David DeVidi
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Editors

The Western Ontario Series
in Philosophy of Science

Logic, Mathematics, Philosophy : Vintage Enthusiasms

Essays in Honour of John L. Bell

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A SERIES OF BOOKS IN PHILOSOPHY OF MATHEMATICS AND NATURAL SCIENCE, HISTORY OF SCIENCE, HISTORY OF PHILOSOPHY OF SCIENCE, EPISTEMOLOGY, PHILOSOPHY OF COGNITIVE SCIENCE, GAME AND DECISION THEORY

VOLUME 75

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John and Mimi Bell in Oxford in 1970, following the conferral of John’s D. Phil.

To the memory of $-M^2$

Preface

If things of Sight such Heavens be
What Heavens are those we cannot see?
Andrew Marvell

This collection of essays was put together to celebrate John Bell's sixtieth birthday on the 25 March 2005. The list of contributors signals some of the important stations of John's career as a mathematician, teacher, colleague and friend: the student days in Oxford; the years of the young Lecturer, pacing the rooming houses of Londinium's Bedsit Land; the years of the Reader, ensconsed in more sedate accommodations; the years of the Canadian Professor in London, Ontario, no longer a philosopher in a mathematics department, but now a mathematician in a philosophy department. And all of them years of books, records and beloved recordings, from low-fi to high-fi, the sounds of Beethoven and Schoenberg, Heifetz and Gould, Parker and Powell, conversations (monologues?) late (very late!) into the night—vintage years of undying, revivifying enthusiasms, not least among them, the enthusiasm for vintages.

The contributions are in no way intended to be commentaries on John's work; they are to be seen rather as presents from some of the people John has influenced, been inspired by or inspired, encouraged, amazed and amused through the years. They include contributions from former fellow students, former and current colleagues, former pupils, collaborators and joint authors, and from friends and admirers. The papers are grouped into a small number of broad categories, though the breadth of topics gives some indication of the range of John's interests and influence, running from mathematics to aesthetics, and from philosophy of science to political theory. What unites the material in this volume is what is characteristic of John's work: when it is mathematical, the topics are chosen because of their philosophical interest; when it is philosophical, it takes, where appropriate, full advantage of illumination from relevant work in mathematics and formal logic.

The attentive reader will no doubt have noticed that a considerable time has elapsed between March of 2005 and the date of publication. Some of the lapse has been in the interests of the quality of the volume itself, as we waited for some of our eminent, and correspondingly busy, contributors to complete their papers. The Editors, less eminent but perhaps not less busy, accept full responsibility for the rest of the delay. We wish to thank the contributors, not just for their papers, but for their patience and efficiency in the face of delays, pesterings and quibbles. We also wish to thank John, Bill Demopoulos (the General Editor of the Western Ontario Series in the Philosophy of Science, in which this volume appears) and the publisher for their tolerance and understanding. We also owe an enormous debt to the meticulous work of Oran Magal on the penultimate draft of the manuscript, and for compiling the Name Index; his diligence and care has saved us from numerous infelicities.

Because of the various delays in its completion, John began referring to the book waggishly as "the memorial volume." But the joke, meant entirely good-heartedly, has now, sadly, a cruel sting. We had intended to dedicate the volume to the fixed point in John's "perpetual motion," Mimi, or $-M^2$ as her dedicatory sobriquet became. However, Mimi died from cancer on 20 November 2009; she had been John's constant companion for over 40 years, from their early days together at LSE.

Two of the editors first got to know the Bells at LSE in the early 1970s. Thanks to John's (literally) prodigious talent, which led to an Oxford Scholarship at the age of 15, he was officially our senior as teacher and supervisor, and indeed, it seemed then, as intellectual and cultural being, although he was scarcely older. We picture John and Mimi in the large LSE Refectory (the standard meeting place, and John's constant resort), at one of the long formica tables, perhaps with a bowl of the stodgy spaghetti or one of the glutinous curries, or just mugs of coffee or tea and a biscuit or a cheese roll. The sense of fun was intense, as was the Bells' delight in the ridiculous, of which the Refectory, with its food and habitués (including ourselves), provided a constant supply. John, of course, was the outwardly dominant one, but Mimi was firmly in charge, a fact which became clearer as one got to know them better. John was never allowed lasting dominance, and Mimi wouldn't permit the conversation to be swamped by John's obsession *du jour*. After a first rebuke, John would try reviving the topic on which he had fixed, especially if there was some *aperçu* which had occurred to him and he was itching to get out; this would be followed by a somewhat exasperated second rebuke, and the cycle would repeat itself. Eventually, Mimi would exclaim "Oh John, you're so irri*tat*ing," and the cycles would be at an end. It was hard work, but Mimi always won, as she did at Scrabble. Many eccentric and odd characters would turn up, and (of course!) gravitate to the Bell circle, if sometimes only briefly. John encouraged this, and delighted in it; in fact, for him the odder and more eccentric, the better. Mimi, however, would usually remain aloof, composed, dignified and mildly sceptical, a rock of reassurance in the unpredictable political and emotional turbulence.[1]

Eventually, the centre of this semi-communal social life shifted from the LSE to the Bells' rented flat in Alexandra Grove, where the lads of humble English origin, having grown up in the monochrome aftermath of the Second World War, were first exposed to new worlds of intellectual and culinary creation. The only occasion when

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[1] A fine sense of many of the LSE characters encountered by the Bells at this time is conveyed by the chapter "London, 1968–73" in John's memoir *Perpetual Motion: My First Thirty Years*, available from his website.

at least one of the Editors saw John genuinely silenced and overawed was when their son, Alex, was born. After witnessing Mimi give birth, John came to a dinner party to which he had been previously invited. He hardly spoke a word all evening and sat amazed by what he had experienced; when he did speak, it was to express his admiration for Mimi and his complete devotion to her and their new son.

Evolving social and academic commitments and careers dispersed us somewhat, and, as the years went by, there was less communal life and social gatherings became rarer; new circles were formed of which we were no longer part. At the end of the 1980s, the persuasive powers of some of John's future colleagues at the University of Western Ontario, following on a decade of Thatcherism, convinced the Bells to swap Londinium, as it became known for purposes of disambiguation, for London, Ontario. This was a massive disruption, involving significant adaptation, no matter how willingly undertaken. Academic life remained relatively familiar, and, for John, the major difficulty now was being surrounded by students and colleagues with philosophical, rather than mathematical, background. But Canada generally, and London in particular, involved considerable cultural bemusement for both the Bells. John greatly enjoyed regaling visitors with lists of pros, cons and constants. John was eventually able to reconcile himself to the strange ways of Canadians; one suspects that he would thrive anywhere, but it has been the good fortune of Canadian philosophy that he has been thriving in Canada. The Bells came to admire greatly Canada's well-run, multi-cultural mix, largely untroubled as it is by the hatreds that had scarred old London in the thirties and which were, by the time the Bell family left, beginning to reassert themselves in the darker corners of British political and social life.

Once they'd arrived in Canada, and escaped a particularly miserly landlord (a source of new obsessions!) by moving to their own home, the Bells kept a busy social schedule, one that included frequent visits from graduate students. It was during this transition period that the third editor got to know the Bells, again first as John's Ph.D. student, then as friend. For the shyer or more reserved of the Canuck students, a visit to the Bell household promised intimidation, what with John's boisterousness on home soil added to his brilliance. But, once arrived at the house, Mimi (again firmly in charge) ensured that everyone recognized how welcome they were. This third editor, who doesn't think of himself as a shrinking violet, fondly remembers spending a good portion of his first party at the Bells in the kitchen helping Mimi prepare the food, a respite from the storm of conversation in the living room. That job involved, among other things, the peeling of vegetables the editor had never seen before, and was a first hint of the fact that every visit to the Bells involved marvellous food, prepared with flair and imagination.

While John pursued his usual academic life, one of enormous productivity and Oxonian "effortless superiority," Mimi's endeavours in new London were more varied. For years, she worked at a local shelter for victims of domestic violence, putting her deeply felt (if not often voiced) political commitments into action; while she didn't talk about this work often, at least at social gatherings, the work was essential and potentially dangerous. In more recent years, she was involved in helping new immigrants and refugees get settled in Canada, steering them through the formidable bureaucracy involved in finding adequate housing, language classes, training and jobs. She was also eventually persuaded to share her culinary expertise with circles outside the guests at her house. For years she taught cooking classes, and she set up and ran her own catering business. She wrote a cookery book (which is partly responsible for the difficulty one of the editors has in keeping his weight under control). She wrote poetry and painted, including work that found its way into exhibitions. All this in addition to raising the Bells' son, Alex, and doing the bulk of the work keeping a frenetic household running.

We cannot, sadly, dedicate this book to Mimi, as John so often did with his own books. Instead, we dedicate it to her memory, with wonderfully fond recollection.

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Part I
History and Philosophy of Mathematics and Logic

Chapter 1
On Logicist Conceptions of Functions and Classes

William Demopoulos

1 Introduction

John Bell arrived in "new" London in 1989, a refugee from the academy under Margaret Thatcher. We soon became good friends, and during the early years of our friendship we collaborated on two papers (Bell and Demopoulos, 1993, 1996). The first of these collaborations was a paper on the foundational significance of results based on second-order logic and Frege's understanding of his Begriffsschrift; the second was on various notions of independence that arise in connection with elementary propositions in the philosophy of logical atomism. I retain fond memories of both collaborations; they proceeded quickly and almost effortlessly. In this contribution to John's Festschrift, I propose to revisit our paper on Frege. That paper was occasioned by (Hintikka and Sandu, 1992), which questioned whether Frege's understanding of second-order logic corresponded, in his framework of functions and concepts, to what we would now regard as the standard interpretation, the interpretation that takes the domain of the function variables to be the full power-set of the domain of individuals. Hintikka and Sandu maintained that it did not on the basis of a number of arguments, all of which they took to show that Frege favored some variety of non-standard interpretation for which the domain of the function variables is something less than the characteristic functions of *all* subsets of the domain over which the individual variables range.

Although Hintikka and Sandu based their contention on many different considerations, the one to which they assigned the greatest weight was that Frege lacked the concept of an arbitrary correspondence from natural numbers to natural numbers (or from real numbers to real numbers).[1] Hintikka and Sandu contrasted their view of

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[1] The textual considerations Hintikka and Sandu advance in support of this interpretive claim were shown in (Burgess, 1993) to rest on simple misreadings of Frege's *Venia docendi*, and in the case of Frege's relatively late paper, "What is a function?," on passages that are far from unequivocal.

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Frege on the concept of a function with that of Dummett who, in *FPL*,[2] had maintained that Frege's notion of a function coincides with the concept of an arbitrary correspondence:

> And it is true enough, in a sense, that, once we know what objects there are, then we also know what functions there are, at least, so long as we are prepared, as Frege was, to admit all "arbitrary functions" defined over all objects (*FPL* p. 177, quoted by (Hintikka and Sandu, 1992, p. 303)).

However in *FPM* Dummett reversed himself on this question, and suggested that Frege implicitly assumed the adequacy of a substitutional interpretation of his function variables:

> ... Frege fails to pay due attention to the fact that the introduction of the [class] abstraction operator brings with it, not only new singular terms, but an extension of the domain. ... [I]t may be seen as making an inconsistent demand on the size of the domain D, namely that, where D comprises n objects, we should have $n^n \leqslant n$, which holds only when $n = 1$, whereas we must have $n \geqslant 2$, since the two truth values are distinct: for there must be n^n extensionally non-equivalent functions of one argument and hence n^n distinct value ranges. But this assumes that the function-variables range over the entire classical totality of functions from D into D, and there is meagre evidence for attributing such a conception to Frege. His formulations make it more likely that he thought of his function-variables as ranging over only those functions that could be referred to by functional expressions of his symbolism (and thus over a denumerable totality of functions), and of the domain D of objects as comprising value-ranges only of such functions (*FPM*, pp. 219–220).

In *FPM* [Chapter 17, *How the serpent entered Eden*] Dummett put forward a further consideration that can be seen as lending support to Hintikka and Sandu's view that Frege favored some sort of non-standard interpretation. In the course of his discussion of the attempted consistency proof of *Gg* § 31, Dummett suggested that Frege erroneously supposed that the consistency of certain restricted interpretations of the function variables extends to the system of *Gg*'s theory of functions and classes in the context of full second-order logic. This unjustified, and unjustifiable, assumption is what Dummett means by "Frege's amazing insouciance regarding the second-order quantifier" [*FPM* p. 218]. Subsequent developments have shown that there is a systematic consideration in favor of Dummett's claim and, a fortiori, in favor of Hintikka and Sandu's suggestion that Frege assumed a non-standard interpretation: some restricted interpretations yield consistent fragments of *Gg*.[3] Hence it might be that Frege was blind to the inconsistency of his system because he considered the question of consistency only from the perspective of such a restricted interpretation and failed to ask whether what holds of it, holds in general.

But although Dummett shares Hintikka and Sandu's conclusion that Frege tended toward a non-standard interpretation, his analysis does not support Hintikka and

[2] I use the following mnemonic abbreviations: *FPL* for (Dummett, 1981), *FPM* for (Dummett, 1991), *Grundlagen* for (Frege, 1884), *Gg* for (Frege, 1903), *FoM* for (Ramsey, 1990), *Principia* for (Whitehead and Russell, 1910), *Tractatus* for (Wittgenstein, 1922).

[3] See for example the system **PV** discussed in (Burgess, 2005, § 2.1).

Sandu's evaluation of Frege's foundational contributions. If we follow Dummett, Frege missed the fact that the consistency of *Gg*, relative to a nonstandard interpretation, does not necessarily extend to its consistency when the logic is given a full interpretation. This is certainly an oversight, but it is not the oversight that is appealed to in those of Hintikka and Sandu's criticisms of Frege that so offended some of their critics, as for example, whether, without having isolated the notion of a standard interpretation, Frege could have even conceptualized results like Dedekind's categoricity theorem.

Apropos of Frege and categoricity, in their response to Hintikka and Sandu, (Heck and Stanley, 1993) observed (what (Heck, 1995) elaborates in detail) that Frege *proved* an analog of Dedekind's theorem using his own axiomatization of arithmetic (one that is only a slight variant of the Peano-Dedekind axiomatization). And in our response, John and I argued that the relevance of the dependence of this and other similar foundational results on the standard interpretation is not entirely straightforward since the actual arguments which support them have the same character, whether one is working in second-order logic or in a suitably rich first-order theory such as Zermelo-Fraenkel set theory. Hintikka and Sandu's claim that Frege could not even have formulated (let alone appreciated) these results because of their dependence on the standard interpretation is therefore incorrect both historically and methodologically. It is incorrect historically because Frege successfully proved a categoricity theorem like Dedekind's. And it is incorrect systematically because essentially the same argument establishes the categoricity of second-order arithmetic in any of the usual systems of set theory. And surely it is implausible that only someone familiar with the categoricity of the Peano-Dedekind Axioms as a theorem of second-order logic has really grasped the theorem or its proof. At most, Frege might be charged with having missed a subtlety concerning the distinction between formal and semi-formal systems; but this is hardly surprising for the period in which he wrote.

In our paper, John and I accepted Dummett's view in *FPL* and based our claim that Frege's interpretation of the function variables was the standard one on the premise that Frege's concept of a function coincides with the set-theoretic notion of an arbitrary correspondence, in which case the domain of the function variables is in one-one correspondence with the power-set of the domain of the individual variables. Our thought was that whatever covert role the neglect of Cantor's theorem might have played in the inconsistency of *Gg*, it is unlikely that Frege sought to ignore the theorem by assuming that the totality of functions, like the totality of expressions, is countably infinite. But we sided with Dummett in *FPM* and supposed that Frege might very well have been misled into assuming that what holds for certain countable interpretations of the function variables holds in general; hence we agreed with Dummett's evaluation of the sense in which Frege missed the significance of the possibility of different interpretations for his program.

More recently, reflection occasioned by reading (Sandu, 2005) has convinced me that the equation of Frege's concept of a function with the notion of an arbitrary correspondence should be reconsidered, and that it might be fruitful to reconsider it from the perspective of Ramsey's interpretation of *Principia*'s propositional functions.

Let us call *classes* the extensions of functions in the *logical* sense of "function," a notion I will explain in the sequel. Then we wish to explore whether the classes determined by such functions correspond to only a fragment of the sets associated with arbitrary correspondences. The assumption that something like this is correct evidently underlies Ramsey's proposed reinterpretation of the first edition *Principia*'s notion of a propositional function in his essay, *FoM*. That essay is usually cited for its separation of the paradoxes and its isolation of the simple hierarchy of *Principia*'s functions of lowest order. It is generally assumed that by this hierarchy Ramsey understood a hierarchy of functions interchangeable with the standard hierarchy of sets, modulo the condition that *Principia*'s hierarchy is stratified rather than cumulative. In fact, Ramsey regards *Principia*'s simple hierarchy of functions of lowest order as inadequate, and he devotes a central chapter of his essay [*FoM*, Chapter IV] to criticizing *Principia*'s notion of propositional function and arguing for its extension. He achieves this extension, and more, by the introduction of the notion of a propositional function in extension, or what I will call an *extensional propositional function*. The consideration of Ramsey's criticisms of *Principia* has led me to the view that the interest of Hintikka and Sandu's paper has less to do with standard vs. non-standard interpretations of second-order logic than with Frege's concept of a function. As I now see it, the chief interest of their paper, although not perhaps their principal aim, is the suggestion that the difference between logical and set-theoretic notions of *function* parallels the well-known contrast between the logical notion of *class* and the mathematical notion of *set*.

My strategy for the balance of the paper is to proceed anti-historically by reviewing Ramsey's notion of an extensional propositional function, its difference from what Ramsey calls the predicative propositional functions of *Principia*, and the uses to which Ramsey put the notion in his defense of a modified logicist position. To motivate the notion of an extensional propositional function, I will begin by briefly recounting the relevant Tractarian background to Ramsey's thought. I will then describe two reconstructions of the Axiom of Infinity, one with, and one without, the notion of an extensional propositional function. Both reconstructions trace back to Ramsey. The reconstruction involving the notion of an extensional propositional function was elaborated in his *FoM*; my discussion of it constitutes the principal novelty of the present paper. Having clarified the predicative and extensional notions, I will return to Frege's functions. I will argue that they have a feature in common with *Principia*'s propositional functions, and that it is plausible to argue, on the basis of this, that they should be distinguished from the notion of an arbitrary correspondence. However the situation is not entirely straightforward, and we will see that there are also reasons to suppose that Frege's notion can be regarded as falling in line with the mathematical (or "extensionalist") tradition.

2 The Tractarian Background to Ramsey's Extensional Propositional Functions

There are three key ideas of the *Tractatus* that form the background to Ramsey's notion of an extensional propositional function: (i) A proposition is significant only insofar as it partitions all possible states of affairs into two classes. This is what (Potter, 2005) has called "Wittgenstein's big idea." It implies that tautologies cannot be regarded as significant propositions, and insofar as the propositions of logic are all tautologies, the propositions of logic are not significant propositions; in particular, they cannot be merely more general than the truths of zoology or any other special science as Russell claimed (in (Russell, 1919, p. 169)). (ii) Objects, which constitute the substance of the world, are constant across alternative possibilities; so also therefore is their number. Potter puts the point well when he says that "what changes in the transition between [possibilities] is how objects are combined with one another to form atomic facts; what the objects are does not change because they are the hinges about which the possibilities turn and hence are constant" (Potter, 2005, p. 72). It follows that there cannot be a significant proposition regarding the number of objects. At best language can "reflect" the cardinality of the world. (iii) Since atomic propositions are logically independent of one another, statements of identity cannot express significant propositions, for if they did, they would establish relations of logical dependence among atomic propositions.

It is a consequence of Wittgenstein's rejection of identity, a rejection with which Ramsey concurred, that the notion of class implicit in Russell's theory of classes must be an "accidental" one, that is, one according to which it is possible that every property might be shared by at least two individuals, with the consequence that there would be nothing to answer to the number 1 [*FoM*, p. 213]. If we could express significant propositions with the use of identity, then among the properties of *a* there would be the property of *being identical with a*, and this would settle the question of the existence of the number 1 on Russell's theory. But acceptance of the third Tractarian thesis argues against a solution which, like this one, treats identity as a possible constituent of a significant proposition. Hence if we follow the *Tractatus* on identity, Russell's theory cannot recover the notion of class that its account of number—and indeed, of the whole of mathematics—requires. This is an objection that is in some ways more fundamental than the standard objection to the apparent contingency of the Axiom of Infinity, since even if that axiom could in some way be made acceptable, according to the present objection, it is still *possible* on Russell's theory that there might not be enough numbers to go around. Hence, for this and other reasons we will soon come to, mathematics cannot be based on *Principia*'s theory of classes.

Ramsey's notion of an extensional propositional function emerged from his attempt to address this and other defects in *Principia*'s theory of classes, such as those surrounding its account of Choice. But perhaps the most important consideration in favor of the notion was that it proved essential to a formulation of Infinity that accords with the first and second Tractarian theses. For, if we follow Russell and simply take Infinity to assert that for every inductive cardinal number n, there is a class whose cardinality is given by n, then we treat the axiom as an accidental truth concerning the number of things of a particular sort. As a consequence, we miss the peculiar character of claims regarding the cardinality of objects—in the present case, of individuals—enunciated by (ii): The existence of objects is a precondition for significant propositions and cannot be a subject with which such propositions deal; the same is true of their number. In order to see how Ramsey's notion of an extensional propositional function leads to a formulation of Infinity that addresses these issues, it will be useful to begin with a formulation of the axiom that has all the ingredients of the formulation of *FoM* *except* for the notion of an extensional function.

3 Infinity Without Extensional Propositional Functions

Potter (2005) discusses an early unpublished fragment of Ramsey's ("The number of things in the world") which contains what Potter calls "Ramsey's transcendental argument for Infinity." Potter discusses this argument at length, and canvasses a number of questions of Ramsey-interpretation which the fragment raises. The transcendental argument evidently precedes Ramsey's discovery of extensional propositional functions. For my purposes, the argument's most interesting feature is the formulation of Infinity it suggests, rather than the considerations in favor of the axiom that it advances. I will refer to this as "Ramsey's early formulation of Infinity"; however, my discussion is not based on Ramsey's unpublished fragment, but on Potter's reconstruction of it. The historical accuracy of Potter's account is of course completely irrelevant to its usefulness for motivating the formulation of Infinity that Ramsey gives in *FoM*. That formulation is my main concern, and I will take it up in the next section.

Let φx be any propositional function, and let Tx be $\varphi x \vee \neg\varphi x$. Then to say that there are at least n individuals write

$$p_n =_{Df} \exists x_1 \ldots \exists x_n Tx_1 \;\&\; \ldots \;\&\; Tx_n,$$

where, here and elsewhere, Ramsey assumes Wittgenstein's "nested variable convention," which says that whenever a variable occurs within the scope of another variable, it is to be assigned a different individual as its value.[4] Thus, for example, "$\exists x \exists y \ldots$" is read, "there is an individual x and *another* individual y ...." Then if there are n individuals $a_1, \ldots, a_n$, this is reflected by the tautology $Ta_1 \;\&\; \ldots \;\&\; Ta_n$.

[4] What I am calling Wittgenstein's nested variable convention is explained in detail in (Wehmeier, 2008) in his discussion of "W-logic." Wehmeier (2009) discusses a variant of W-logic ("R-logic") which originated from Ramsey's initial misunderstanding of Wittgenstein's convention.

Now let p_ω be the "logical product" of the p_n, for all finite n. Then p_ω is Ramsey's early formulation of Infinity, where what Ramsey means by the logical product of the p_n can be gleaned from what he says about logical sums:

> A logical sum is not like an algebraic sum; only a finite number of terms can have an algebraic sum, for an "infinite sum" is really a limit. But the logical sum of a set of propositions is the proposition that these are not all false, and exists whether the set be finite or infinite (*FoM* p. 219, n. 1).

Ramsey clearly takes this to apply mutatis mutandis to the notion of a logical product, in which case we are to understand by p_ω the proposition that every member of the p-series is true. This is expressed by the infinite conjunction of the p_n, which is what Ramsey means by the proposition that every member of the p-series is true. Hence, if there are infinitely many individuals, this is reflected by the tautology $Ta_1\ \&\ \ldots\ \&\ Ta_n\ \&\ \ldots$, which asserts, or appears to assert, the infinity of the number of objects of the type of individuals, and is to be contrasted with a claim about the number of things there are of some particular kind.

To construct an "Axiom of Infinity" regarding some sort of thing, we also proceed by first describing a series of propositions such as

$$\begin{aligned} q_1 &=_{Df} \exists x(x \text{ is a hydrogen atom}) \\ q_2 &=_{Df} \exists x \exists y(x \text{ and } y \text{ are hydrogen atoms}) \\ &\ \vdots \end{aligned}$$

where once again Wittgenstein's convention regarding nested variables is assumed. In this case, each q_n is an ordinary empirical proposition concerning not objects in general but the number of things of a particular kind, and q_ω (the product of all the q_n) says that there are infinitely many of them. The *Tractatus* imposes no prohibition against the significance of a proposition like q_n since it is a genuine possibility that there are at least n hydrogen atoms, even if there are in fact only m of them for m less than n. It is likewise a genuine possibility that there are *infinitely many* hydrogen atoms. All such claims mark genuine possibilities in the sense demanded by the Tractarian notion of a significant proposition.

There is a well-known distinction of Carnap's that bears on the evaluation of Ramsey's early proposal. Carnap (1931, p. 41) distinguished between two types of logicist reduction. Let us call a reduction *type (a)* if it defines all concepts of a mathematical theory in terms of those of logic, and *type (b)* if, in addition to providing definitions of a mathematical theory's concepts in terms of logical concepts, it derives the axioms of the theory from purely logical axioms. It is worth remarking that although Russell may have sometimes expressed the view that a type (a) reduction suffices for logicism, Frege was always clear on the need to secure both type (a) and type (b) reductions.

In *FoM* [pp. 166–167], which was written after the fragment containing Ramsey's early formulation of Infinity, there is an anticipation of Carnap's distinction between type (a) and type (b) reductions. Ramsey gave the distinction an interesting interpretation when, under the clear influence of Wittgenstein, he remarked that while a type (a) reduction might illuminate the *generality* of mathematics, a type (b) reduction would address what is truly distinctive about it: its *necessity*. But if Ramsey and Wittgenstein are the source of the view that necessity is the chief characteristic of mathematical propositions that requires explanation, it is important to remember that there is not a trace of an interest in necessity in Frege's *Grundlagen* or in the first edition *Principia*. Both works stress the generality of mathematics and the possibility that this might be illuminated by assimilating it to the generality of logic. Ramsey's aim was to illuminate the necessity of mathematics by assimilating it to the necessity of logic. But he also saw that there are at least two ways in which this might be accomplished. We can try reducing the propositions of mathematics to those of logic after the fashion of a type (b) reduction of the sort Frege tried, unsuccessfully, to achieve. Alternatively, we might attempt to show that the propositions of mathematics have the same kind of necessity that we find in the propositions of logic, *not* by deriving mathematics from logic, but by an analysis of the necessity mathematical propositions exhibit. Ramsey sought to explicate this characteristic by proposing that the propositions of mathematics, like those of logic, are "tautologies," an idea that evidently has its basis in the *Tractatus*. However, by a tautology Ramsey did not mean something that is *truth-table decidable*, as is clear from his discussion of the Axiom of Choice where he mentions with approval the possibility of "a tautology, which could be stated in finite terms, whose proof was, nevertheless, infinitely complicated and therefore impossible for us" [*FoM*, p. 222]. We will return to Ramsey's notion of *tautology*.

How do these considerations bear on Ramsey's early formulation of Infinity? There are two objections to this formulation, one more telling than the other. The less telling objection is that the existential claims expressed by the propositions of the p-series fall short of possessing the necessity of logical propositions. The fact that $Tx_1 \ \& \ \ldots \ \& \ Tx_n$ becomes a tautology when there are individuals $a_1, \ldots, a_n$ which accord with Wittgenstein's convention does not show that p_n is a tautology, and therefore does not show that the axiom is necessary. In a domain of $m < n$ objects, p_n is simply false, a fact which on Ramsey's formulation is represented by the absence of values for the variables of p_n that accord with Wittgenstein's nested variable convention. But to this objection Ramsey can respond that it was not his intention to show that p_n, let alone p_ω, is a tautology. Rather, the point of the formulation was to capture the idea that if there are n individuals, then p_n is *witnessed* by the tautology, $Ta_1 \ \& \ \ldots \ \& \ Ta_n$. The objection mistakenly assumes that the only way to establish the logical necessity of Infinity is to show that it holds in every "universe of discourse." This is one sense in which a proposition may be seen to be a logical proposition, but it is not the sense Ramsey's formulation was intended to capture. A proposition can be one whose truth is witnessed by a tautology, and thus be a logical proposition in Ramsey's sense, without holding in every universe of discourse. And this is precisely the case with Infinity.

The second objection is harder to articulate and will only become clear after we have examined the formulation of Infinity in *FoM* that is based on Ramsey's notion of an extensional propositional function. As a first approximation, the difficulty is that the tautological character of the propositional function T_x which figures in Ramsey's p-series is not integrated into the theory of functions and classes as it would be on a formulation of Infinity that was derived from an analysis of *class* and *propositional function*. For this reason, Ramsey's formulation carries little conviction as an account of the necessity that a fundamental axiom like Infinity is supposed to exhibit. Since, according to Ramsey's reading of the *Tractatus*, the explanation of this characteristic is the central task of a philosophy of mathematics, it seems likely that the source of his dissatisfaction with his early proposal, and the reason he never published it, was the recognition that it fails to present a convincing account of Infinity's necessity.

4 Infinity with Extensional Propositional Functions

In *FoM* Ramsey rejected the notion of propositional function that we associate with the 1910 *Principia*. Such functions are, in Ramsey's phrase, "predicative" in the sense that the proposition φa which the propositional function φ assigns to a says or *predicates* the same thing of a as the proposition φb, which φ assigns to b, does of b. This connection with predication is essential to any *logical* notion of the functions which determine classes, and it stands in contrast with the idea that a function is an *arbitrary* correspondence. According to Ramsey's new conception, the way to accommodate the broader notion of set which is implicit in the idea of an arbitrary correspondence is by conceiving of a propositional function as an arbitrary mapping, in the present case from individuals to propositions, with φa and φb merely the values of φ for a and b as arguments. Here "arbitrary" means both that the mapping is not constrained to preserve a property in the sense that the propositions φa and φb are not required to say the same thing of a and b, *and* that it allows all combinatorial possibilities of functional pairings of individuals with propositions. On Ramsey's view, it is only *propositions* that involve properties which are predicable of individuals; the *classes* which propositional functions determine should not be constrained by the demand that their elements share a common property.[5]

When this purely extensional notion of a propositional function is adopted, there is, of course, no difficulty with the existence of unit classes and the number 1, since

[5] By an abuse of notation which uses the same symbol φ in two very different ways—both for a mapping from individuals to propositions and for a predicate of the language—a predicative propositional function φ takes an individual a to a proposition of the form φa, i.e., to a proposition which is expressed by a sentence consisting of the concatenation of the predicate φ with the constant a. The class determined by φ is the class of all a such that φa is true. If φ is extensional but not predicative, the class determined by φ is the class of all individuals a which φ maps to truths. Under an extensional understanding of propositional functions, there is not in general a correspondence between propositional functions and predicates of the language, so that the association with propositions is in this sense "arbitrary." It is also arbitrary in the stronger sense of allowing all possible pairings of individuals with truth values.

every object has a mapping to propositions that is unique to that object.[6] Notice however that, of the two features of extensional propositional functions, it is only the second, namely the fact that such functions exhaust all combinatorial possibilities, that the solution to this difficulty depends upon, and this can readily be accommodated within a framework that considers only pairings of individuals with truth values. The "propositional" aspect of propositional functions, the fact that they map individuals to propositions, doesn't enter into the solution of the problem raised by the possibility that every property is shared by at least two individuals.

Ramsey says of the Axiom of Choice that given his notion of an extensional propositional function, "it is the most evident tautology … [and not something that] can be the subject of reasonable doubt" [*FoM*, p. 221]. He then proceeds to show how, on *Principia*'s understanding of *propositional function* and *class*, the axiom, though not a contradiction, is also not a tautology. Ramsey does not pause to explain why on his account the axiom is an evident tautology, but as John once observed in conversation, after the existence of singletons has been shown to follow from the concept of an extensional propositional function, Choice, in the form "If K is a family of disjoint non-empty classes, then K has a choice class," is clearly true. For if classes are determined by extensional functions, it is evident that every class in K can be shrunk to a singleton; the sum of all such singletons is the required choice class. Like the difficulty with the number 1, the foregoing justification of Choice depends only on the fact that extensional propositional functions exhaust all combinatorial possibilities.

Clearly, what Ramsey means by the tautologousness of Choice is not captured by the idea that it holds in all universes of discourse, still less that it is truth-table decidable. Ramsey perceived that there are "interpretations" of *Principia*, by which he meant understandings of the notion of *propositional function*, of which the one favored by Whitehead and Russell is just one example, under which Choice can be shown to be false. And in a remark made in the course of a discussion of Infinity and the possibility of saying something about the cardinality of the world given his adherence to the second Tractarian idea to which we called attention in Section 2, Ramsey shows his appreciation of the possibility of falsifying a fundamental axiom by "imagining a universe of discourse, to which we may be confined, so that by 'all' we mean all in the universe of discourse" [*FoM*, p. 224]. From this we may conclude that Ramsey's understanding of the "tautologousness" of Choice is that relative to his extensional understanding of "propositional function" and the intended meaning of "all," and relative perhaps as well to the intended meanings of the propositional connectives, Choice is evidently true.[7] Ramsey expresses this by saying that, under these conditions, the axiom is "an evident tautology," to draw attention to the fact that its obvious demonstration in the finite case proceeds by inspecting all the com-

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[6] For example the singleton of a is determined by a propositional function which maps a to an arbitrarily selected truth and maps every other individual to a falsehood.

[7] Sandu misses this point about the tautologousness of Choice when he criticizes Ramsey's contention that Choice is a tautology and argues that since "… there are models of set theory in which the Axiom of Choice is false, … it cannot, therefore, be a tautology" (Sandu, 2005, p. 252).

binatorially possible relations between classes and their elements. In the general case, we are incapable of carrying out such an inspection, but this does not affect the truth of the principle on an understanding of *propositional function* that admits all combinatorial possibilities.

The situation is different with Ramsey's analysis of the Axiom of Infinity. Here, in addition to the fact that Ramsey's functions exhaust all combinatorial possibilities, the propositional character of extensional propositional functions is an indispensable component of the solution to the problem the axiom presents. To understand Ramsey's account of Infinity, suppose we try expressing the idea that there are at least two things by the proposition

$$\exists x \exists y \neg(\varphi)(\varphi x \equiv \varphi y), \tag{$*$}$$

where it is assumed that propositional functions are to be understood predicatively, after the manner of Whitehead and Russell. Now consider a universe of discourse U containing precisely two individuals a and b which have all their properties in common. For Ramsey there is nothing absurd in the idea that two things might share all their properties and thus be indistinguishable but distinct; and since he rejects identity he will not allow an appeal to the property, *being identical with a* to insure the truth of $(*)$ in U under such a predicative understanding of propositional functions. Under these circumstances, $(*)$ will fail to reflect the fact that a and b comprise a two-element universe. But for Ramsey that it should even be *possible* that $(*)$ can fail in this way shows that, under its predicative interpretation, $(*)$ purports to express a general truth, and hence, a significant proposition; it cannot therefore be the correct expression of the idea that there are at least two things.

Let us now consider what happens when propositional functions are understood extensionally. Assuming Wittgenstein's nested variable convention, $(*)$ *must* be true in any two-element universe such as U. For if we consider all possible mappings φ, there must be one among them that assigns a to the negation of whatever proposition it assigns b. Hence the function $\neg(\varphi)(\varphi x = \varphi y)$ will map (a, b) to the negation of a contradiction, and therefore on Ramsey's understanding of *propositional function*, the truth of $(*)$ in U will be witnessed by a tautology. This contrasts with the predicative interpretation under which $(*)$ can fail in a two-element universe, so that even in cases where it holds, it does so under a predicative understanding of *propositional function* only in virtue of a contingent fact about individuals and their properties. It also contrasts with Potter's reconstruction of Ramsey's early formulation, since the fact that $(*)$ is witnessed by a tautology is not an ad hoc stipulation, but a consequence of Ramsey's extensional understanding of the nature of a propositional function.

In light of the foregoing considerations, let us define the propositional function

$$T(x, y) =_{Df} (\varphi)(\varphi x = \varphi y).$$

As we have seen, when propositional functions are understood extensionally, the function $T(x, y)$ maps to a tautology when the values of x and y are the same and to a contradiction otherwise. Thus if propositional functions are understood extensionally, then if there are two individuals, and we understand $\exists x \exists y T(x, y)$ to say, "There is an x and *another* y such that T(x,y)," then $\exists x \exists y T(x, y)$ reduces to a contradiction, while $\exists x \exists y \neg T(x, y)$ reduces to a tautology. This suggests a *new* p-series defined as follows:

$$p_1 =_{Df} \exists x T(x, x)$$
$$p_2 =_{Df} \exists x \exists y \neg T(x, y)$$
$$p_3 =_{Df} \exists x \exists y \exists z \neg T(x, y) \ \& \ \neg T(x, z) \ \& \ \neg T(y, z)$$
$$\vdots$$

For each n, p_n is true in a universe of n individuals. The witnesses to the propositions of the series alternate between a tautology, when a member of the series is true, and a contradiction, when it is false. Ramsey's new formulation of the Axiom of Infinity is given by the logical product $p_{\aleph_0}$ of the propositions of this series. Observe that if there are $\aleph_0$ individuals, $p_{\aleph_0}$ is witnessed by a product of tautologies, and if fewer than $\aleph_0$, its falsity is witnessed by a contradiction.

To appreciate Ramsey's achievement, consider how his proposal differs from the following reduction of truths to tautologies. (For simplicity, we restrict ourselves to quantifier-free molecular formulas, since this suffices to illustrate the point of difference with the alternative reduction to which I wish to call attention.) Replace a quantifier-free molecular formula φ by its equivalent disjunctive normal form. Next replace any literal of φ's disjunctive normal form which is true by $x = x$, and replace a literal by $x \neq x$ if it is false. Then the resulting formula is a truth function of "tautologies" and "contradictions," and it reduces to one or the other according to whether it is true or false. This is clearly artificial since it enables us to "reduce" to tautologies and contradictions many propositions which are merely true or false. But *FoM*'s proposal regarding Infinity does not simply replace truths with tautologies, and falsehoods with contradictions; it *derives* the tautologous or contradictory character of a witness to a proposition of the p-series from an analysis of the notion of a propositional function: a proposition of the p-series is witnessed by a tautology or contradiction as a consequence of the cardinality of the domain *and the extensional character of propositional functions*. In this respect Ramsey's reduction procedure stands in marked contrast with one which merely stipulates the tautologousness of the witness to the truth of a proposition.

Does the formulation of Infinity based on Ramsey's reinterpretation of the notion of a propositional function provide a logical justification in the usual sense? It does not. Ramsey's achievement consists in showing how the witness to the truth of Infinity reduces to a product of tautologies when the cardinality of the domain of individuals is infinite. In conformity with the first of the Tractarian theses noted earlier, the truth of the axiom is not expressed by a genuine proposition; rather, the cardinality of the world is "shown"—or as I prefer to say, "witnessed"—by a tautology. This is not the provision of a logical justification for Infinity that purports to show that it is true in every universe of discourse, but an explanation of how, *as a consequence of the notion of an extensional propositional function*, a witness to the truth of the axiom reduces to a product of tautologies in any domain in which the axiom holds, and how a witness to its falsity reduces to a contradiction in any in which it does not hold. By integrating the idea that Infinity, if true, is witnessed by a tautology into the notion of a propositional function, it addresses the second of the two objections we noted in our discussion of the formulation of the unpublished fragment.

5 Extensional Propositional Functions and Logicism

Hintikka and Sandu's discussion of Fregean concepts shares a number of similarities with Ramsey's discussion of the 1910 *Principia*'s propositional functions: Like Ramsey, Hintikka and Sandu regard the notion of *class* on which the early logicists relied as inadequate by comparison with the extensionalist tradition's notion of *set*; and both Hintikka and Sandu and Ramsey base their analyses of the inadequacy of classes on a failure of the early logicists to conceive of a function as an arbitrary correspondence. But there are also significant differences. By contrast with Hintikka and Sandu, Ramsey took himself to be contributing to a more defensible version of logicism. He hoped to show that his formulation of an appropriate extensionalist replacement of *Principia*'s predicative propositional functions would address an internal difficulty with the work, one surrounding the adequacy of its account of mathematical propositions.

Ramsey's extensional propositional functions attempt to marry two seemingly incompatible ideas: they preserve the letter of the logicist thesis that classes are determined by functions, but they also invoke the combinatorial notion of set by representing a propositional function as an arbitrary functional pairing of individuals with propositions. As a consequence, one gives up the Russellian idea that a propositional function determines a class in terms of an antecedently available property. But Ramsey's extensionalism is in some respects congenial to Frege. This is because Frege's functions satisfy a condition of extensionality in the sense that functions whose courses-of-values coincide are not distinguished, and because Fregean functions form a simple hierarchy which is not constrained by ramification. Despite these points of agreement with an extensionalist viewpoint, Frege's assimilation of concepts to functions which map into truth values is usually understood to share with the 1910 *Principia* the idea that the correspondence is not arbitrary, but is constrained by the principle that if a function maps two objects to The True, they must fall under a common concept. Hence Frege's generalization of the function concept to include those that map to truth values is, in Ramsey's sense of the term, a generalization to a notion of function that is just as *predicative* as Russell's.

If I have understood him, Sandu's suggestion in (Sandu, 2005) is that his paper with Hintikka should be understood as arguing that the predicativeness of Frege's functions is sufficient to show that they do not exhaust all mappings between objects