Giovanni Sommaruga
Editor

The Western Ontario Series
in Philosophy of Science

Foundational Theories of Classical and Constructive Mathematics

Springer

Foundational Theories of Classical and Constructive Mathematics

THE WESTERN ONTARIO SERIES IN PHILOSOPHY OF SCIENCE

A SERIES OF BOOKS IN PHILOSOPHY OF MATHEMATICS AND NATURAL SCIENCE, HISTORY OF SCIENCE, HISTORY OF PHILOSOPHY OF SCIENCE, EPISTEMOLOGY, PHILOSOPHY OF COGNITIVE SCIENCE, GAME AND DECISION THEORY

VOLUME 76

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Foundational Theories of Classical and Constructive Mathematics

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ISSN 1566-659X
ISBN 978-94-007-0430-5 e-ISBN 978-94-007-0431-2
DOI 10.1007/978-94-007-0431-2
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2011922884

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The present book project grew out of the Swiss Society for Logic and Philosophy of Science (SSLPS) annual meeting on “Foundational theories of mathematics” which was held in Freiburg (Switzerland) on October 11/12 2006. John Bell and Gerhard Jäger, both participating in different functions in this meeting, responded to this book project with great enthusiasm and fueled its evolution with recurrent positive feedbacks. I’m happy to have had the opportunity over the years to discuss with them Foundations of Mathematics (FOM) and many other hot topics as well as some less hot ones.

I’m grateful to Oxford University Press for permission to reprint Section I.2 of Penelope Maddy’s book *Naturalism in Mathematics* (Oxford: Clarendon Press, 1997) and also to the publishing company Polimetrica for permission to reprint Solomon Feferman’s article which first appeared in G. Sica’s book *What is Category Theory?* (Monza: Polimetrica, 2006). I’d like to thank Norman Sieroka for helping me to translate several papers from (ancient) Word to (modern) LaTeX. I’d also like to thank Lucy Fleet, Senior Assistant to the Editorial Director—Humanities of Springer, for her patience and her refreshing confidence that this book would eventually see the light of day. I’m very grateful to the IT Service of the ETH Zurich and in particular to its collaborator Dieter Hennig for his very valuable support in my rather dilettante LaTeX-engineering.

Last and most I’d like to thank my wife Andrea Gemma and my little son Beniamino for not having banned me from home before regaining normality, and even more so for their loving care with which they endured and accepted my mixture of mental absence and busy bee existence during the more productive phases of this book’s production.

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Introduction

Giovanni Sommaruga

Part I A Retrospective: Some Remarks on the Historiography of FOM[1]

Section 1 : The Meaning of 'FOM' According to Mostowski, Parsons and Wang

Before sketching the development of 120 years of studies in FOM, the following questions ought to be addressed: What makes the studies and results considered, analysed, discussed etc. by Andrzej Mostowski, Charles Parsons and Hao Wang foundational studies or results, that is contributions to FOM? What are the features or characteristics of foundational research according to these specialists and historiographers on FOM?

The most important or at least characteristics highly valued and shared by all these specialists on FOM are the following ones:

1. Some types of conceptual analysis
   For Wang, the business of research in FOM is essentially conceptual analysis. He distinguishes 2 essential ingredients of conceptual analysis in FOM (H.W.: in mathematical logic): (i) reduction, (ii) formalisation. The purpose of this conceptual analysis is to make a concept or a set of concepts or a theory more sharp, more precise.

   concerning (i): Wang characterises reduction as "one way of simplifying a concept [...] by reducing more components to less or by simplifying each separate aspect".[2] He continues to further divide reduction into (i.1) local reduction and (i.2) regional or global reduction (in H.W.'s terms "whole"

[1] This first part of the introduction is entirely based on the historical surveys of studies in Foundations of Mathematics (FOM): A. Mostowski (1965), C. Parsons (2006) and H. Wang (1958).

[2] Wang (1958, p. 468).

reduction). Among the types of local reduction he distinguishes the reduction by definition, the reduction by deduction (i.e. applying the axiomatic method) and something he calls uniform local reduction which is the interpretation of a formalism or formal system by another one. And by a regional or global reduction he means the reduction of (a region or branch or) the whole of mathematics to a (another) region or branch. Wang observes that regional or global reductions are often of greater interest to philosophers than local ones, but at the same time they are often not sound.

concerning (ii): Wang characterises formalisation as putting a concept or a set of concepts into a formal system which makes all the implicit assumptions explicit. He subdivides formalisation into thorough formalisation (a thorough axiomatisation containing predicate logic and the concepts to be formalised which are thereby "implicitly defined") and partial formalisation (no or only partial axiomatisation and other concepts than the ones to be formalised occur).

Hao Wang concludes his reflections on conceptual analysis in FOM with two claims: Claim A: Formalisation rather than reduction is the appropriate method in foundational studies, as the latter are primarily interested in irreducible concepts. Claim B: Formalisation as a method has been mainly practiced in mathematical logic (far more so than in any other branch of mathematics) for the last (and the first) 80 years of studies in FOM.[3]

2. An orientation towards the basic distinction between constructive methods and non-constructive (or classical) methods in mathematics.[4]

   In his discussion of this basic distinction, Wang refers to Bernays' 5 shades of constructive and non-constructive methods in mathematics: in order of decreasing constructivity (i) anthropologism (or finitism in the narrower sense), (ii) finitism (in the broader sense), (iii) intuitionism, (iv) predicative set theory (or predicativism), and (v) classical set theory (or platonism). Wang makes a series of comments on these different domains, a particular and very modern comment being the following one: The domains (i)–(v) should not be treated as rival domains among which one has to choose one (for life), but they should rather be treated "as useful reports about a same grand structure which can help us to construct a whole picture that would be more adequate than each taken alone".[5] He identifies a central irreducible concept of each of these domains: of (i) the concept of feasibility, of (ii) the concept of constructivity, of (iii) the concept of (constructive) proof, of (iv) the concept of (natural) number, and of (v) the concept of set. Then he remarks that a sharp and precise definiteness of these only vaguely characterised domains (i)–(v) may be obtained by a conceptual

[3] Reduction and formalisation also play a fairly important role in Mostowski's (1967), however in a more implicit way. Parsons in his (2006) treats the axiomatic method (reduction) and formalisation explicitly in the 1st paragraph.

[4] Mostowski calls the divide: infinitistic or set-theoretical vs. finitistic or arithmetical; Parsons calls the divide: platonism vs. constructivism.

[5] Wang (1958, p. 472).

analysis, and in particular by a formalisation of these 5 central concepts. These formalisations might form the hard core of studies in FOM. But, completely in agreement with Mostowski, he adds that at times various ramifications and cross-overs may be more important than the hard core itself.[6] Hao Wang subsequently presents a short historical characterisation of each of these domains (i)–(v) within the development of research on FOM. And Parsons dedicates a whole paragraph (§3) to these various domains.

3. An intimate relation with mathematical logic in its various subdisciplines (set theory, model theory, proof theory and computability theory)
4. (For Mostowski and Parsons) being a contribution to one of the original 3 movements (schools) in the philosophy of mathematics: formalism (Hilbert's program), intuitionism, and logicism, or later on to their more technical successors: meta-mathematics, constructivism, and set theory resp.[7]
5. (For Mostowski and Parsons) being a solution or a partial solution to a major philosophical problem, such as e.g. the completeness problem, the problem of set-theoretical paradoxes, the decision problem, the problem of impredicative definition etc.
6. (Esp. for Parsons) being a contribution to the solution of the problem of justifying mathematical statements or principles (the so-called epistemological point of view in FOM)

NB. These features or characteristics of studies or results on FOM (according to Mostowski, Parsons and Wang) are obviously not mutually exclusive.

Section 2 : The First 80 Years of Studies in FOM According to Mostowski and Wang

According to Mostowski and Wang,[8] the 1st phase of studies in FOM starts in the 1880s.[9] Its most important elements are: Cantor's so-called naive set theory and the subsequent formalisation of the central concept of set (i.e. the various axiomatisations of Cantorian set theory) as a reaction to the appearance of set-theoretical paradoxes; Frege's classical 1st order logic as a formalisation of all usual methods of mathematical argument of a strictly logical nature; and finally, the 3 well-known movements (schools) in the philosophy of mathematics:

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[6] Parsons draws the dividing line in a slightly different way from Wang: According to him predicativism belongs to the side of platonism rather than to the one of constructivism in a large sense.

[7] Note that all the three specialists in FOM agree (more or less; cf. the following footnote) that the original 3 movements in the philosophy of mathematics failed and somehow came to an end (by about the 1930s).

[8] Wang presumably exempts intuitionism from the criticism and treatment to which he subjects logicism and formalism; see below. And the same may hold for Parsons as well.

[9] The following historical sketch of the development of studies in FOM concerns what is usually called "the history of ideas" in a broad sense.

1. Logicism as a reduction of mathematics to logic
2. Formalism (called finitism by Wang) as a reduction of mathematics to finitist mathematical methods; it is an endeavor towards a formalisation of the central concept of constructivity.
3. Intuitionism as an endeavor towards a formalisation of the central concept of (constructive) proof

Logicism and formalism are, according to Wang, global reductionist projects; and moreover, they both are failures on several accounts. Hilbert's formalism has, however, in a 2nd phase of studies in FOM given rise to uniform local reductions which are of far greater interest than Hilbert's original global reduction. And Mostowski writes implicitly—and ambiguously—about the end or the decline of the three movements in the philosophy of mathematics in the late 1920s and their great impact on more formal and technical developments in a 2nd phase.

Mostowski identifies a new phase of FOM studies beginning in the 1930s with three particularly influential works, namely: K. Gödel's "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I" (1931), A. Heyting's "Die formalen Regeln der intuitionistischen Logik" (1930), and A. Tarski's "Der Wahrheitsbegriff in formalisierten Sprachen" (1936).

In his presentation of Gödel's incompleteness theorems, Mostowski formulates and sketches the proofs of the 1st and the 2nd incompleteness theorems, he discusses some important assumptions of these proofs and mentions some highly influential effects on the subsequent development of foundational studies: first, Gödel's 2nd incompleteness theorem was used as a powerful tool for investigating the relative strength of various axiomatic theories; second, his paper contains various results regarding the decision problem(s); and third, the method he invented of comparing intuitively true properties of mathematical objects with properties expressible in the formal system under consideration turned out to be extremely fruitful in meta-mathematics. His division of reasoning in intuitive meta-mathematics and formal mathematics was a very useful tool for establishing properties of formal systems, e.g. consistency, completeness and decidability.

An important strand in this 2nd phase of foundational studies is the development of intuitionism and *constructivism* in logic and mathematics. Heyting's just mentioned paper is the starting point of this strand: In this paper Heyting presents a formalisation of intuitionistic 1st order logic. Gödel proved a bit later that classical logic can be represented in the intuitionistic logic, and Tarski again a few years later that open subsets of a topological space form a matrix in which all provable formulas of intuitionistic propositional logic are valid. Subsequently, various interpretations were proposed to prove soundness and completeness of intuitionistic logic and intuitionistic arithmetic: (a) Tarski's topological interpretation was extended to yield a soundness and completeness proof for intuitionistic 1st order logic. (b) The Beth or tree models constitute another modification of the classical notion of model providing an adequate interpretation of intuitionistic 1st order logic. Both these interpretations suit pure intuitionistic logic very well, but seem less suitable for the interpretation of intuitionistic arithmetic. Mostowski raises the question: What is the purpose of an interpretation of a formal system? and answers the question as follows: It is supposed to give a precise meaning to concepts which are either incompletely explained or taken as primitive terms in the resp. formal system. The fundamental concept in intuitionism is, as Heyting showed, that of construction which is used by the intuitionists without explication. In (c) Kleene's realizability interpretation of intuitionistic arithmetic, Kleene proposes to explicate this concept by identifying it with partial computable functions. By means of his interpretation, he succeeds in proving the soundness of intuitionistic arithmetic, but Rose later disproved completeness. Hence Kleene's realizability interpretation does not provide an adequate interpretation of intuitionistic arithmetic: There must be intuitionistically acceptable "constructions" which are not reducible to partial computable functions. The principle of (d) Gödel's functional interpretation is similar to Kleene's, but Gödel used a much wider class of admissible constructions, namely the class of primitive recursive functionals. Gödel not only proved the soundness of intuitionistic arithmetic w.r.t. his functional interpretation, but also the relative consistency of intuitionistic arithmetic w.r.t. his axiomatic theory of primitive recursive functionals. Mostowski points out that all the interpretations (a)–(d) try to explain intuitionistic (logical or arithmetical) concepts in classical terms, and that an intuitionist would of course be far more interested in an interpretation explaining classical concepts in intuitionistic terms. Mostowski then puts the constructivist response to the set-theoretical paradoxes in a larger perspective, i.e. he describes other constructive, less "extreme" attempts (than the intuitionistic one) to solve the problem of the paradoxes. One sort of attempt employs quite arbitrary (not even necessarily constructive) means. One attempt of this first sort is computable analysis which restricts all mathematical notions and in particular those occurring in mathematical analysis to computable functions. Other attempts of this first sort concern extensions of computable analysis such as Grzegorczyk's elementarily definable analysis or hyper-arithmetical analysis studied by Kreisel. The second kind of attempt, not merely of mathematical but also of philosophical interest, consists of the theories called strictly finitistic by Mostowski: it is their aim to examine constructive mathematical objects by constructive means. One attempt of this second kind is recursive arithmetic the main idea of which is to develop mathematics as a formal system operating exclusively with equations. Another attempt of this second kind is Markov's algorithmic mathematics which reduces all other mathematical notions to the one of algorithm and which implicitly accepts and uses intuitionistic logic.

Another very important strand in the second phase of research in FOM is the unfolding of *computability theory*. Out of the decision problem for a denumerable class C of objects grew the need to define a class of arithmetical functions whose values can be computed in a finitistic way. Thus, the concept of a computable function from a set of integers to the set of integers served to make precise the concept of definability. In the 1930s several definitions of this concept of computable function were proposed which turned out to all be equivalent (cf. also the Church-Turing Thesis CT). The concept of computable function served likewise to define the concept of a recursively enumerable (r.e.) set. Large parts of computability theory were further developed by Kleene: He introduced the concept of relative computability, he defined the degrees of computability based on this concept as well as the arithmetical hierarchy of sets of integers, and he contributed to the study of recursive well-orderings which are part of a constructivistic program attempting to reconstruct parts of classical set theory in computable terms. The hyper-arithmetical hierarchy of sets of integers was but an extension of the arithmetical hierarchy into the constructive transfinite. And the analytical hierarchy of sets of integers was a further extension of the hyper-arithmetical hierarchy.

Another big subject in computability theory (other than hierarchies) is that of functionals which was first introduced by Gödel: He extended the concepts of primitive recursiveness and computability of functions and sets to objects of higher logical types called functionals. Whereas arbitrary functionals are highly infinitistic objects, Gödel considered the very narrow class of primitive recursive functionals, Kleene the much larger class of partial computable functionals, and Spector an intermediate class of so-called Bar-recursive functionals. The main drives for departing from the ideal simplicity of computable functions and sets and heading towards more and more infinitistic objects (objects of higher hierarchies and functionals) were according to Mostowski on the one hand to round off the theory of computability, and on the other to find objects which would be useful for the realisation of Hilbert's program (of consistency). Mostowski doubts whether these more infinitistic objects still fit into a constructivist philosophical program.

A special area of research in computability theory (or closely related to it) is the one of decision problems. Hilbert's original decision problem was: Is there a method allowing to decide effectively whether any 1st order formula is provable or not? The decision problem: Is there a method allowing to decide effectively whether any given formula of 1st order logic is satisfiable in some domain? could be called (the semantical version of) a Hilbert-type decision problem. Now, several partial Hilbert-type decision problems were found to have positive solutions, and Mostowski mentions several classes of 1st order formulas which are decidable. The decision problem: Is there a method allowing to decide effectively whether a given formula (in the theory's language) is provable in a theory T or not? could be called a Skolem-type decision problem. Skolem and Tarski designed a method, called the elimination method, to tackle this problem, and they used it successfully to solve positively the Skolem-type decision problem for various theories such as e.g. the theory of real closed fields. There are however important negative solutions of Hilbert-type decision problems, the most prominent being Church's negative solution of Hilbert's original decision problem in the 1930s, that is Church's proof of the undecidability of full 1st order logic. The basic method, called the reduction method, used in the proofs of undecidability (negative solutions to a Hilbert-type decision problem) is the reduction of a decision problem for a set of formulas K to the decision problem for another set K_0 for which the solution is known to be negative. Church also solved in the negative the Skolem-type decision problem for 1st order arithmetic. And Rosser even proved the essential undecidability of 1st order arithmetic (that is, the undecidability of every consistent extension of 1st order arithmetic). Mostowski finally distinguishes a third sort of decision problem: Is there a method allowing one to decide effectively whether any given 1st order formula is true in a given model M or not? which could be called a model-type decision problem. The Robinsons proved that in most cases the undecidability of a model M can be obtained if one shows the integers and usual arithmetical operations on integers to be definable in M.

According to Mostowski, the strand of (abstract) *set theory* is of special importance in the history of studies in FOM. In 1940 Gödel made a contribution to the consistency problem of hypotheses in set theory which had a deep influence on meta-mathematical work in the following 20 years. Gödel constructed a model of set theory in which the set-theoretical axioms, the Axiom of Choice AC and the Continuum Hypothesis CH, are valid by extending the arithmetical hierarchy into the (Cantorian) transfinite. A set which can be constructed at one of the finite or transfinite levels of this extended arithmetical hierarchy was called by Gödel "constructible". The constructible sets form this model denoted by **L** and they form a hierarchy. The family of constructible sets represents a realisation of the predicative foundation of mathematics. Gödel's Axiom of Constructibility ACon is the following one: Every set is constructible (where a set in Gödel's sense is a transitive and ground set), for short, $\mathbf{V} = \mathbf{L}$. ACon is considered as a highly dubious statement (even by Gödel). The effect of ACon is to give the not sharply defined concept of an arbitrary subset of a given infinite set a very definite limitation and interpretation. There seems to exist the possibility of 2 equally acceptable set theories: an axiomatic set theory + ACon, and an axiomatic set theory + not-ACon. Now, ACon is not only consistent relative to the other axioms of set theory, it also implies e.g. the Generalised Continuum Hypothesis GCH or the well-ordering theorem. A big philosophical question is: Is ACon true? Since ACon is also provably independent of the other axioms of set theory, there exist indeed 2 mutually contradictory systems of set theory. Mostowski wonders whether the choice is a matter of taste or whether there are compelling reasons for choosing the one set theory rather than the other. The just mentioned question touches on the fundamental problem of truth of set-theoretical hypotheses.

The problem of inconsistencies in Cantor's naïve set theory sparked off a number of axiomatic set theories all of which trying to modify Cantor's original theory in such a way that the inconsistencies disappear. Cantor distinguishes in his theory between "consistent" and "inconsistent sets". While Zermelo-Fraenkel's axiomatic set theory ZF simply ignored the inconsistent sets, the Bernays-Gödel axiomatic set theory BG mimics this distinction by assuming not only sets (as ZF does), but also classes. Whereas there is not much difference in mathematical content between ZF and BG, there are considerable differences between ZF or BG and extensions of these systems: in particular, those extensions adopting Tarski's axiom of the existence of inaccessible cardinals or Levy's axiom schema of the existence of various kinds of inaccessible cardinals, or even a very strong Axiom of Infinity: There are compact regular ordinals $\mu > \omega$. These extensions are essential extensions of ZF and BG and they cannot be proven to be consistent relative to ZF or BG. The big question of course is on which grounds these strong or very strong axioms of infinity can be taken to be consistent (in order not to introduce inconsistencies again through the back door).

Moreover, Cantor (as well as Frege) used in his naive set theory a naive Comprehension Principle CP of set existence: Whenever F is a formula (with one free variable), there exists a set S consisting of all elements a satisfying F. Since CP turned out to be inconsistent, subsequent axiomatic set theories tried to make set theory consistent by modifying CP in 3 different ways: (a) by not accepting CP for all formulas F; (b) by restricting the variability of a; (c) by imposing at the same time restrictions (a) and (b). Chwistek's and Ramsey's Simple Type Theory STT accepts (c). The ZF and BG set theories accept (b). And an axiomatic system due to Quine and referred to by NF accepts (a). Mostowski surmises that the axioms of set theory have not reached their definitive form yet. Another axiom of set theory which at the beginning stirred up a lot of philosophical debate, namely AC, was in the 2nd phase of foundational studies investigated concerning its relative consistency as well as its independence. In the 1960s Cohen introduced a new method allowing him to establish the independence of AC and GCH from practically every axiomatic system of set theory built along the ZF-lines. The success of his method is based on the new meta-mathematical concept of forcing. This concept of forcing is of considerable interest also apart from its applications. Furthermore, Cohen's forcing method suggested the study of an essential ingredient of it, namely of the generic sets. These sets seem to satisfy intuitions underlying Brouwer's intuitionistic conception of sets (of integers), and they can be defined not only for set theory, but also for arithmetic and other theories. Cohen's proof that there are (at least) 2 consistent and mutually incompatible set theories launched some more philosophical questioning: (i) Will mathematics accept the existence of these 2 incompatible set theories? or (ii) Will mathematics try to find new axioms which will eliminate one of them? or (iii) Will mathematics try to limit itself to more finitistic domains? The issue between formalists (option (i)), platonists (option (ii)) and intuitionists (option (iii)) is still open.

Yet another strand in the 2nd phase of development of studies in FOM is *proof theory* which is rooted in Hilbert's program (formalism). The main inputs in proof theory were given by Herbrand and Gentzen. Herbrand's main result contains a certain reduction (although obviously not a complete one) of 1st order logic to propositional logic. It shows that if a formula F is provable in 1st order logic, then there exists a proof of it consisting exclusively of subformulas of F. This result greatly simplifies the study of formal proofs. Herbrand's results were rediscovered and greatly improved by Gentzen who devised a new logical system equivalent to the one of the Hilbert school, but much more flexible. The flexibility of Gentzen's approach is obvious from the fact that it is applicable not only to classical logic, but also to many non-classical logics, esp. to intuitionistic logic. Gentzen's other great result is his conception of a consistency proof of arithmetic based on transfinite induction. Herbrand's and Gentzen's work clearly belong to what Mostowski calls the finitistic or arithmetical trend in the studies of FOM. Subsequent work in proof theory, however, borrowed many ideas from the infinitistic or set-theoretical trend, as witnessed by Bernays' general consistency theorem in which set-theoretical semantical notions are consciously imitated in finitistic terms. Mostowski emphasizes that Herbrand's and Gentzen's methods enable certain particular cases of infinitistic, set-theoretical constructions to be made finitistic. There is thus an intertwining of the 2 trends in the studies on FOM.

A last strand in the 2nd phase of studies in FOM is *model theory* (or "logical semantics") which is the study of relations between expressions of a formal language and mathematical objects (or, more precisely, between sentences of a formal language and a class of objects called models): one of the fundamental relations here is that of satisfaction. NB. The nature of these sentences as well as the nature of these models is fairly arbitrary which makes for the great flexibility of model theory. The systematic development of model theory is due to Tarski, started in the early 1950s, and became henceforth one of the most important parts of research in FOM.

Tarski began by showing that all semantical concepts can be reduced to the fundamental concept of a value of a formula (or sentence). He then used this concept to precisely define other important semantical concepts such as e.g. the concepts of satisfiability, validity, logical consequence and that of definability in a given model **M**. Starting from the observation that under certain conditions it is possible to replace the semantical relation: the value of a sentence F in **M** is $\mathcal{V}$ (= true) (or: model **M** satisfies sentence F), by the arithmetical relation (*) f is the Gödel number of a sentence F satisfied by **M**, Tarski raises two questions: (i) Is the arithmetical relation (*) definable in **M**? (ii) If (*) is not definable in **M**, what new relations should be added to **M** to ensure the definability of (*) in the extended model? The answer to question (i) is Tarski's well-known undefinability theorem: The set of sentences true in **M** is not definable in **M**. Tarski's undefinability theorem applied to PA yields Gödel's 1st incompleteness theorem. Question (ii) cannot be answered in a uniquely determined way: Various relations can be added to the model **M** in such a way that the arithmetical relation (*) becomes definable in **M**. A remarkable result here is the following theorem connecting model theory with the theory of inductive definitions: There exist types of inductive definitions which are not reducible to ordinary inductive definitions in a purely arithmetical way.

A particular, typically model-theoretical and highly important problem is the so-called completeness problem. Mostowski presents two formulations of it: (a) Consider an uninterpreted formal system described in a purely syntactic way. Try to find a semantical interpretation, i.e. a model for it satisfying all and only the sentences of that system. (b) Assume given an interpreted language. Try to find a formal system with purely syntactic proof rules allowing one to prove exactly the true sentences of the language. Two different methods were devised to solve the completeness problem of type (a) for 1st order logic. According to the first method which has an algebraic character and is due to Sikorski, Tarski and Rieger, this completeness problem is solved if one shows the existence of a maximal filter A with the property (**) for every formula F with one free variable, if the sentence $\exists x F(x)$ belongs to A then so does at least one sentence of the form $F(t)$. (As a matter of fact, this solution is a consequence of the fundamental theorem of Boolean filter theory.) According to the second method clearly influenced by Gentzen-style formalisations of logic and due to Beth, Hintikka and Schütte, the completeness problem for 1st order logic is solved by looking systematically for a possible counter-example to a given sentence F. Mostowski observes that underlying the completeness problem is a philosophical question concerning the relations between formal systems and their interpretations or models; but despite this philosophical origin the completeness problem has found many purely mathematical applications especially in algebra.

Mostowski continues to sketch two main results of the model theory for the (elementary) language L of 1st order logic. Before doing so, he introduces three important model-theoretical concepts: a model $\mathbf{M_1}$ is a submodel of a model $\mathbf{M_2}$; a model $\mathbf{M_1}$ is an elementary submodel of a model $\mathbf{M_2}$; two models $\mathbf{M_1}$ and $\mathbf{M_2}$ are elementarily equivalent. (1) the different Skolem-Löwenheim theorems, that is, the downward Skolem-Löwenheim theorem and the upward Skolem-Löwenheim theorem. The upward Skolem-Löwenheim theorem is a logical consequence of the compactness theorem (which in turn is a consequence of Gödel's completeness theorem for 1st order logic). Two particularly interesting applications of the Skolem-Löwenheim theorems are on the one hand the characterisation of the so-called spectra of sets X of sentences of L, and on the other hand the discovery of new methods for the solution of the completeness problem: One method due to Vaught is based on the concept of elementary equivalence of two models, and the other one due to A. Robinson on the new concepts of diagram $\mathbf{D}(\mathbf{M})$ of a model $\mathbf{M}$ and model-completeness (not to be confused with the concept of completeness). (2) the (semantical version) of the Craig interpolation lemma. On the one hand this lemma was used by Addison to explore the field of logical and set-theoretical separation principles, on the other hand it was applied by Beth in the theory of definitions. Mostowski then turns to the model theory for non-elementary (higher-order and infinitistic) languages; he briefly discusses the language $\mathtt{Q_a}$, the language $\mathtt{L_a^{II}}$ of weak 2nd order logic, the language $\mathtt{L^{II}}$ of strong 2nd order logic, the sequential 2nd order language $\mathtt{L_\emptyset^s}$, the weak and strong higher order languages $\mathtt{L_a^{(n)}}$ and $\mathtt{L^{(n)}}$, the infinitistic languages $\mathtt{L}_{\omega_1,\omega_\emptyset}$ and $\mathtt{L}_{\omega_\mu,\omega_\nu}$ as well as their resp. models. The main difference between the model theory for the language L and the model theory for the languages $\mathtt{Q_a} - \mathtt{L}_{\omega_\mu,\omega_\nu}$ is the failure of the compactness theorem in most of the latter (namely in $\mathtt{L_\emptyset}$, $\mathtt{L_a^{II}}$, $\mathtt{L^{II}}$, $\mathtt{L_\emptyset^s}$ and most of the languages $\mathtt{L}_{\omega_\mu,\omega_\nu}$). The downward Skolem-Löwenheim theorem is valid (with some modifications) w.r.t. all the languages $\mathtt{Q_a} - \mathtt{L}_{\omega_\mu,\omega_\nu}$, whereas the upward Skolem-Löwenheim theorem fails for almost all these languages (due to the fact that it follows from the compactness theorem). Because of the failure of the upward Skolem-Löwenheim theorem, the structure of the spectra in these languages is much more complex than in the case of language L. The study of analogues of the completeness theorem of 1st order logic for non-elementary languages has produced many interesting problems but only few solutions to these problems.

To draw attention to the great flexibility of model theory mentioned above, Mostowski points to a special algebraic construction in model theory, i.e. to the construction of a model as a direct product of a certain family of models. A special case of this very fruitful model construction is the model called the reduced direct product of a certain family of models. Feferman and Vaught applied the direct product model construction to several decision problems. Major applications of the new concept (or construction) of reduced direct product are: (1) a new and simple proof of the compactness theorem for 1st order logic; (2) the following theorem in abstract set theory: If A is a σ-multiplicative filter, then the reduced direct product of well-ordered models is itself a well-ordered model. This theorem forms the basis for a number of results on denumerably additive filters in Boolean algebras of all subsets of a set; (3) in arithmetic, it provides a simple method for constructing non-standard models of arithmetic; (4) in the theory of real numbers, Robinson used it to construct a non-standard model of analysis containing infinitesimals, and he moreover showed how to get in this way a completely rigorous theory—called non-standard analysis—which is equivalent to classical or standard analysis.

Section 3 : The Subsequent 40 Years of Studies in FOM According to Parsons

In his account of the subsequent 40 years of studies in FOM, Charles Parsons sketches first the more technical mathematical-logical development and then the more philosophical development.

Since he considers computability theory and model theory to have become almost purely mathematical, with hardly any foundational-philosophical impact, they drop out of his account completely.

As for proof theory, Parsons distinguishes two new proof-theoretical programs: (i) The analysis of strong subsystems of classical analysis (2nd order arithmetic) by means which could still be thought of as constructive, but are much more powerful and abstract than the means applied in the 2nd phase of development of FOM. (ii) The attempt to reconstruct classical analysis predicatively: it was shown by Harvey Friedman, Stephen Simpson and others that suitable reformulations of standard theorems of analysis can be proved in weak systems of analysis (cf. the method of reverse mathematics).

In Parsons' opinion the most striking foundational results were obtained in set theory: By means of Cohen's forcing method many more independence results were found in set theory and its applications. This discovery of new important independent statements sparked off a search for new set-theoretical axioms along the lines suggested by Gödel in the 1940s. And some progress has been made in this search by developing the consequences of two sorts of new axioms: (a) strong axioms of infinity, (b) special cases of the game-theoretical Axiom of Determinacy AD. It was discovered in particular that strong axioms of infinity implying PD (i.e. the assumption asserting that the axiom AD holds for projective sets of real numbers) have the convenient feature that their consequences in 2nd order arithmetic cannot be altered by forcing. And W. Hugh Woodin's approach to CH aims at extending this result to a higher level. This can be conceived of as an important step in the solution of the problem of whether CH has a determinate truth-value.

On the philosophical side, Parsons draws attention to a number of new philosophical conceptions of foundational interest:

(1) A kind of neo-logicism: This conception was inaugurated in the early 1980s by Crispin Wright who defined what he called Frege Arithmetic FA by 2nd order logic plus Hume's Principle HP plus a Fregean number operator $(N_x Fx)$ and

then proved the axioms of 2nd order PA from FA using Frege's definitions (this is nowadays called Frege's theorem since it was essentially already proved by Frege). Wright's and Bob Hale's proposal is to take FA as basic arithmetic and to argue that the notion of cardinal number is a logical notion and that HP is a logical principle. This proposal has led to a lot of discussion and debate about the status of abstraction principles like HP. Parsons notes that "[t]he program of axiomatizing parts of mathematics by abstraction principles is of independent logical interest, and work has been done on analysis, and preliminary work on set theory".[10]

(2) A kind of default platonism: Parsons presumes that "[t]aking the language of classical mathematics at face value, as implying the existence of abstract mathematical objects, even forming uncountable and still larger totalities, and allowing reasoning using both the law of excluded middle and impredicative definitions, is probably a default position among philosophers and logicians".[11] He doubts that any strong (decisive) philosophical arguments can be given for a stronger kind of platonism than the default one, a conception he dubs "robust platonism". Wang and Penelope Maddy accept default platonism as "the limit of what one should claim about the determinateness of the reality described by mathematical theories".[12] This somehow corresponds to the application to mathematics of Quine's naturalistic position, however, without Quine's privileging of empirical science.

(3) One way of rejecting default platonism is by adopting a constructivist stance (constructivism): Parsons observes that in the 40 years of studies in FOM under consideration constructivism has declined significantly as a general approach to FOM competing with classical mathematics. The most remarkable constructivist appearances in this phase of the development of studies in FOM are on the one hand Per Martin-Löf's constructive type theory CTT, and on the other hand Errett Bishop's and Douglas Bridges' constructive approach to mathematics.

(4) Another way of rejecting default platonism is by adopting a nominalist stance (nominalism): The traditional way refuses to take the language of mathematics at face value and tries to reformulate it in such a way that commitment to abstract mathematical objects is avoided. A more radical way was worked out by Hartry Field who rejected "the view that statements of classical mathematics, taken at face value with regard to meaning, are true and even that mathematics aims at truth. He sought to account for the apparent objectivity of mathematics by viewing it instrumentally, as a device for making inferences within scientific theories".[13]

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[10] Parsons (2006, p. 49).

[11] Ibid. p. 49.

[12] Ibid. p. 51.

[13] Ibid. p. 50.

(5) Structuralism: Two related intuitions about modern mathematics are fairly common: (a) modern mathematics is a study of (abstract) structures; (b) "mathematical objects have no more of a nature than is expressed by the basic relations of a structure to which they belong".[14]

The structuralist conception of mathematical objects is an elaboration of intuition (b). Its relation to default platonism is ambiguous and admits of at least 2 different positions: (i) eliminative structuralism: it refuses default platonism' s taking the language of mathematics at face value and proposes paraphrases eliminating reference to mathematical objects or at least to the most typical mathematical objects. (ii) non-eliminative structuralism: it rather constitutes an ontological gloss on default platonism and uses the structuralist conception as an explication of what the reference to mathematical objects in mathematical language amounts to.

"One version of structuralism would allow sets as basic objects. This would be a natural way of developing the first intuition [viz. (a)] , understanding structures as set-theoretic constructs. But a general structuralist view of mathematical objects would naturally aim not to exempt sets from structuralist treatment. At this point modality has been introduced."[15] With his system of Modal-Structural mathematics MS, Geoffrey Hellman worked out a version of eliminative structuralism based on this idea.

Parsons ends off his sketch of structuralism with the following critical remark: What the (eliminative) structuralist constructions accomplish depends on the status of 2nd order logic. And this question arises equally for neologicism and for nominalism. There has been much debate concerning this question.

(6) Naturalism and a Gödelian epistemological view : Whereas the philosophical conceptions (2)–(5) often have a strong ontological character, the conceptions (1) and (6) are of a strong epistemological type.

In the early 1970s Paul Benacerraf raised the following problem: If default platonism is true, how is it possible to have mathematical knowledge? Parsons generalises this problem in the following way: Is it possible to provide an epistemology for mathematics which is naturalistic? After stating that not much of philosophical-foundational interest has resulted from these questions, he turns to a Gödelian epistemological view which he takes to be possibly more interesting: "Gödel's view apparently was that much of mathematics (including some higher set theory) could be seen to be evident in an *a priori* way, not contaminated by evidence derived from application in empirical science. However, particularly in higher set theory axioms could obtain additional justification through the theories constructed on their basis, and such justification would be possible for stronger axioms, such as the stronger large cardinal axioms that have been proposed, where a convincing intrinsic justification is not available". [16] Parsons considers this Gödelian epistemological view to be of

[14] Ibid. p. 50.

[15] Ibid. p. 50/51.

[16] Ibid. p. 52.

great interest for the justification of assumptions applied in the accepted solution of the classical problems of descriptive set theory (e.g. in the justification of the axiom AD) as well as for the justification of any possible solution to the problem of CH to be expected in the future.

Part II The Present Perspective : Analytical Summaries of the Present Contributions[17]

Section 1

In his contribution **Foundational Frameworks** Geoffrey Hellman starts off by characterising the sort of questions asked in FOM: they are questions of justification (as opposed to questions of discovery), and moreover questions of an ideal, an on principle possible justification (as opposed to questions of actual justification). Thus, FOM according to Hellman is neither hermeneutics of mathematics, i.e. claiming to tell what working mathematics really is (reminiscent of Shapiro's "philosophy-last principle"), nor a cultural revolution of mathematics, i.e. advocating the replacement of working mathematics by a certain favored mathematical system or scheme (reminiscent of Shapiro's "philosophy-first principle").

Hellman then continues to enumerate important desiderata for any foundations (foundational frameworks) of mathematics (FOM): there are on the one hand the following traditional desiderata: (1) standards of proof, (2) means of expressing mathematical structures and their interrelations, (3) identification and explanation of the logical and mathematical primitives; on the other hand the modern (or post-modern) desiderata: (4) preservation of past gains, (5) accomodation of multiple approaches, i.e. providing for pluralism, (6) extendability of universes of discourse for mathematics. NB. Only some of these desiderata are actually requirements. A minimal requirement for a foundational framework is the following: providing a resolution of the conflicting tendencies of "creative progress" (desideratum 6 and perhaps desideratum 5) and striving for "all-embracing completeness" (desiderata 1–3). (Zermelo)

After laying down the desiderata of a foundational framework, Hellman goes about assessing set theory and category theory in terms of these desiderata.

First, he considers the prevailing set theory ZFC and many of its variants:

desiderata 1-3: ZFC is a major success story, may be the least w.r.t. desideratum 1(c), which is the most philosophical one. 1(c) concerns the epistemological sense of the foundations of mathematics.

desideratum 5: this desideratum is not met very well by ZFC: if a fixed-universe ontology is assumed for ZFC, then respecting other foundations of mathematics becomes problematic.

[17] These analytical summaries of the various contributions have all been approved by their resp. authors.

desideratum 6: On a fixed hierarchy view of ZFC, the problem of providing for extendability without exceptions is especially intractable: In a certain sense, ZFC does comply with extendability, namely by treating models of (consistent) theories as sets and recognizing sets and models of arbitrarily higher cardinality. But the unresolved problems are:

- A possible distortion of intended meanings
- The problem raised by the presumed fixed universal, hence maximal background of sets and ordinals (which thereby precludes the possibility of still further objects, other than sets and ordinals)

Second, Hellman deals with category and topos theory (CT & TT):

desiderata 5–6: CT & TT are through Bell's "many toposes" perspective a great success story (unlike set theory). Characteristic of this perspective is a plurality of universes.

desiderata 1–3: Here is where the most serious problems with CT & TT arise. 2 main category-theoretic axiom systems have been proposed with a foundational role in mind, namely: ETCS (the Elementary Theory of the Category of Sets) and CCAF (the Category of Categories As a Foundation).

as for CCAF: desideratum 2: considered as a system of Fregean-style axioms. These axioms are satisfied by "structures" (or at least "interrelated things"). Hence there is some dependence on a background explicating satisfaction of sentences by structures, and this background is not CT itself (it may be a dependence on a background set theory or on a higher-order logic or on a mereology + plural quantification). The conclusion is that w.r.t. the desiderata 1–6, CCAF is not yet adequate.

as for ETCS (or ECTS + R (Replacement)): desiderata 5–6: ETCS(+R) inherits some of the same problems affecting ZFC (cf. ibid.). desiderata 1–3: First, it seems that ETCS(+R) can only be really understood given a prior understanding of general notions such as "collection", "operation" or "relation" (Feferman). Second, a fundamental dilemma concerning categorical foundations of mathematics is the following: if desiderata 2–3 are met by categorical set theory, then desiderata 5–6 will not be met (due to limitations analogous to those of membership-based set theory); if however desiderata 5–6 are met by categorical set theory, then desiderata 2–3 won't be met. The conclusion in this case is that w.r.t. desiderata 1–6, ETCS(+R) is not adequate (yet) either.

Hellman goes on to assess modal-structural mathematics (MS) in terms of the desiderata of a foundational framework.

The modal-structural mathematics MS interpretation of mathematics has 2 components:

1. A hypothetical component: a translation pattern sending any ordinary mathematical sentence to a sentence asserting what would hold in any structure of appropriate type that there might be.
2. A corresponding categorical component: structures of the relevant type are (logico-mathematically) possible.

The Core System of MS is the following:

- The (monadic) logic of plurals
- 2 comprehension principles of mereology
- As an improvement over his original presentation *Mathematics without Numbers*, this presentation adopts both an Extendability Principle for structures for set theories (or precursors) and certain instances of a Modal Reflection Principle suitable to the MS framework, based on the idea that the logico-mathematical possibilities of structures should be "indescribable" by 1st or 2nd order sentences in a specific sense explained by him. From these 2 principles applied within the realm of finite structures, Hellman then can derive the compossibility of infinitely many objects and a "set" of them as needed for reconstructing classical analysis. Thus, it is not necessary to postulate an axiom of infinity separately as it is in set theory and category theory.

desiderata 1–2: MS' coverage of mathematics as practiced is almost as "complete" as that of set theory and category and topos theory; that is, these desiderata are about as well satisfied by MS as by ZFC and CT & TT.

desideratum 1(c): MS provides an interesting epistemological alternative to the acceptance of actual infinities without being as restricted or restrictive as strict finitism.

desiderata 5–6: These are MS' fortes. None of the alternative foundations of mathematics seems to be capable of satisfying them as well as MS.

desideratum 3: Hellman has done some work to make MS meet it; but more remains to be done.

Hellman rejects a strong foundationalism, i.e. which seeks to found all of mathematics on certain or self-evident assumptions, but he adopts a "modest, well-tempered foundationalism", i.e. the search for foundations providing a measure of epistemic order and a balance of unity and diversity (and perhaps, as Hellman puts it, even some insight into the nature of mathematics).

Bob Hale begins his contribution **The Problem of Mathematical Objects** by distinguishing two senses of 'foundations of mathematics' FOM:

1. The logical sense of FOM: foundations consist of a single, unified set of principles from which all or at least a large part of mathematics can be derived.
2. The epistemological sense of FOM: foundations consist of an account explaining how standard mathematical theories can be known to be true or at least be justifiably believed. Foundations in this sense cannot be mathematical theories, but have to be philosophical accounts of how working mathematics is getting known.

Hale's interest is clearly in the epistemological sense of FOM—independently of whether the search for such a foundation is a legitimate ("right") or possible endeavor.

Hale's point of departure is what he calls the problem of mathematical objects, and more specifically the problem whether one can be justified in believing in an infinity of objects of any kind. He subsequently distinguishes two approaches towards a solution of this problem:

(a) The so-called object-based approach: This is an approach arguing directly that it is possible to have access to or knowledge of an at least potentially infinite sequence of objects.
(b) The so-called property-based approach: This sort of approach argues indirectly for an infinity of objects by making the latter depend on an underlying infinity of properties.

According to Hale, Charles Parsons' study of mathematical intuition presents the most clear and convincing example of the object-based approach:

After distinguishing between intuition of objects and intuition that *p* where *p* is a proposition, Parsons makes it clear that the objects of intuition have to be restricted to the concrete and the quasi-concrete (and do not extend to the abstract). He claims that intuition of quasi-concrete stroke-string types can ground propositional knowledge concerning the system of stroke-string types, and that it can provide knowledge of analogues of the elementary Dedekind-Peano axioms. Hence it is possible to have intuitive knowledge of the existence of potentially infinitely many objects.

Hale starts his criticism of Parsons' object-based approach by pointing out that some of the (Parsons') Dedekind-Peano axioms are general and that it is hard to see how intuition of objects can yield knowledge of general truths. He carries on by emphasizing another, more fundamental problem in Parsons' approach: The difficulty concerns what is taken to be required for the existence of a stroke-string type: Is it (i) that there exists at least one token of that type? (ii) that it exists totally independently of any actual, possible or imaginative instantiation? or (iii) that there could exist at least one token of that type?

(i) already requires knowing that there are infinitely many concrete objects. This option is no good. (ii) implies rampant Platonism. This option runs counter to Parsons' attempt to exhibit intuitive knowledge of quasi-concrete stroke-string types. (iii) seems to imply that given any perceived or imagined stroke-string token, a single stroke-string extension of it is imaginable. But spelling out what 'imaginable' means just seems to get Parsons into further troubles.

Since Hale cannot see how any object-based approach could get around some appeal or other to some such grasp of possibilities, he infers that such an approach cannot work.

Question: Can a property-based approach to the problem of mathematical objects do better? Hale's intention is to show that indeed it can. G. Frege sought to prove the existence of successors for all finite numbers given Hume's principle (HP). His purported proof exemplifies according to Hale the so-called property-based approach. For Frege, numbers are essentially numbers belonging to concepts (i.e. a number exists only if there is a concept which it essentially belongs to), and Hale interprets this as saying: numbers essentially belong to properties. Given HP, Frege not only succeeds in proving that the sequence of finite numbers is infinite, but also in proving the Dedekind-Peano axioms in 2nd order logic (the latter is nowadays called Frege's theorem). Whereas Hale considers these proofs to be of considerable philosophical importance, other philosophers of mathematics have raised doubts concerning that importance. The first doubts or objections to be tackled by Hale are Michael Dummett's. According to Hale's analysis, Dummett objects:

i. That Frege can procede in his *Grundlagen der Arithmetik* only on the assumption that there exist inifinitely many objects, but that this assumption cannot be grounded in logic only.
ii. That there is some sort of a vicious circle in Frege's procedure (in Frege's definition of number or in HP) due to the fact that the numbers themselves are taken to belong to the domain over which the individual quantifiers range. Hale claims that his objection can be interpreted in two ways: the alleged circularity can be definitional or it can be epistemological.

Hale dismisses the definitional vicious circularity objection as it seems to imply that for any specific kind of object lying in the range of some (individual) quantifiers one must possess the concept of that kind of object (e.g. here the concept of number). But this, so he argues, is not the case. Hale's rejoinder to Dummett's epistemological vicious circularity objection as well as to Dummett's objection i. is basically the same: According to Dummett, Frege makes the assumption that numerical terms have reference. For Hale there is no such assumption, at least not when HP is put forward as an implicit definition. HP is an instance of a general abstraction principle whose instances are biconditionals which are to be so understood that their truth is consistent with their ingredient abstract terms (e.g. in HP the number terms) lacking reference. In the particular case of HP, there are instances the truth of whose right-hand sides is indeed a matter of logic.

Dummett's objection could be strengthened and would be quite closely related to an objection of Charles Parsons':

iii. Even if there is no explicit assumption to the effect that there exist infinitely many objects (or which boils down to the same thing, that number terms have reference), there is such an assumption implicit in the use of HP: HP quantifies over properties and makes the assumption of the existence of infinitely many objective properties.

Hale admits that if the assumption of existence of infinitely many properties were as problematic as the assumption of existence of infinitely many objects, Dummett's strengthened objection would be devastating. He, moreover, concedes that the first assumption would indeed be as problematic, if properties were conceived of in an extensional way (that's essentially Parsons' objection: there is no philosophical advantage of higher-order logic over set theory). Hale concludes first that the strengthened objection is justified if directed against Frege, and second that any property-based approach appears to be a waste of time if properties are indeed treated extensionally.

But as Hale puts it: "[i]f one thinks instead of properties as individuated non-extensionally, there is at least some chance of philosophical advantage" of higher-order logic over set theory and there is hope for a property-based approach. The assumption of the existence of infinitely many properties may be significantly weaker and so epistemologically less problematic than the assumption that there are infinitely many objects; it may, Hale suggests, even be weak enough to form part of a foundation of mathematics. A big task of the property-based approach will be to clarify what a non-extensional conception of properties amounts to. He provides a few hints at what such a conception might look like.

In her contribution **Set Theory as a Foundation**, Penelope Maddy recalls that set theory as a FOM goes back to the founders of set theory and has become part of contemporary orthodoxy in the philosophy of mathematics and even in mathematics itself. She briefly shows how natural numbers, integers etc. up to the reals and complex numbers can be represented as set theoretic entities and points out that all its standard theorems can be proved from ZFC. This is a remarkable mathematical fact. The philosophical question is what this fact means. What does it show? Maddy presents and discusses 6 interpretations of this fact. They will be called the 6 senses of (set theory as a) 'foundations of mathematics' FOM.

1. The metaphysical sense (attributed to a so-called strong reading of Frege's project): The current set theoretic versions of numbers, functions, spaces etc. show what numbers, functions, spaces, etc. really are; they exhibit the true nature of the various mathematical entities.

   Benacerraf objected to this metaphysical interpretation of FOM that there is not a unique or clearly priviledged identification of natural numbers with certain (pure) sets, but that many different identifications seem equally good. And the same holds in an analogous way for identifications of integers, reals, functions etc.

   According to Maddy, there is a weaker reading of Frege's according to which Frege was merely concerned about the just mentioned mathematical fact without any associated metaphysical ambition.
2. The ontological sense (Quine's ontological reduction): The current set theoretic versions of numbers, functions, spaces etc. admit of an ontological economy: it suffices for a mathematical ontology to merely accept the existence of these current set theoretic versions of the various mathematical entities. Quine advocated such an "ontological reduction", i.e. a replacement of a world view countenancing both numbers, functions, spaces and sets, with a world view countenancing only the sets.
3. The methodological sense (e.g. Moschovakis): the current set theoretic versions of numbers, functions, spaces, etc. are set theoretic surrogates of mathematical entities sharing with them the same mathematically relevant features.

Maddy appears to view a certain order in these senses. These senses are all mutually exclusive and there's an order of decreasing strength in them: the metaphysical sense is the philosophically strongest sense, the ontological sense is somewhat weaker, and the methodological sense is the philosophically weakest one (making the least philosophical presuppositions).

To each of these senses is attached a certain benefit (and it appears that such a benefit ought to be attached): in the case of the metaphysical sense, it is a metaphysical insight; in the case of the ontological sense, it is an ontological economy; in the