Computational Methods in Applied Sciences

Manolis Papadrakakis
George Stefanou
Vissarion Papadopoulos
Editors

Computational Methods in Stochastic Dynamics

ECCOMAS
European Community on Computational Methods in Applied Sciences

Springer

Computational Methods in Stochastic Dynamics

Computational Methods in Applied Sciences

Volume 22

Series Editor

E. Oñate
International Center for Numerical Methods in Engineering (CIMNE)
Technical University of Catalunya (UPC)
Edificio C-1, Campus Norte UPC
Gran Capitán, s/n
08034 Barcelona, Spain
onate@cimne.upc.edu
www.cimne.com

For other titles published in this series, go to
www.springer.com/series/6899

Manolis Papadrakakis · George Stefanou
Vissarion Papadopoulos
Editors

Computational Methods in Stochastic Dynamics

Springer

Editors
Manolis Papadrakakis
National Technical University of Athens
Institute of Structural Analysis and
Seismic Research
9 Iroon Polytechniou Str.
Zografou Campus
15780 Athens
Greece
mpapadra@central.ntua.gr

Vissarion Papadopoulos
National Technical University of Athens
Institute of Structural Analysis and
Seismic Research
9 Iroon Polytechniou Str.
Zografou Campus
15780 Athens
Greece
vpapado@central.ntua.gr

George Stefanou
National Technical University of Athens
Institute of Structural Analysis and
Seismic Research
9 Iroon Polytechniou Str.
Zografou Campus
15780 Athens
Greece
stegesa@mail.ntua.gr

ISBN 978-90-481-9986-0 e-ISBN 978-90-481-9987-7
DOI 10.1007/978-90-481-9987-7
Springer Dordrecht Heidelberg London New York

Cover design: SPi Publisher Services

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

At the dawn of the twenty-first century, the considerable influence of inherent uncertainties on structural behavior has led the engineering community to recognize the importance of a stochastic approach to structural problems. Issues related to uncertainty quantification and its influence on the reliability of the computational models, are continuously gaining in significance. In particular, the problems of dynamic response analysis and reliability assessment of structures with uncertain system and excitation parameters have been the subject of extensive research over the last two decades as a result of the increasing availability of powerful computing resources and technology. This book focuses on advanced computational methods and software tools which can highly assist in tackling complex problems in stochastic dynamic/seismic analysis and design of structures. The selected chapters are authored by some of the most active scholars in their respective areas and represent some of the most recent developments in this field.

This edited book is primarily intended for researchers and post-graduate students who are familiar with the fundamentals and wish to study or to advance the state of the art on a particular topic in the field of computational stochastic structural dynamics. Nevertheless, practicing engineers could benefit as well from it as most code provisions tend to incorporate probabilistic concepts in the analysis and design of structures. The book consists of 16 chapters which are extended versions of papers presented at the recent COMPDYN 2009 and SEECCM 2009 Conferences. The chapters can be grouped into several thematic topics including dynamic response variability and reliability of stochastic systems, risk assessment, stochastic simulation of earthquake ground motions, efficient solvers for the analysis of stochastic systems, dynamic stability and stochastic modeling of heterogeneous materials.

In Chapter 1, G.I. Schuëller gives an overview of the well established deterministic model reduction techniques (Guyan reduction, component mode synthesis) and approaches for efficient uncertainty propagation in structural dynamics. The recent advances in the combination of these two fields are reviewed and methodologies for the consideration of uncertainty when using model reduction schemes are presented. The capabilities of a Karhunen-Loève expansion of the substructure matrices and of a random combination of substructures are investigated. It is shown that the proposed approaches can combine high accuracy with a substantial reduction of the computational cost compared to Monte Carlo simulation (MCS) of the full model.

In Chapter 2, S. Krenk and J. Høgsberg present a design principle for resonant control of a SDOF system by second order filters. The basic idea is the introduction of a resonant force with frequency tuning that results in splitting the original resonant mode into two modes with equal damping ratio. The design procedure is developed for resonant displacement and acceleration feedback, respectively, based on a combination of "equal modal damping" and approximately equal response amplitudes of the two modes. The procedure is extended to collocated resonant control of the lower modes of MDOF systems. An explicit design approach for the control parameters of MDOF systems is developed that includes the effect of the higher modes via a quasi-static approximation. The efficiency of resonant damping is illustrated by application to a benchmark example for stochastic wind load on a high-rise building.

S.K. Au presents a novel approach for improving importance sampling in the estimation of the first passage probability of nonlinear hysteretic systems subjected to stochastic earthquake excitations (Chapter 3). Instead of using fixed design point excitations, the importance sampling distribution is constructed using a stochastic process, called "adapted process". A stochastic control algorithm is developed for the adapted process where the objective function reflects the expected energy needed to next yield as well as the exit velocity. The variance reduction efficiency of the "optimal" control law is investigated through a numerical example involving a SDOF elasto-plastic structure. It is shown that the proposed approach leads to substantial reduction of the "unit COV" of the importance sampling estimator while maintaining the required computational effort at a reasonable level (comparable to that of Markov Chain Monte Carlo-based approaches).

Another approach to the solution of the first passage problem for earthquake engineering applications is provided by M. Barbato in Chapter 4. This chapter describes the representation of the complex envelope process for transient response of linear structures and illustrates the use by the first passage problem. The Vanmarcke and modified Vanmarcke approximate solutions to the first passage problem are compared with the classical Poisson approximation and MCS results for an idealized linear elastic model of a steel building. The failure condition is expressed in terms of inter-storey drifts outcrossing specified deterministic thresholds. The retrofit of the benchmark structure with viscous dampers is also considered, allowing to illustrate the use of the closed-form approximations of the failure probability for non-classically damped linear elastic systems. The results presented show that the two Vanmarcke approximations can improve considerably the estimates of the failure probability for the first passage problem when compared with the simpler Poisson approximation.

A computationally efficient method for the stochastic analysis of large linear structural systems subjected to seismic actions is developed by P. Cacciola and G. Muscolino in Chapter 5. The method is based on modal analysis for the evaluation of stochastic response along with a modal correction approach, which includes the contribution of the neglected modes. Furthermore, a previously developed method for the evaluation of spectrum-compatible evolutionary power spectra of ground motions is presented. Some numerical results show the accuracy and

efficiency of the proposed technique for determining the spectral moments of the response.

In Chapter 6, S.M. Elachachi et al. investigate the effect of soil spatial variability (correlation length, coefficient of variation) on the reliability of buried networks subjected to earthquake loading. The reliability analysis is performed using a Response Surface Model (RSM) and the reliability index is calculated for serviceability and ultimate limit states. The dynamic response of a buried pipe subjected to natural ground motion records is computed taking into account a longitudinal variability of the properties of the soil. It is shown that the magnitude of the induced stresses and thus the reliability of the pipe are mainly affected by four parameters: a soil-structure length ratio, a soil-structure stiffness ratio, a structure-joint stiffness ratio (relative flexibility) and the magnitude of soil variability (coefficient of variation).

The problem of reliability-based optimization of structural systems under stochastic loading is treated by H. Jensen et al. in Chapter 7. In the proposed approach, a standard gradient-based algorithm with line search is used, allowing to explore the space of the design variables most efficiently. Subset simulation is adopted for the purpose of estimating the corresponding failure probabilities. The gradients of the failure probability functions are estimated by an approach based on the local behavior of the performance functions that define the failure domains. Numerical results show that only a moderate number of reliability estimates has to be performed during the entire design process. A numerical example with deterministic system properties is provided to show the effectiveness of the proposed approach.

Meta-models are a powerful tool for the solution of complex realistic problems in stochastic dynamics. L. Pichler et al. present a mode-based meta-model for the reliability assessment of linear dynamical systems (Chapter 8). The reliability analysis is carried out through the application of the meta-model for estimating the modal properties which are then subsequently required to perform a mode-superposition analysis. The proposed approach is proved to be accurate and computationally efficient in both cases of deterministic and stochastic loading. The verification of the results for the reliability assessment carried out with the meta-model is performed with a full finite element (FE) analysis and shows a good agreement of the response over the whole duration of the stochastic ground acceleration.

The effect of uncertain system properties on structural response and reliability is examined by G. Stefanou and M. Fragiadakis in Chapter 9. An efficient approach combining MCS with translation process theory is presented for assessing the nonlinear stochastic response and reliability of a steel moment-resisting frame subjected to transient seismic actions. The structure is modeled with a mixed fiber-based beam-column element, whose kinematics is based on the natural mode method. The adopted formulation leads to the reduction of the computational cost required for the calculation of the element stiffness matrix, while increased accuracy compared to traditional displacement-based elements is achieved. The uncertain parameters of the problem are the Young modulus and the yield stress, both described by homogeneous non-Gaussian translation stochastic fields. Under the assumption of a pre-specified power spectral density function of the stochastic fields, a parametric

investigation is carried out providing useful conclusions regarding the influence of the correlation length of the stochastic fields on the response variability and reliability of the frame.

Although reliability analysis methods have matured in recent years, the number of available techniques to evaluate the risk of complex structural systems is limited. In Chapter 10, F. Petrini et al. investigate the problem of Aeolian risk assessment of slender structures in the framework of performance-based wind engineering. A classification of the main sources of uncertainty is attempted and the importance of some of them (epistemic uncertainty of the aerodynamic coefficients, model uncertainty of aeroelastic forces, influence of model uncertainty on risk assessment) is examined with reference to an example case, a long span suspension bridge.

The concept of evolutionary power spectrum is important in the context of stochastic simulation of earthquake ground motions. Chapter 11 by D. Schillinger and V. Papadopoulos presents a novel approach called "method of separation" for the accurate estimation of evolutionary power spectra of non-homogeneous strongly narrow-band stochastic fields encountered in various engineering applications. The method is developed for power spectra separable in space-frequency but can also efficiently treat a class of non-separable spectra. The method of separation is based on the geometrical similarity of functions. It leads to the estimation of evolutionary power spectra from a series of samples based only on sample mean square and on an estimate of the homogeneous Fourier power spectrum. The limitations of existing methods (short-time Fourier, wavelet and Wigner-Ville transforms) are avoided and a very good localization in both space and frequency is achieved. Numerical examples including analytical benchmark spectra as well as spectra corresponding to measured geometric imperfections of I-section beams illustrate the superiority of the proposed approach in comparison to other established techniques.

An energy-based envelope function is proposed in Chapter 12 by S. Sgobba et al. enabling to characterize the time-varying amplitude of strong ground motion and to generate realistic stochastic accelerograms corresponding to a given earthquake scenario. This envelope function is directly related to the Arias intensity of the ground motion and has a functional form similar to that of a lognormal probability density function. In conjunction with suitable peak factors, the envelope function may also be used to predict the distribution of PGA values corresponding to a given earthquake scenario.

The development of robust and efficient solution algorithms is crucial for the treatment of large-scale realistic problems involving uncertainties. A novel numerical algorithm for the solution of stochastic partial differential equations (SPDEs) arising in the context of stochastic mechanics is presented by H. Matthies and E. Zander in Chapter 13. The algorithm exploits the natural tensor product structure between basis vectors describing the physical/deterministic behavior and a basis describing the stochastic response of the system. A sparse representation of input and output is achieved using singular value decomposition (SVD). This sparse format is preserved throughout the computation and is also used to reduce the amount of computation. A numerical example involving the 1D diffusion equation is presented

to illustrate the efficiency of the approach. It is shown that for a sufficiently accurate representation of the solution, the required storage is reduced by 90% and by 50–80% as intermediate storage is concerned.

The problem of dynamic stability is treated in Chapter 14. J. Náprstek and C. Fischer investigate the dynamic stability behavior of a non-linear auto-parametric system with three degrees of freedom (used as a simple mathematical model of a slender structure on elastic subsoil exposed to strong vertical excitation). Certain intervals of the excitation frequency are identified where the semi-trivial solution loses its dynamic stability and the strong horizontal response components become important. Post-critical states of both deterministic and chaotic type are thoroughly analyzed with respect to excitation frequency and amplitude. It is observed that the response amplitudes can remain in acceptable limits when adequate excitation (in terms of frequency and amplitude) acts throughout a short time interval. It is mostly not the case when the excitation has an infinite duration and the response is approaching to the steady state.

The last two chapters of the book are devoted to the rapidly evolving area of stochastic modeling of heterogeneous materials. A. Ibrahimbegovic et al. (Chapter 15) discuss the failure analysis of civil engineering structures built of heterogeneous materials. The structured FE mesh approach (consisting of constant stress triangular element that can contain two different phases and phase interface) is proved to be much more efficient than the non-structured mesh representation of material heterogeneities for the case of 2D two phase materials. This feature is of special interest for probabilistic analysis, where a large amount of computation is needed in order to provide the corresponding statistics. One such case of probabilistic failure analysis is also considered in this chapter, where the geometry of the phase interface remains uncertain since it is obtained as the result of the Gibbs random process. This computation is further used to provide the appropriate probabilistic description of material parameters of phenomenological model of localized failure in terms of correlated random fields. Subsequent Monte Carlo computations of failure phenomena in simple tension test performed with such probabilistic phenomenological model clearly show the capability of the presented approach to recover the size effects anywhere within a range between the two classical bounds which are Continuum Damage Mechanics and Linear Fracture Mechanics.

The book closes with a stochastic mechanics approach by D. Schwarzer and C. Proppe for the calculation of the statistics of eigenfrequencies in heterogeneous beams made of metal foam (Chapter 16). The spatial variation of the linear elastic material properties of metal foam is quantified using random fields generated by the spectral representation and Karhunen-Loève expansion methods. The probability distribution and the correlation structure of the random fields are derived from data. MCS of eigenvibration problems of beams of different length are performed. The computed eigenfrequencies are compared to experimental data. From this comparison, it can be concluded that the heterogeneity of the material microstructure affects the macroscopic properties. The mean values of the eigenfrequencies are smaller than those of beams with homogeneous material of similar mean material

properties. A scatter of the eigenfrequencies is also observed, a fact that has to be taken into account when using parts consisting of metal foam.

The book editors would like to express their deep gratitude to all contributors for their active participation in the COMPDYN 2009 and SEECCM 2009 Conferences and for the time and effort devoted to the completion of their contributions to this volume. Special thanks are also due to the reviewers for their constructive comments and suggestions which enhanced the quality of the book. Finally, the editors would like to thank the personnel of Springer for their most valuable support during the publication process. It is hoped that this book will convey the reader to the excitement, advances and challenges the computational stochastic dynamics community faces in the near future.

June 2010

Manolis Papadrakakis
George Stefanou
Vissarion Papadopoulos

Contents

Model Reduction and Uncertainties in Structural Dynamics

Gerhart I. Schuëller
in Cooperation with Barbara Goller and Helmut J. Pradlwarter

Abstract Model reduction procedures for the purpose of reducing computational efforts in structural analysis are already well developed and widely used to compute the dynamic response of complex structural systems. For example the Guyan reduction might essentially reduce the size of the structural matrices, while component mode synthesis provides a means to consider first simpler substructures and to use its modal properties for deriving a reduced global structural model for computing the dynamic response. Moreover, component mode synthesis allows for an assembly of large finite element models consisting of substructures established by different working groups and hence has significant advantages in the design cycle.

The above mentioned, frequently used finite element (FE) reduction procedures assume perfect, i.e. deterministic knowledge of all structural properties. However, the assumption of deterministically known structural properties is in most practical, real world cases, not realistic. Many items (e.g. stiffness, mass and damping parameters, etc.) incorporated in the mathematical FE model of a structure are uncertain, i.e. are either not precisely known or might reveal unpredictable random behavior.

This paper will give an overview of the well established deterministic reduction techniques and approaches for the efficient uncertainty propagation. Finally, the recent advances in the combination of these two fields are reviewed and methodologies for the consideration of uncertainty when using model reduction schemes are presented. The model reduction schemes considering uncertainties, which usually involve some simplifying assumptions, similar to their deterministic counterparts, are compared with the reference solution as obtained by direct Monte Carlo simulation (MCS), where the deterministic FE reduction scheme is applied together with randomly generated structural properties.

G.I. Schuëller (✉)
Institute of Engineering Mechanics, University of Innsbruck, Technikerstr. 13,
A-6020 Innsbruck, Austria
e-mail: mechanik@uibk.ac.at

M. Papadrakakis et al. (eds.), *Computational Methods in Stochastic Dynamics*,
Computational Methods in Applied Sciences 22, DOI 10.1007/978-90-481-9987-7_1,

1 Introduction

Model reduction is a commonly used technique in structural dynamics to facilitate the analysis of complex structural systems. The primary purpose of model reduction techniques are the establishment of models which are simpler to manage and faster to analyze. Especially in a first stage of the design, approximate responses are often sufficient for further design decisions. Nowadays, and surely in the near future, structural analysis is almost exclusively performed by applying Finite Element software. Automated mesh generation usually produce a detailed fine mesh which involves a large number of degrees of freedom to represent local stress concentration with sufficient fidelity. In structural dynamics, however, the response of many degrees of freedom can be safely represented by so called master degrees of freedom without noteworthy loss in accuracy. For large complex structures, the modeling of substructures and its global assembly provides many advantages, computationally and organizationally. For this purpose, various component mode synthesis procedures have been developed. These numerical methods speed up the development cycle and are especially useful in cases where many modifications of a structure need to be analyzed.

The above mentioned methods are also very useful in case the robustness of the response prediction should be investigated. This issue of robustness arises from uncertainties regarding the adequacy of the mathematical (FE) model and from the parameter uncertainties applied within the FE model. Unfortunately, the adequacy of a FE-model for its specified purpose is generally difficult to validate and often requires expensive measurements for an objective decision process. The study of the robustness of the response predictions by incorporating uncertainties of the input parameters, however, is a feasible task. Structural engineers are expected to know which parameters are uncertain and to have some information on the variability of these uncertain quantities, e.g. lower and upper bounds, mean value, standard deviation and similar. Translating the available information into probability distributions allows to study the resulting uncertainties of the response.

The model reduction techniques developed in the context of deterministic structural analysis provide useful techniques when evaluating the resulting uncertainties of the structural response. Since uncertainty propagation is generally a computationally intensive task, these techniques can be employed to keep the efforts computationally feasible and also within reasonable time constraints. These techniques need to be adjusted to the special needs in stochastic structural mechanics where the focus will be on stochastic structural dynamics. To study the response of uncertain linear dynamical structural systems, most attention has been paid to the so called random eigenvalue problem. Although, its efficient solution does not cover non-linear structural problems, it is certainly of central interest for many practical applications where larger linear structural FE model are used. There exist a large variety of approaches of which representative methods are selected to provide an informative overview on these methods.

2 Deterministic Procedures to Reduce Models

2.1 General Remarks

The increasing need of many industrial fields for highly accurate predictions gives rise to the development of finite element models of high complexity. The size of these models may lead to enormous computational efforts which may be even beyond the limits of hardware resources. Hence, it is necessary to reduce the computational costs by establishing surrogate models which approximate the full model and truncate unnecessary information. Another approach consists of the division of the model into substructures. The most commonly used approaches for model reduction will be discussed in the following subsections.

2.2 Guyan Reduction

Guyan reduction [21] is a substructuring approach which reduces the size of the model in a smaller problem by relating the displacements of the so-called slave degrees of freedom (DOFs) to the master DOFs. Hence, the governing equation in static analysis, $\mathbf{Ku} = \mathbf{F}$, can be written as

$$\begin{bmatrix} \mathbf{K}_{SS} & \mathbf{K}_{SM} \\ \mathbf{K}_{MS} & \mathbf{K}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{u}_S \\ \mathbf{u}_M \end{bmatrix} = \begin{bmatrix} \mathbf{F}_S \\ \mathbf{F}_M \end{bmatrix}, \tag{1}$$

where the indices S and M refer to the slave and master DOFs, respectively. Under the condition that the slave DOFs are unloaded, the displacements of the master DOFs are given by

$$\mathbf{u}_M = \left[\mathbf{K}_{MM} - \mathbf{K}_{MS}\mathbf{K}_{SS}^{-1}\mathbf{K}_{SM}\right]^{-1} \mathbf{F}_M . \tag{2}$$

The slave DOFs are related to the master DOFs through

$$\mathbf{u}_S = -\mathbf{K}_{SS}^{-1}\mathbf{K}_{SM}\mathbf{u}_M . \tag{3}$$

Hence, the static analysis can be partitioned into two smaller problems, where the solutions for $\mathbf{u}_S$ and $\mathbf{u}_M$ involve no approximations.

In dynamics, the application of the Guyan reduction to the determination of the structural response $\mathbf{u} = \mathbf{x}e^{i\omega t}$ of an undamped system with harmonic excitation $Fe^{i\omega t}$ of the form

$$\begin{bmatrix} \mathbf{K}_{SS} - \omega^2\mathbf{M}_{SS} & \mathbf{K}_{SM} - \omega^2\mathbf{M}_{SM} \\ \mathbf{K}_{MS} - \omega^2\mathbf{M}_{MS} & \mathbf{K}_{MM} - \omega^2\mathbf{M}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{x}_S \\ \mathbf{x}_M \end{bmatrix} = \begin{bmatrix} \mathbf{F}_S \\ \mathbf{F}_M \end{bmatrix} \tag{4}$$

yields the following solution for the slave DOFs

$$\mathbf{x}_S = -\left[\mathbf{K}_{SS} - \omega^2 \mathbf{M}_{SS}\right]^{-1} \left[\mathbf{K}_{SM} - \omega^2 \mathbf{M}_{SM}\right] \mathbf{x}_M. \tag{5}$$

This equation enforces a matrix inversion at each frequency step ω which leads to a high computational effort. Hence, an approximation of Eq. 5 is performed such that the contribution of the mass matrix is neglected, which leads to

$$\mathbf{x}_S = -\mathbf{K}_{SS}^{-1} \mathbf{K}_{SM} \mathbf{x}_M. \tag{6}$$

The application of Guyan reduction to dynamical systems therefore introduces some approximation errors. Approaches for the improvement of the Guyan condensation method in dynamics can be found in e.g. [5].

2.3 Component Mode Synthesis

The basic idea of component mode synthesis (CMS) is the division of the full model into a set of substructures and to describe these component models in terms of some basis vectors. By enforcing equilibrium and compatibility along component interfaces, the substructures are assembled to the global model, which exhibits a considerably smaller size than the original problem. The most common approaches in CMS, which was originally introduced by Hurty in 1965 [23], are the fixed interface method with constraint modes (Craig-Bampton method) [9–11] and the free interface method with attachment modes [12, 13].

2.3.1 Fixed Interface Craig-Bampton Method

The structural matrices are partitioned according to the boundary degrees of freedom (index B) and the interior degrees of freedom (index I) which leads to the following form of the undamped equation of motion of one component α

$$\begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{IB} \\ \mathbf{M}_{BI} & \mathbf{M}_{BB} \end{bmatrix}^{(\alpha)} \begin{bmatrix} \ddot{\mathbf{u}}_I \\ \ddot{\mathbf{u}}_B \end{bmatrix}^{(\alpha)} + \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_{BB} \end{bmatrix}^{(\alpha)} \begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{bmatrix}^{(\alpha)} = \begin{bmatrix} \mathbf{F}_I \\ \mathbf{F}_B \end{bmatrix}^{(\alpha)}. \tag{7}$$

The physical coordinates $\mathbf{u}^{(\alpha)}$ can be represented by component generalized coordinates, denoted $\mathbf{p}^{(\alpha)}$ using the transformation matrix $\mathbf{B}^{(\alpha)}$

$$\mathbf{u}^{(\alpha)} = \mathbf{B}^{(\alpha)} \mathbf{p}^{(\alpha)}. \tag{8}$$

In the Craig-Bampton method, the matrix $\mathbf{B}^{(\alpha)}$ is composed of fixed interface modes and constraint modes, where the fixed interface modes are obtained solving the eigenvalue problem

$$\mathbf{K}_{II} \boldsymbol{\Phi} = \mathbf{M}_{II} \boldsymbol{\Lambda} \boldsymbol{\Phi}. \tag{9}$$

The constraint modes are defined as the static reactions of the interior DOFs due to a unit displacement of one boundary DOF and zero displacement of the remaining boundary DOFs, and are given by

$$\boldsymbol{\Psi} = \begin{bmatrix} -\mathbf{K}_{II}^{-1}\mathbf{K}_{IB} \\ \mathbf{I}_{BB} \end{bmatrix}. \tag{10}$$

Hence, the transformation matrix $\mathbf{B}^{(\alpha)}$ in the Craig-Bampton method is given by

$$\mathbf{B}^{(\alpha)} = \begin{bmatrix} \boldsymbol{\Phi}_k & -\mathbf{K}_{II}^{-1}\mathbf{K}_{IB} \\ \mathbf{0} & \mathbf{I}_{BB} \end{bmatrix}, \tag{11}$$

where the index k refers to the number of kept normal modes.

The transformation of the mass and stiffness matrix into the component modal space yields the component modal mass and stiffness matrix

$$\boldsymbol{\mu}^{(\alpha)} = \mathbf{B}^{(\alpha)T}\mathbf{M}^{(\alpha)}\mathbf{B}^{(\alpha)} = \begin{bmatrix} \mathbf{I}_{kk} & \mathbf{m}_{IB} \\ \mathbf{m}_{BI} & \mathbf{m}_{BB} \end{bmatrix}, \quad \boldsymbol{\kappa}^{(\alpha)} = \mathbf{B}^{(\alpha)T}\mathbf{K}^{(\alpha)}\mathbf{B}^{(\alpha)} = \begin{bmatrix} \boldsymbol{\Lambda}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_{BB} \end{bmatrix} \tag{12}$$

The matrices $\mathbf{m}_{BB}$ and $\mathbf{k}_{BB}$ are the constraint modal mass and stiffness matrices and the matrix $\mathbf{m}_{IB}$ is a coupling matrix. The matrix $\boldsymbol{\Lambda}_k$ is a diagonal matrix composed of the eigenvalues that are retained. In case of many boundary DOFs as it appears in applications with line and surface coupling, the size of the FE-model can be dominated by the boundary DOFs and it might be desirable to reduce the size of the constraint and coupling matrices. One approach is given by the use of so-called characteristic constraint modes which form a highly reduced basis for the representation of the constraint modes [7].

As a next step, the substructure matrices $\alpha = 1, \ldots N_c$ are assembled to the global mass and stiffness matrix in the modal space.

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mu}^{(2)} & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\mu}^{(N_c)} \end{bmatrix}, \qquad \boldsymbol{\kappa} = \begin{bmatrix} \boldsymbol{\kappa}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\kappa}^{(2)} & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\kappa}^{(N_c)} \end{bmatrix} \tag{13}$$

The displacements at the juncture DOFs of two components α and β which have a common boundary are constrained by

$$\mathbf{u}_B^{(\alpha)} = \mathbf{u}_B^{(\beta)}. \tag{14}$$

Hence, a relation to impose the coupling constraint has to be established such that

$$\mathbf{p} = \mathbf{S}\mathbf{q}, \tag{15}$$

where the vector $\mathbf{p}$ contains the complete set of generalized components using all components of $\mathbf{p}^{(\alpha)}$, $\alpha = 1, \ldots N_c$ and the vector $\mathbf{q}$ contains the set of independent generalized coordinates (i.e. the redundant coordinates are excluded). The matrix $\mathbf{S}$ is then used to establish the global mass and stiffness matrices

$$\mathbf{M}_{\text{red}}^{\text{glob}} = \mathbf{S}^T \boldsymbol{\mu} \mathbf{S}, \quad \mathbf{K}_{\text{red}}^{\text{glob}} = \mathbf{S}^T \boldsymbol{\kappa} \mathbf{S}. \tag{16}$$

The reduced eigenvalue problem can now be solved

$$\mathbf{K}_{\text{red}}^{\text{glob}} \boldsymbol{\Phi}_{\text{red}} = \mathbf{M}_{\text{red}}^{\text{glob}} \boldsymbol{\Lambda}_{\text{red}} \boldsymbol{\Phi}_{\mathbf{red}}. \tag{17}$$

2.3.2 Component Modes for Unconstrained Components

An alternative approach to define the transformation matrix $\mathbf{B}^{(\alpha)}$ is given by the use of the free normal modes in combination with the attachment modes. The free interface modes are obtained by solving the eigenvalue problem which involves the full component matrices (i.e. corresponding to interior and boundary DOFs). The attachment modes are defined as the static deformation of the interior DOFs due to a unit force applied at one boundary DOF and with the remaining DOFs unloaded. In case the component exhibits rigid body motion, it must be sufficiently constrained and the rigid body modes are then added separately. An alternative approach of defining attachment modes in combination with rigid body motion consists of the use of inertia relief modes, where the details can be found in [31,43].

Although the discussion of the inclusion of damping is beyond the scope of this presentation it should be noted that also the damping matrix can be included in the CMS approach (see [4, 22]), where improvements of these original versions can be found, e.g. in [14]. In case of general viscous damping, complex component mode synthesis is required [37, 42], whereas applications in context of non-linear hysteretic elements are shown in [40].

2.4 Meta-Models

Apart from substructuring techniques the use of surrogate models offers the possibility to reduce the computational efforts considerably while still achieving highly accurate approximations. Response surfaces were introduced by Box and Wilson in 1951 [6] and developed into an indispensable tool for many scientific and technical applications in the development, improvement and optimization of the design. The scope of such a surrogate model is the representation of a relationship between several input variables and an output quantity. There is a wealth of different approaches available where the most commonly used meta-models are based on linear or polynomial regression, on a least-squares formulation or on kriging and radial basis functions. Comparisons of different approaches and examples for applications

can be found in e.g. [15, 20, 24, 45, 54]. A thorough treatment of various approaches for the formulation of response surfaces can be found, e.g. in [8, 34].

Similarly to the response surface method, a so-called *meta-model*, which represents a functional relation between the input parameters and the model properties, has been developed in order to approximate the eigenfrequencies and eigenvectors of large dynamical systems (see e.g. [39]). This meta-model can be used for the assessment of the variability of the structural response and also for optimization and reliability analysis.

The basic idea of this approach is the approximation of an eigenvector $\phi_k^{(j)}$ in the j-th simulation as a linear combination of the neighboring modes of a reference solution

$$\phi_k^{(j)} = \sum_{i=k-m_k}^{k+m_k} \alpha_{ki}^{(j)} \phi_i^{(0)} , \tag{18}$$

where the parameters $\boldsymbol{\alpha}_k = \{\alpha_{k,i}\}_{i=k-m_k}^{k+m_k}$ represent the coefficients of the linear combinations of the reference modes $\boldsymbol{\Phi}^{(0)}$ and are obtained by projecting the vector $\phi_k^{(j)}$ onto the subspace spanned by the reference eigenvectors. The assumption is that in case of small variability of the input parameters in combination with well separated modes the eigenvectors differ only slightly from the nominal solution which was also shown in e.g. [53]. The corresponding eigenfrequency is given by

$$\omega_k^{(j)} = (1 + \Delta)\, \omega_k^{(0)}. \tag{19}$$

Hence, the modal characteristics of each simulation can be described by $\alpha_{ki}^{(j)}$, Δ and the reference solutions $\omega_k^{(0)}$, $\boldsymbol{\Phi}^{(0)}$. With respect to the relationship between these quantities and the structural input parameters, a linear approximation suffices to be sufficiently accurate. The matrix representing the functional relation is based on a calibration set and can be calculated using several procedures, such as the least-squares method, regression, optimization or soft computing methods.

3 Stochastic Analysis

3.1 Uncertainty Modeling

Engineers are aware that their established models represent only an approximation to reality and that there are various sources of uncertainties in the modeling process, such as the uncertainties in the model parameters and the simplifying assumptions made to reduce the complexity of the model. Hence, in order to give confident predictions of the target structural responses the consideration of uncertainties becomes an indispensable step in the design process. It has been recognized already in the

1920s that the modeling process should account for a quantitative assessment of the uncertainties [32]. Comprehensive surveys of the developments of theoretical aspects and their practical applicability can be found, e.g. in [46–49].

Most uncertainty quantification schemes are based on the description of the uncertainties as random variables, stochastic processes or random fields. A random variable X is described by its probability density function $f(x)$ (PDF) and the probability that the parameter takes a value within a certain interval $[a, b]$ is given by the integral

$$P(a < X < b) = \int_a^b f(x)\,dx, \tag{20}$$

where this formula can be extended to multi-dimensional space. Typical examples for random variables are the Young's modulus, the thickness, etc.

A random variable $f(x,t)$ which varies in time is called a stochastic process. Examples are given by the uncertainties associated with wind, earthquakes, etc. A dependence of the uncertain quantity on the spatial coordinate is modeled by a random field, e.g. a spatially varying Young's modulus.

The variation of a random variable in time or space can be represented by a Karhunen-Loève (KL) expansion which is given for a Gaussian random variable in the form

$$X(\varphi) = x^{(0)}(\varphi) + \sum_{j=1}^{m} \xi_j x^{(j)}(\varphi), \tag{21}$$

where the variable φ may refer to the time or a spatial coordinate. The functions $x^{(0)}(\varphi)$ and $x^{(j)}(\varphi)$ are deterministic functions and ξ_j is a realization of a standard normal distribution. The definition of the basis functions $x^{(j)}(\varphi)$ is performed such that only a few terms m are necessary to represent $X(\varphi)$ with high accuracy. In structural dynamics, the Karhunen-Loève expansion is mainly used for the representation of random fields and random processes, applications in the characterization of the structural response can be found in e.g. [55], where a reduction in phase space is performed by using KL-expansion.

In addition to approaches which consider uncertainties on the physical level, i.e. by considering the scatter of the physical model parameters, an approach has recently been proposed which takes uncertainties on a global level into account. This method is referred to as a non-parametric approach [51, 52] and defines the matrix entries of the reduced numerical model as random variables. This reduced numerical model is obtained with the truncated modal basis of the full nominal model.

3.2 Simulation Techniques

Several approaches have been developed in order to capture these different types of uncertainties. The most versatile approach to capture uncertainties is the Monte

Carlo simulation (see e.g. [16]). However, due to the associated large computational efforts, already the analysis of moderate sized problems may meet the limits of computational feasibility. Hence, large research efforts have been ongoing towards the minimization of the analysis time.

One way to accomplish this goal is the development of advanced simulation algorithms which are directed towards "controlled" sampling, i.e. towards guiding the samples into the domain of interest, which is e.g. the failure domain in case of reliability analysis. In case the domain of interest is known or can be computed by gradient based algorithms, methods such as importance sampling (see e.g. [50]) or line sampling [29] are very efficient procedures. In other cases, as e.g. for strongly non-linear responses, subset simulation [2] is an accurate and powerful technique.

The second approach in order to reduce the computational costs is the parallelization of the analysis. The importance of parallel computing in stochastic structural mechanics has long been emphasized (see e.g. [17]). In the field of structural dynamics, Johnson et al. [25] investigated the performance of massively parallel platforms for the Monte-Carlo simulation of stochastic dynamics problems. Applications of parallel processing in context with uncertainty quantification in structural dynamics can be found in e.g. [26, 54] and referring to reliability analysis in e.g. [38].

The third approach is concerned with the minimization of the computational costs of one simulation. This can be achieved by using local approximation methods based on the perturbation method or Neumann expansion series which are widely used in computational stochastic analysis [28]. The perturbation method is based on a Taylor expansion of the structural matrices, the load vector and the vector of the structural response. The computational efforts grow with the number of uncertain parameters and with the order of the expansion. In case of small coefficients of variation, a linear approximation of the response turns out to predict the structural response sufficiently accurately. The Neumann expansion takes advantage – similarly to the perturbation approach – from a reference solution and is based on the approximation of the solution using a few basis vectors which span the preconditioned stochastic Krylov subspace as shown in [35]. Hence, the Neumann method constitutes a special case of a stochastic reduced basis method as proved in [27].

Another stochastic reduced basis method which approximates the solution in the random dimension – and not in space as described above – is given by the Polynomial Chaos decomposition scheme [18]. The key ingredient of the chaos expansion is to represent the random entities by an orthogonal basis defined in a Hilbert space. Applications of Polynomial Chaos in dynamics, specifically for the random eigenvalue problem, are shown in e.g. [19, 56]. A comparison of different subspace projection techniques can be found in [44].

The goal of reducing the computational costs of one single simulation can also be reached by using the substructure techniques described in Sect. 2. Hence, recent advances and new developments in the combination of uncertainty quantification methods and model reduction techniques are addressed in the following.

4 Model Reduction Taking into Account Uncertainties

4.1 Introduction

Component mode synthesis represents an appealing framework for the quantification of uncertainty in build-up structures due to the computational savings that occur in each simulation [3]. In addition, the substructures are usually statistically independent since they are manufactured by different companies and finally assembled. Hence, this makes it possible to independently analyze the components with the option of applying different uncertainty propagation procedures to the components.

Component mode synthesis is also particularly advantageous for the application of perturbation approaches in order to estimate the variability at the global level. Due to the almost linear relationship between the structural parameters and the component modal properties, as well as between the component modal properties and global modal properties, it is possible to accurately approximate the variability of the global modal properties and frequency response functions by linear approximations. This local modal/perturbational approach was proposed in [30]. Another approach for the assessment of the variability of the eigenfrequency and eigenvectors using the advantages of substructuring is presented in [41], where a special form of CMS is proposed which is oriented towards an efficient computation of eigenvectors and eigenvalues.

A further possibility for the consideration of uncertainty that is offered by CMS methods is based on the Karhunen Loève expansion of the substructure matrices where the sparse structure of the modal substructure matrices leads to a computationally efficient uncertainty propagation method. This novel approach will be discussed in detail in the next section.

In addition, a second approach will be proposed that uses the advantages of substructuring by randomly combining independently simulated substructures. Hence, in addition to the probability density functions of the physical parameters, also the random combination, expressed by discrete uniform distributions for the set of substructure matrices, introduces variability in order to capture the uncertainty by which the structure is affected. Finally, the combination of the approximation methods and substructuring constitutes a straight forward combination of advanced techniques for uncertainty propagation and model reduction schemes. These approaches will be described in the following.

4.2 Karhunen-Loève Expansion of the Substructure Matrices

4.2.1 Introduction

The idea of this approach consists of capturing uncertainties not on the *physical level* but rather in the *modal space* of each substructure. The special form of the Craig-Bampton substructure matrices with its high sparsity (see Eq. 12) makes it

computationally advantageous to define the entries of the substructure matrices in the modal space as random variables and to reproduce the variability of the physical parameters on the matrix level.

In order to calibrate the uncertain reduced substructures, samples of substructures are generated by considering the scatter of the physical parameters. Based on this sample set of each substructure, the second order statistics of the entries of the reduced matrices can be estimated. The vectors of the mean values and the covariance matrices are then used to represent the entries using a Karhunen-Loève expansion as introduced in Sect. 3 for random processes and random fields. As opposed to random processes and random fields the Karhunen-Loève representation does not discretize the process in time or space but in the random dimension by transforming a number of correlated matrix entries into a smaller number of uncorrelated variables (i.e. principal components).

4.2.2 Calibration Set for the Structural Matrices

As a first step, a set of substructure matrices $\boldsymbol{\mu}_j^{(\alpha)}$ and $\boldsymbol{\kappa}_j^{(\alpha)}$ with the corresponding transformation matrix $\mathbf{B}_j^{(\alpha)}$, for $j = 1, \ldots, J$ is generated, where J denotes the number of simulations forming the calibration set. This yields the vectors $\hat{\mathbf{x}}_j^{\mu}$ and $\hat{\mathbf{x}}_j^{\kappa}$ which are composed of the non-zero entries of the modal mass and stiffness matrix of one simulation, where the symmetry of the matrices leads to a reduced number of entries to be stored.

When generating the calibration set for the modal mass matrix it should be noted that in contrast to the matrices $\mathbf{m}_{BB}$ and $\mathbf{k}_{BB}$ the coupling matrix $\mathbf{m}_{IB}$ is a function of the normal modes. Hence, the entries of this matrix obtained from two different simulations are only physically comparable if the sequence of the mode shapes is the same in both simulations. Hence, mode pairing has to be adopted, which can be performed using e.g. the widely applied modal assurance criterion (MAC) that has been introduced in [1].

4.2.3 Calibration Set for the Matrix of Eigenvectors and Constraint Modes

The number of non-zero entries of the transformation matrix $\mathbf{B}^{(\alpha)}$ is too high in order to efficiently use the vectors $\hat{\mathbf{x}}_j^{\Phi}$ and $\hat{\mathbf{x}}_j^{\Psi}$ for the calibration set since the computation of the eigensolution of the covariance matrix may lead to an infeasible computational effort for larger systems. Hence, the idea is to represent the random eigenvectors $\boldsymbol{\Phi}_j$ and constraint modes $\boldsymbol{\Psi}_j$ by linear combinations of basis vectors, i.e.

$$\boldsymbol{\Phi}_j = \boldsymbol{\Phi}^{\mathrm{ref}} \hat{\mathbf{Q}}_j, \quad \boldsymbol{\Psi}_j = \boldsymbol{\Psi}^{\mathrm{ref}} \hat{\mathbf{P}}_j \tag{22}$$

These basis vectors $\boldsymbol{\Phi}_j$ and $\boldsymbol{\Psi}_j$ are an approximation to the full space of eigenvectors and constraint modes, which are obtained in every simulation. The coefficients

$\hat{\mathbf{Q}}_j$ and $\hat{\mathbf{P}}_j$ are obtained by solving a least squares problem. The construction of the reference subspaces $\boldsymbol{\Phi}^{\text{ref}}$ and $\boldsymbol{\Psi}^{\text{ref}}$ is based on the extraction of important components of the set of random eigenvector bases and random constraint mode bases. The extraction of the important component can be carried out by projecting the eigenvectors and constraint modes of the current simulation onto the subspace $\mathbf{X}$, i.e.

$$\Pi_{\mathbf{X}}(\phi_j) = \sum_{i=1}^{k} \frac{< \phi_j, \mathbf{x}_i >}{\|\mathbf{x}_i\|^2} \mathbf{x}_i, \tag{23}$$

The initialization of the subspace can be performed using the orthonormal eigenvectors of the first simulation. The same procedure is employed for the constraint modes. If the eigenvector or constraint mode can not be represented with a certain accuracy by the current basis vectors, then the subspace is augmented.

4.2.4 Karhunen-Loève Representation of the Matrices

The vectors forming one simulation $\hat{\mathbf{x}}_j^{\mu}$, $\hat{\mathbf{x}}_j^{\kappa}$, $\hat{\mathbf{x}}_j^{Q}$ and $\hat{\mathbf{x}}_j^{P}$ of each substructure (the index α has been omitted) are assembled to one simulation vector $\hat{\mathbf{x}}_j^{(\mu,\kappa,Q,P)}$. The vectors $\hat{\mathbf{x}}_j^{(\mu,\kappa,Q,P)}$, $j = 1, \ldots, J$ form the calibration set of each substructure and are used for the estimation of the mean vector $\bar{\mathbf{x}}^{(\mu,\kappa,Q,P)}$ and the covariance matrix $\mathbf{C}^{(\mu,\kappa,Q,P)}$. An eigenvalue solution of the covariance matrix yields the basis vectors for reproducing the uncertainty in the calibration set. The most important directions are associated with the largest eigenvalues which leads to the following representation of the matrix entries for any $k > J$:

$$\mathbf{x}_k^{(\mu,\kappa,Q,P)} = \bar{\mathbf{x}}^{(\mu,\kappa,Q,P)} + \sum_{i=1}^{N_{KL}} \xi_i \sqrt{\lambda_i} \phi_i, \tag{24}$$

where λ_i and ϕ_i denote the ith eigenvalue and eigenvector of the covariance matrix with $N_{KL} \ll N$, and N is the length of the vector $\mathbf{x}_j^{(\mu,\kappa,Q,P)}$.

4.2.5 Assembly of the Global Matrices

The representation of the substructure matrix entries by a KL-expansion allows for an efficient simulation of the matrix entries. Hence, in each simulation, the vector $\mathbf{x}^{(\mu,\kappa,Q,P)}$ is simulated as described above. Then, the modal mass matrix $\boldsymbol{\mu}$ and the modal stiffness matrix $\boldsymbol{\kappa}$ can be established according to the structure in Eq. 12. These substructure matrices are assembled to the global reduced matrices according to the Craig-Bampton reduction scheme (see Eqs. 13–16).

An example-based verification of this approach where the results are compared with direct Monte Carlo simulation will be presented in Sect. 5.

4.3 Random Combination of Substructures

In the approach presented here, a random sampling strategy is applied in order to obtain a statistical assessment of the quantity of interest using only a moderate sample size of e.g. 30–50 samples. The idea consists of the generation of a pool of random substructure matrices and to combine these substructure matrices randomly.

More specifically, $J^{(\alpha)}$ structural matrices of the substructure α are generated and an equal probability is assigned to each realization j

$$p_j^{(\alpha)} = \frac{1}{J^{(\alpha)}}. \tag{25}$$

The global structural matrices are composed of the substructures $\alpha = 1, \ldots, N_C$, where N_C denotes the number of components. These samples are generated using direct Monte Carlo simulation. In order to represent the variability with a few samples it can also be advantageous to use Latin Hypercube sampling. The random combination of the substructures introduces some additional variability in the structural response which makes the approach particularly advantageous in case of many substructures. This method aims to imitate the randomness associated with random sampling. Once the set of substructures is selected, the global reduced structural matrices are assembled by applying the Craig-Bampton method (see Eqs. 13–16). It should be noted that this approach can also be used if mode switching occurs and is thus also applicable for the higher frequency range. An example-based verification of this approach will be given in Sect. 5.

4.3.1 Stochastic Reduced Basis Approximation (SRBA)

The simulation of the set of substructures needed for the above approach may lead to large computational efforts if dealing with complex substructures. The most time-consuming part concerns the solution of the eigenvalue problem. Therefore it is advantageous to apply approximate solution schemes. Among these approximation methods, the Stochastic Reduced Basis Approximation (SRBA) introduced in [36], which constitutes a reduced basis formulation for the efficient solution of large-scale eigenvalue problems, will be in the focus of this manuscript. The eigenvalue problem is given by

$$\left[\mathbf{K_0} + \sum_{i=1}^{p} \theta_i \mathbf{K}_i\right] \boldsymbol{\phi}(\boldsymbol{\theta}) = \lambda(\boldsymbol{\theta}) \left[\mathbf{M_0} + \sum_{i=1}^{p} \theta_i \mathbf{M}_i\right] \boldsymbol{\phi}(\boldsymbol{\theta}), \tag{26}$$

where $\mathbf{K_0}$ and $\mathbf{M_0}$ denote the nominal mass and stiffness matrices, $\sum_{i=1}^{p} \theta_i \mathbf{K}_i$ and $\sum_{i=1}^{p} \theta_i \mathbf{M}_i$ are the perturbations in the mass and stiffness matrix and $\boldsymbol{\phi}(\boldsymbol{\theta})$ and $\lambda(\boldsymbol{\theta})$ are the modal properties of the perturbed system. In the present application, these structural matrices do not refer to the full system, but to the substructures of the investigated structure.

The basic idea of this approach consists of the representation of the basis vectors for approximating the random eigenvalue solution by the eigenvectors of the nominal solution and their first order derivatives. Therefore, the approximation for a perturbed eigenvector $\boldsymbol{\phi}$ can be written as

$$\boldsymbol{\phi}(\boldsymbol{\theta}) = \zeta_1(\boldsymbol{\theta})\boldsymbol{\phi}_0 + \zeta_2(\boldsymbol{\theta}) \sum_{i=1}^{p} \theta_i \frac{\partial \boldsymbol{\phi}}{\partial \theta_i}, \tag{27}$$

where $\boldsymbol{\theta}$ denotes the vector of the uncertain parameters of the structural matrices, $\boldsymbol{\phi}_0$ is the eigenvector obtained from the baseline eigenvalue problem and $\frac{\partial \boldsymbol{\phi}}{\partial \theta_i}$ are the partial derivatives of the eigenvector evaluated at the nominal solution. In order to compute the undetermined coefficients $\zeta_1(\boldsymbol{\theta})$ and $\zeta_2(\boldsymbol{\theta})$, the approximation of the eigenvectors in Eq. 27 is used in the random eigenvalue problem as stated in Eq. 26. This leads to a 2×2 eigenvalue problem of the form

$$\mathbf{K}_R(\boldsymbol{\theta})Z(\boldsymbol{\theta}) = \lambda(\boldsymbol{\theta})\mathbf{M}_R(\boldsymbol{\theta})Z(\boldsymbol{\theta}), \tag{28}$$

where the explicit formulation of the elements of the reduced matrices $\mathbf{K}_R(\boldsymbol{\theta})$ and $\mathbf{M}_R(\boldsymbol{\theta})$ can be found in [36].

4.4 Linear Interpolation of Substructure Matrices

Another approach for the simulation of structural matrices by means of a set of previously generated structural matrices consists in the representation of structural matrices as a linear combination of matrices. Hence, instead of taking one simulated matrix for each substructure randomly from the sample pool, these matrices are approximated using two previously generated matrices. In this way, a meta-model for constructing efficiently structural matrices and for solving the corresponding eigenanalysis in a reduced space can be constructed.

4.4.1 Generation of the Sample Pool

First, a set of input points and their associated substructure matrices $\mathbf{K}(\boldsymbol{\theta})$ and $\mathbf{M}(\boldsymbol{\theta})$ are generated as discussed in Sect. 4.3, where the n-dimensional vector $\boldsymbol{\theta}$ represents a point in the standard normal space. For these simulations and for the nominal solution $\boldsymbol{\theta}^{(0)} = \mathbf{0}$ the classical general eigensolutions have been computed.

$$\mathbf{K}^{(j)}\boldsymbol{\Phi}^{(j)} = \mathbf{M}^{(j)}\boldsymbol{\Phi}^{(j)}\boldsymbol{\Lambda}^{(j)}, \quad 0 \leq j \leq J. \tag{29}$$

As opposed to the approach in Sect. 4.3, a new simulation is not generated by taking *one* simulation of each substructure from the pool, but by using *two* samples of each substructure.

4.4.2 Approximation of Structural Matrices and Eigenvectors

Any random point $\boldsymbol{\theta}^{(k)}, k > J$ lying in a two dimensional subspace spanned by two randomly selected points from the sample pool can be represented as

$$\boldsymbol{\theta}^{(k)} = \xi_1 \boldsymbol{\theta}^{(m)} + \xi_2 \boldsymbol{\theta}^{(n)}, \tag{30}$$

where $\boldsymbol{\theta}^{(m)}$ and $\boldsymbol{\theta}^{(n)}$ are two previously sampled points, i.e. $1 < m, n < J$. The radius R of the point $\boldsymbol{\theta}^{(k)}, k > J$ is distributed according to a chi-square distribution and the angle α, which defines the direction in this two-dimensional subspace follows a uniform distribution in the interval $[0,\ 2\pi]$. Hence, Eq. 30 is written as

$$\boldsymbol{\theta}^{(k)} = (\cos\alpha \mathbf{v}_1 + \sin\alpha \mathbf{v}_3)\, R, \tag{31}$$

where $\mathbf{v}_1$ is the normalized vector $\boldsymbol{\theta}^{(m)}$ and defines with $\mathbf{v}_3$ an orthonormal basis for the subspace (see Fig. 1).

The associated structural stiffness and mass matrices are now defined as

$$\mathbf{M}(\boldsymbol{\theta}^{(k)}) := \mathbf{M}^{(k)} = \mathbf{M}^{(0)}[1 - \xi_1 - \xi_2] + \xi_1 \mathbf{M}^{(m)} + \xi_2 \mathbf{M}^{(n)}, \tag{32}$$

$$\mathbf{K}(\boldsymbol{\theta}^{(k)}) := \mathbf{K}^{(k)} = \mathbf{K}^{(0)}[1 - \xi_1 - \xi_2] + \xi_1 \mathbf{K}^{(m)} + \xi_2 \mathbf{K}^{(n)}, \tag{33}$$

where $\mathbf{M}^{(m)}$ and $\mathbf{K}^{(m)}$ denote $\mathbf{M}(\boldsymbol{\theta}^{(m)})$ and $\mathbf{K}(\boldsymbol{\theta}^{(m)})$, respectively. Moreover, the following linear approximation makes sense, because it approximates the eigensolution in the close neighborhood of the established solutions for $\boldsymbol{\Phi}^{(j)}, j \in [0, J]$ very well.

$$\boldsymbol{\Phi}(\boldsymbol{\theta}^{(k)}) \approx \tilde{\boldsymbol{\Phi}}^{(k)} = \boldsymbol{\Phi}^{(0)}[1 - \xi_1 - \xi_2] + \xi_1 \boldsymbol{\Phi}^{(m)} + \xi_2 \boldsymbol{\Phi}^{(n)} \tag{34}$$

The quality of the approximation for points $\boldsymbol{\theta}^{(k)}$, which are distant from the two points $\boldsymbol{\theta}^{(m)}$ and $\boldsymbol{\theta}^{(n)}$, must be investigated by numerical tests.

In the following, however, it is not assumed that these values are accurate, but that they establish the correct subspace, in which the vectors $\tilde{\boldsymbol{\Phi}}^{(k)}$ describe the subspace

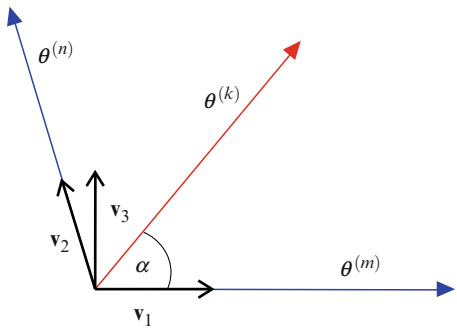

Fig. 1 Representation of $\theta^{(k)}$ by the two samples $\theta^{(m)}$ and $\theta^{(n)}$

spanned by the eigenvectors of the matrices $\mathbf{M}^{(k)}$ and $\mathbf{K}^{(k)}$. This assumption implies that the correct eigenvectors are a linear combination of the vectors in the matrix $\tilde{\boldsymbol{\Phi}}^{(k)}$, which can be represented by

$$\boldsymbol{\Phi}^{(k)} = \tilde{\boldsymbol{\Phi}}^{(k)} \mathbf{Q}^{(k)} \tag{35}$$

This transformation matrix $\mathbf{Q}^{(k)}$ can be determined by the reduced eigensolution

$$\mathbf{K}_R^{(k)} \mathbf{Q}^{(k)} = \mathbf{M}_R^{(k)} \mathbf{Q}^{(k)} \boldsymbol{\Lambda}^{(k)} \tag{36}$$

where the reduced matrices read

$$\mathbf{K}_R^{(k)} = \tilde{\boldsymbol{\Phi}}^{(k)T} \mathbf{K}^{(k)} \tilde{\boldsymbol{\Phi}}^{(k)} \tag{37}$$

$$\mathbf{M}_R^{(k)} = \tilde{\boldsymbol{\Phi}}^{(k)T} \mathbf{M}^{(k)} \tilde{\boldsymbol{\Phi}}^{(k)}. \tag{38}$$

The reduced eigenvalue problem provides also the eigenvalues.

5 Numerical Example

5.1 Problem Statement

As a numerical example a satellite structure is used. This model is divided into the following ten substructures (shown in Fig. 2 [33]):

- Substructure 1: adapter
- Substructure 2: central cylinder
- Substructure 3: lower platform
- Substructures 4–9: panel no. 1–6
- Substructure 10: upper platform

The connections between the substructures are modeled using elastic springs leading to a total number of 187 springs with 3 translational and 3 rotational DOFs. On each of the six panels, concentrated masses are added in order to model some equipment. The uncertainties of the structural parameters, i.e. of spring stiffnesses, concentrated masses, thicknesses, Young's moduli and densities, are considered, where truncated Gaussian distributions with a coefficient of 20% (spring stiffnesses) and 2.5% (concentrated masses, thicknesses, Young's moduli and densities), respectively, are assumed. A fine mesh is adopted to model the build-up structure leading to approximately 36,000 DOFs. This size of the structural matrices makes it possible to explore also the scalability of the proposed algorithms in addition to investigating their accuracy.

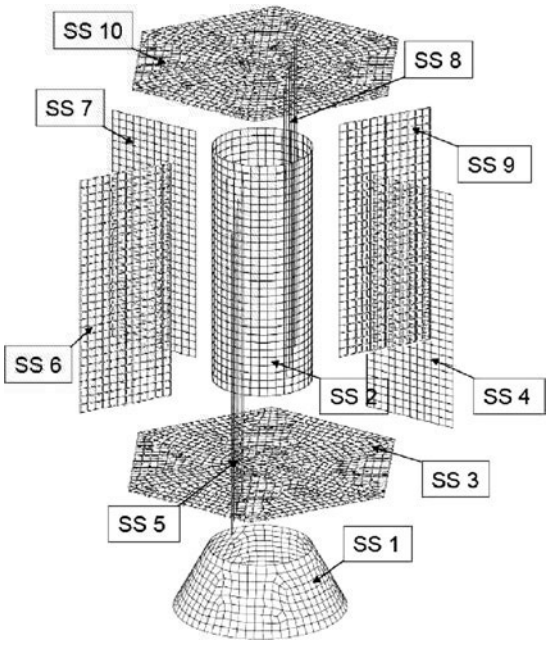

Fig. 2 Numerical example [33] split into ten substructures

As a reference solution 1,000 Monte Carlo samples are generated and compared with the four methods presented in the previous section. The quantities of interest are the statistics of the eigenvalues. For the methods that are based on a random combination of substructures, a set of 50 substructure matrices is generated and combined randomly in order to assess the statistical description of the output quantities of interest. In order to capture the randomness with few samples, Latin Hypercube sampling is adopted.

5.2 Results

5.2.1 Karhunen-Loève Expansion of the Substructure Matrices

Figure 3 shows the comparison of the histograms of the first six eigenfrequencies. The second and third mode have closely spaced eigenfrequencies, for which reason only one peak is visible. The mean values of the six eigenfrequencies (apart mode no. 5) can be predicted well using the KL-decomposition of the substructure matrix entries. The standard deviation leads to an underestimation of the reference scatter as obtained with direct MCS of the full model.