k \to \infty$. Thus, $L^1(I,X)$ is a Banach space. $\square$

By the definition of Bochner integrability, the simple functions are dense in $L^1(I,X)$ and, by the remarks following the definition, the step functions are dense. It follows easily that the infinitely differentiable functions of compact support are also dense in $L^1(I,X)$.

We shall be particularly interested in the case when $I = \mathbb{R}_+ := [0,\infty)$. If $f \in L^1(\mathbb{R}_+,X)$, an application of the Dominated Convergence Theorem shows that

$$\int_0^{\infty} f(t)\, dt = \lim_{\tau \to \infty} \int_0^{\tau} f(t)\, dt. \tag{1.3}$$

When $f \in L^1_{loc}(\mathbb{R}_+,X)$ (i.e., f is Bochner integrable on $[0,\tau]$ for every $\tau \in \mathbb{R}_+$), the limit in (1.3) may exist without f being Bochner integrable on $\mathbb{R}_+$. If the limit exists, we say that $\int_0^{\infty} f(t)\, dt$ *converges as an improper* (or *principal value*) *integral*, and we define

$$\int_0^{\infty} f(t)\, dt := \lim_{\tau \to \infty} \int_0^{\tau} f(t)\, dt.$$

When $f \in L^1(\mathbb{R}_+, X)$, i.e. $\int_0^\infty \|f(t)\| \, dt < \infty$, we say that the integral is *absolutely convergent* .

For $1 < p < \infty$, let $L^p(I, X)$ denote the space of all measurable functions $f : I \to X$ such that

$$\|f\|_p := \left(\int_I \|f(t)\|^p \, dt \right)^{1/p} < \infty.$$

Let $L^\infty(I, X)$ be the space of all measurable functions $f : I \to X$ such that

$$\|f\|_\infty := \operatorname*{ess\,sup}_{t \in I} \|f(t)\| < \infty.$$

Note that the spaces $L^p(I, \mathbb{C})$ $(1 \leq p \leq \infty)$ are the usual Lebesgue spaces which we shall denote simply by $L^p(I)$. With the usual identifications, each of the spaces $L^p(I, X)$ becomes a Banach space. The proofs of completeness are similar to the scalar-valued cases.

The proof of Theorem 1.1.4 shows that the simple functions are dense in $L^p(I, X)$ for $1 < p < \infty$ (so $L^p(I, X)$ can also be defined in a similar way to the Bochner integrable functions). It follows that the step functions, and the infinitely differentiable functions of compact support, are also dense. By considering such functions first and then approximating, one may show as in the scalar-valued case, that if $f \in L^p(I, X)$ and

$$f_t(s) := \begin{cases} f(s-t) & \text{if } s - t \in I, \\ 0 & \text{otherwise,} \end{cases}$$

then $t \mapsto f_t$ is continuous from $\mathbb{R}$ into $L^p(I, X)$ for $1 \leq p < \infty$.

We have presented the theory above in the case when I is an interval in $\mathbb{R}$ (or, for Fubini's theorem, I is a rectangle in $\mathbb{R}^2$). Almost all the integrals which appear in this book will indeed be over intervals in $\mathbb{R}$ (or repeated integrals in $\mathbb{R}^2$). However, the entire theory works, with no changes in the proofs, when I is a measurable set in $\mathbb{R}^n$ (or in $\mathbb{R}^m \times \mathbb{R}^n$, in Fubini's theorem). Since the step functions are dense in each of the spaces $L^p(I \times J, X)$ for $1 \leq p < \infty$, it is easy to see from Fubini's theorem that there is an isometric isomorphism between $L^p(I \times J, X)$ and $L^p(I, L^p(J, X))$ given by $f \mapsto g$, where

$$(g(s))(t) := f(s, t).$$

This enables many properties of the spaces $L^p(I, X)$ when I is a rectangle in $\mathbb{R}^n$ to be deduced from the case $n = 1$.

Finally in this section, we introduce notation for spaces of continuous and differentiable functions. Let I be an interval in $\mathbb{R}$. We denote by $C(I, X)$ the vector space of all continuous functions $f : I \to X$. For $k \in \mathbb{N}$, we denote by $C^k(I, X)$ the space of all k-times differentiable functions with continuous kth derivative, and we put $C^\infty(I,X) := \bigcap_{k=1}^\infty C^k(I,X)$. When I is compact, $C(I,K)$ is a closed subspace of $L^\infty(I,X)$ and therefore a Banach space with respect to the supremum norm $\|\cdot\|_\infty$.

When I is not compact, we let $C_c(I,X)$ be the space of all functions in $C(I,X)$ with compact support, and $C_c^\infty(I,X) := C_c(I,X) \cap C^\infty(I,X)$. Thus $C_c^\infty(I,X)$ is a dense subspace of $L^p(I,X)$ for $1 \le p < \infty$. When $I = \mathbb{R}_+$ or $I = \mathbb{R}$, we shall also consider the space $C_0(I,X)$ of all continuous functions $f : I \to X$ such that $\lim_{|t|\to\infty, t\in I} \|f(t)\| = 0$ and the space $\mathrm{BUC}(I,X)$ of all bounded, uniformly continuous functions $f : I \to X$. These are both Banach spaces with respect to $\|\cdot\|_\infty$, and $C_0(I,X) \subset \mathrm{BUC}(I,X) \subset L^\infty(I,X)$.

When $X = \mathbb{C}$, we shall write $C(I)$ in place of $C(I,\mathbb{C})$, etc., and we shall extend this notation to cases when I is replaced by an open subset Ω of $\mathbb{R}^n$. Note that $C_c^\infty(\Omega)$ coincides with the space $\mathcal{D}(\Omega)$ of *test functions* on Ω (see Appendix E), and we shall use both notations according to context. Furthermore, when Ω is any locally compact space, we shall let $C_0(\Omega)$ be the Banach space of all continuous complex-valued functions on Ω which vanish at infinity, with the supremum norm. When K is any compact space, we shall let $C(K)$ be the Banach space of all continuous complex-valued functions on K, with the supremum norm.

1.2 The Radon-Nikodym Property

In this section we consider properties of functions F obtained as indefinite integrals. If $f : [a,b] \to X$ is Bochner integrable, we say that $F : [a,b] \to X$ is an *antiderivative* or *primitive* of f if

$$F(t) = F(a) + \int_a^t f(s)\,ds \qquad (t \in [a,b]).$$

Given a function $F : [a,b] \to X$ and a partition π, $a = t_0 < t_1 < \ldots < t_n = b$, let

$$V(\pi, F) := \sum_{i=1}^n \|F(t_i) - F(t_{i-1})\|.$$

Then F is said to be *of bounded variation* if

$$V(F) := V_{[a,b]}(F) := \sup_\pi V(\pi, F) < \infty,$$

where the supremum is taken over all partitions π of $[a,b]$.

We say that F is *absolutely continuous* on $[a,b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|F(b_i) - F(a_i)\| < \varepsilon$ for every finite collection $\{(a_i,b_i)\}$ of disjoint intervals in $[a,b]$ with $\sum_i (b_i - a_i) < \delta$. We say that F is *Lipschitz continuous* if there exists M such that $\|F(t) - F(s)\| \le M|t-s|$ for all $s,t \in [a,b]$. Clearly, any Lipschitz continuous function is absolutely continuous.

Proposition 1.2.1. *Let $F : [a,b] \to X$ be absolutely continuous. Then F is of bounded variation. Moreover, if $G(t) := V_{[a,t]}(F)$, then G is absolutely continuous on $[a,b]$.*

Proof. Take $\varepsilon > 0$, and let δ be as in the definition of absolute continuity of F. Then $\sum_i V_{[a_i,b_i]}(F) \le \varepsilon$ whenever $\{(a_i,b_i)\}$ is a finite collection of disjoint subintervals of $[a,b]$ with $\sum_i (b_i - a_i) < \delta$. In particular, F is of bounded variation on any subinterval of length less than δ. Since $[a,b]$ is a finite union of such intervals, F is of bounded variation on $[a,b]$. Moreover,

$$\sum_i |G(b_i) - G(a_i)| = \sum_i V_{[a_i,b_i]}(F) < \varepsilon.$$

Thus, G is absolutely continuous. □

A point $t \in [a,b]$ is said to be a *Lebesgue point* of $f \in L^1([a,b],X)$ if $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \|f(s) - f(t)\|\, ds = 0$. It is easy to see that any point of continuity is a Lebesgue point, and the following proposition shows that almost all points are Lebesgue points.

Proposition 1.2.2. *Let $f : [a,b] \to X$ be Bochner integrable and $F(t) := \int_a^t f(s)\, ds$ $(t \in [a,b])$. Then*

a) *F is differentiable a.e. and $F' = f$ a.e.*

b) $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \|f(s) - f(t)\|\, ds = 0$ *t-a.e.*

c) *F is absolutely continuous.*

d) $V_{[a,b]}(F) = \int_a^b \|f(s)\|\, ds$.

Proof. To show a) and b) let g_n be step functions such that

$$f(t) = \lim_{n\to\infty} g_n(t) \;\text{ a.e. and }\; \lim_{n\to\infty} \int_a^b \|f(t) - g_n(t)\|\, dt = 0.$$

For $h > 0$,

$$\begin{aligned} \left\| \frac{1}{h}\int_t^{t+h} f(s)\, ds - f(t) \right\| &\le \frac{1}{h}\int_t^{t+h} \|f(s) - f(t)\|\, ds \\ &\le \frac{1}{h}\int_t^{t+h} \|f(s) - g_n(s)\|\, ds \\ &\quad + \frac{1}{h}\int_t^{t+h} \|g_n(s) - g_n(t)\|\, ds + \|g_n(t) - f(t)\|. \end{aligned}$$

Since g_n is a step function and $s \mapsto \|f_n(s) - g_n(s)\|$ is Lebesgue integrable, it follows from Lebesgue's theorem [Rud87, Theorem 8.17] that

$$\begin{aligned} \limsup_{h \downarrow 0} \left\| \frac{1}{h} \int_t^{t+h} f(s)\, ds - f(t) \right\| &\leq \limsup_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|\, ds \\ &\leq 2\|g_n(t) - f(t)\| \end{aligned}$$

for all $t \in [a, b] \setminus \Omega_n$ and some null set Ω_n. Taking the limit as $n \to \infty$ yields the right-differentiability of F and

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|\, ds = 0$$

for all $t \in [a, b] \setminus \bigcup_{n \in \mathbb{N}} \Omega_n$. The left-hand limits are similar.

For c), let $\varepsilon > 0$. There exists $\delta > 0$ such that $\int_\Omega \|f(s)\|\, ds < \varepsilon$ whenever $\mu(\Omega) < \delta$. If $\{(a_i, b_i)\}$ is a finite collection of disjoint subintervals of $[a, b]$ with $\sum_i (b_i - a_i) < \delta$, then taking $\Omega = \bigcup_i (a_i, b_i)$, we deduce that

$$\sum_i \|F(b_i) - F(a_i)\| = \sum_i \left\| \int_{a_i}^{b_i} f(s)\, ds \right\| \leq \int_\Omega \|f(s)\|\, ds < \varepsilon.$$

To prove the statement d), observe first that, for any partition π of $[a, b]$,

$$V(\pi, F) = \sum_i \left\| \int_{t_{i-1}}^{t_i} f(s)\, ds \right\| \leq \int_a^b \|f(s)\|\, ds.$$

Thus, $V(F) \leq \int_a^b \|f(s)\|\, ds$. Conversely, given $\varepsilon > 0$, we may choose a step function g such that $\int_a^b \|f(s) - g(s)\|\, ds < \varepsilon$. There is a partition π of $[a, b]$ such that g is constant on each interval (t_{i-1}, t_i). Then

$$\begin{aligned} \int_a^b \|f(s)\|\, ds - V(F) &\leq \int_a^b \|f(s)\|\, ds - V(\pi, F) \\ &\leq \int_a^b \|g(s)\|\, ds + \varepsilon - \sum_i \left\| \int_{t_{i-1}}^{t_i} f(s)\, ds \right\| \\ &= \sum_i \left(\left\| \int_{t_{i-1}}^{t_i} g(s)\, ds \right\| - \left\| \int_{t_{i-1}}^{t_i} f(s)\, ds \right\| \right) + \varepsilon \\ &\leq \int_a^b \|f(s) - g(s)\|\, ds + \varepsilon \\ &\leq 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of d). $\square$

In the scalar case, the fundamental theorem of calculus [Rud87, Theorem 8.18] states that any absolutely continuous function $F : [a, b] \to \mathbb{C}$ is differentiable a.e., $f := F'$ is Lebesgue integrable, and $F(t) - F(a) = \int_a^t f(s)\, ds$ for all $t \in [a, b]$. We will see below (Example 1.2.8) that the fundamental theorem does not hold for Lipschitz continuous functions with values in arbitrary Banach spaces. The following weaker statement holds for all Banach spaces.

Proposition 1.2.3. *Let $F : [a, b] \to X$ be absolutely continuous, and suppose that $f(t) := F'(t)$ exists a.e. Then f is Bochner integrable and $F(t) = F(a) + \int_a^t f(s)\, ds$ for all $t \in [a, b]$.*

Proof. Since $f(t) = \lim_{n\to\infty} n(F(t + 1/n) - F(t))$, it follows from Corollary 1.1.2 that f is measurable. Let $G(t) := V_{[a,t]}(F)$, so $G : [a, b] \to \mathbb{R}$ is absolutely continuous by Proposition 1.2.1. Hence G is differentiable a.e. and $G' \in L^1[a, b]$. Since

$$\|F(t + h) - F(t)\| \le V_{[t,t+h]}(F) = G(t + h) - G(t),$$

$\|f(t)\| \le G'(t)$ a.e. Hence $\|f\| \in L^1[a, b]$, so f is Bochner integrable by Theorem 1.1.4. For $x^* \in X^*$,

$$\begin{aligned} \langle F(t), x^*\rangle &= \langle F(a), x^*\rangle + \int_a^t \langle f(s), x^*\rangle\, ds \\ &= \left\langle F(a) + \int_a^t f(s)\, ds, x^* \right\rangle \end{aligned}$$

by the scalar fundamental theorem of calculus. By the Hahn-Banach theorem, $F(t) = F(a) + \int_a^t f(s)\, ds$. □

Let I be any interval in $\mathbb{R}$. A function $F : I \to X$ is said to be *absolutely continuous* if it is absolutely continuous on each compact interval of I. We now consider the property that every absolutely continuous function $F : I \to X$ is differentiable a.e. It is easy to see that this property is independent of the interval I, so it is a property of X alone.

Proposition 1.2.4. *For any Banach space X the following are equivalent:*

(i) *Every absolutely continuous function $F : \mathbb{R}_+ \to X$ is differentiable a.e.*

(ii) *Every Lipschitz continuous function $F : \mathbb{R}_+ \to X$ is differentiable a.e.*

Proof. Clearly, (i) implies (ii). Assume that statement (ii) holds and let $F : \mathbb{R}_+ \to X$ be absolutely continuous. By Proposition 1.2.1, F is locally of bounded variation and G is absolutely continuous where $G(t) := V_{[0,t]}(F)$. Let $h(t) := G(t) + t$. Then h is strictly increasing, $h(0) = 0$, and $h(\mathbb{R}_+) = \mathbb{R}_+$. Moreover,

$$\|F(t) - F(s)\| \le G(t) - G(s) \le h(t) - h(s)$$