Operator Theory
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214

Jan Janas
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Editors

Spectral Theory and Analysis

Conference on Operator Theory, Analysis and Mathematical Physics (OTAMP) 2008, Bedlewo, Poland

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Jan Janas
Institut of Mathematics
Polish Academy of Sciences
ul. Sw. Tomasza 30/7
31-027 Krakow, Poland
najanas@cyf-kr.edu.pl

Pavel Kurasov
Department of Mathematics
Lund Institute of Technology
221 00 Lund, Sweden
kurasov@maths.lth.se

Ari Laptev
Department of Mathematics
South Kensington Campus
Imperial College London
London SW7 2AZ, UK
a.laptev@imperial.ac.uk
and
Department of Mathematics
Royal Institute of Technology
100 44 Stockholm, Sweden
laptev@math.kth.se

Sergei Naboko
Department of Mathematical Physics
St. Petersburg State University
Ulyanovskaya 1
198904 St. Petersburg, Russia
naboko@snoopy.phys.spbu.ru

Günter Stolz
Department of Mathematics
University of Alabama
452 Campbell Hall
Birmingham, AL 35294-1170, USA
stolz@math.uab.edu

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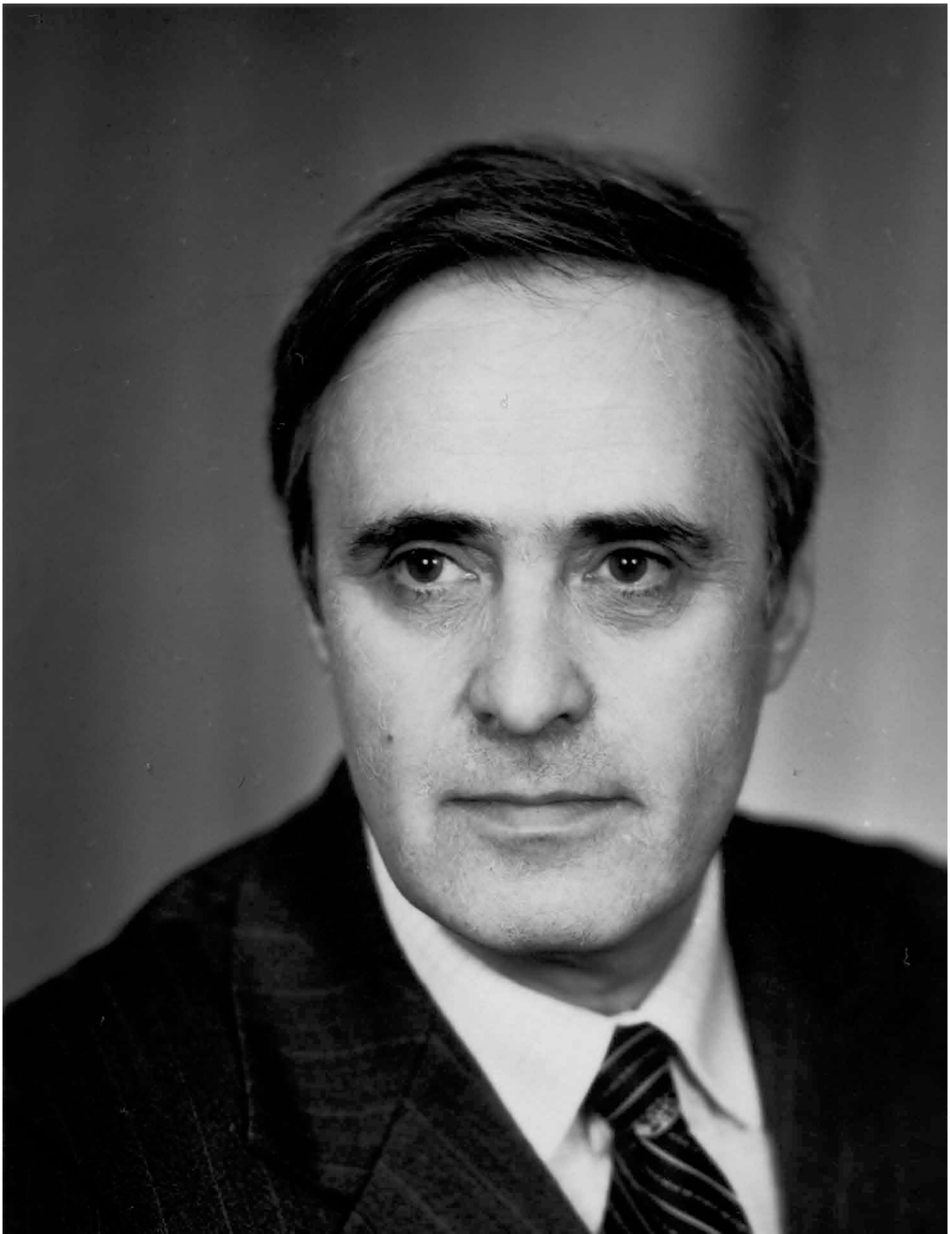
This volume is dedicated to the
memory and achievements of

Mikhail Shlemovich Birman

an outstanding mathematician of the
20th century in Mathematical Physics



Mikhail Shlemovich Birman 1928–2009

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Introduction

This volume appears as an outcome of the conference Operator Theory, Analysis and Mathematical Physics – OTAMP2008, held at Mathematical Research and Conference Center in Bedlewo near Poznan. The volume contains few review articles as well as original research papers presented at the conference or appeared as a result of inspiring discussions during the meeting.

The titles of conference talks followed the subjects of four special sessions:

- random and quasi-periodic differential operators
- orthogonal polynomials
- Jacobi and CMV matrices
- quantum graphs

All contributions to this volume are devoted to different chapters of operator theory with a focus towards applications in mathematical physics. Several articles are in the area of spectral theory of Schrödinger operators having emphasis on problems with magnetic fields. Another subject well-represented concerns spectral theory for non-self-adjoint problems. Spectral analysis is not restricted to just linear and self-adjoint problems.

This volume is devoted to the memory of Mikhail Shlemovich Birman – one of the most outstanding scientists of the last century. The influence of his ideas on the development of mathematical physics in the whole world and especially in Saint Petersburg will continue for decades, several of the authors contributed to this volume have been his students and will carry over his special attitude towards science to new generations to come.

Preparing this volume we remembered another remarkable mathematical physicist Israel Gohberg who always supported OTAMP conferences by including proceedings into the series *Operator Theory: Advances and Applications* and helping us with selection of outstanding contributors and interesting subjects in operator theory.

We would like to thank the European Science Foundation (ESF) for a generous financial support which allowed to transfer the OTAMP conference into a major event in the area of mathematical physics in 2008. We are grateful to all people working at Mathematical Research and Conference Center in Bedlewo for creating a stimulating scientific atmosphere and help before, during and after the conference.

Birmingham-Krakow-London May 2010
Lund-St. Petersburg-Stockholm The Editors

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Floquet-Bloch Theory for Elliptic Problems with Discontinuous Coefficients

B.M. Brown, V. Hoang, M. Plum and I.G. Wood

Abstract. We study spectral properties of elliptic problems of order $2m$ with periodic coefficients in L^∞. Our goal is to obtain a Floquet-Bloch type representation of the spectrum in terms of the spectra of associated operators acting on the period cell. Our approach using bilinear forms and operators in H^{-m}-type spaces easily handles discontinuous coefficients and has the merit of being rather direct. In addition, the cell of periodicity is allowed to be unbounded, i.e., periodicity is not required in all spatial directions.

Mathematics Subject Classification (2000). 35J10, 35j30, 35J99, 35P10.

Keywords. Floquet-Bloch, $2m$th-order elliptic, spectral theory.

1. Introduction

One of the most important partial differential operators in quantum physics is the Schrödinger operator

$$-\Delta + V(x), \quad x \in \mathbb{R}^d.$$

In many application areas the potential $V(\cdot)$ is periodic with respect to a lattice in $\mathbb{R}^d$. An extension of this equation to include a magnetic term gives rise to the magnetic Schrödinger operator

$$(-i\nabla - A(x))^2 + V(x), \quad x \in \mathbb{R}^d$$

where now both the potential V and the magnetic potential A are periodic with respect to the underlying lattice. Further examples of elliptic partial differential operators which have periodic coefficients may also be found in the periodic Dirac operator, the fields of periodic acoustics and photonic crystals. In all these cases of periodic coefficients, the main way of studying the spectrum of a suitable self-adjoint realisation is via the so-called Floquet-Bloch theory where essentially the spectral properties of the operator in $\mathbb{R}^d$ are read off from the behaviour on a fundamental cell of periodicity resulting in the well-known band-gap structure. For further information, see [2–5, 7, 8, 10, 11, 13–15] and the references quoted therein.

Often, in particular when the potentials are non-smooth, it is advantageous to study these problems using an associated bilinear form, and this is the approach that we take. We work with an mth-order elliptic Hermitian sesquilinear form on $H^m(\mathbb{R}^d)$ with periodic and bounded coefficients which we also allow to be discontinuous. This is equivalent to studying a selfadjoint operator L (associated with the bilinear form) in the dual space $H^{-m}(\mathbb{R}^d)$. Our motivation for this is found in the study of crystals where in practice often two (or more) materials are combined to form a periodic structure, which results in piecewise constant periodic coefficients. For future investigation of wave-guide properties we will require periodicity of the coefficients only in some spatial directions, i.e., we allow unbounded periodic cells. Hence in general no Bloch waves are available, and we have to use other techniques replacing the usual Bloch wave expansion, e.g., we prove that the Floquet transform (which is known to be an isometric isomorphism between L^2-spaces) is an isometric isomorphism also between H^{-m} spaces (see [10] for further mapping properties of the Floquet transformation).

Our result gives the well-known decomposition of the spectrum of periodic differential operators, developed, e.g., in [10], [14], [4], now also in the case of discontinuous coefficients (including the principal ones) and unbounded periodicity cell. A corresponding result is stated in [6], lacking however a detailed and self-contained proof, which we will give in this paper in a rather direct way.

We shall further show that the spectrum of L coincides with the spectrum of a suitable operator $\widetilde{\mathrm{L}}$ in $L^2(\mathbb{R}^d)$ associated with the bilinear form, which is constructed in a standard way. A direct study of the spectrum of $\widetilde{\mathrm{L}}$ by the "usual" Floquet-Bloch theory in $L^2(\mathbb{R}^d)$ seems to be problematic due to lack of smoothness in the coefficients.

2. Definitions and preliminary results

In the following, $H^m(\mathbb{R}^d)$ will always denote the Sobolev space of functions which are square Lebesgue-integrable over $\mathbb{R}^d$ with square integrable derivatives up to order m. We denote the usual norm by

$$\|u\|^2_{H^m(\mathbb{R}^d)} = \sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\mathbb{R}^d)}\,.$$

Let

$$B : H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \to \mathbb{C}$$

be a closed Hermitian sesquilinear form. We write $d = d_1 + d_2$ and use variables $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. We assume B is given in the form

$$B[u,v] := \sum_{|\rho|,|\sigma|\le m} \int_{\mathbb{R}^d} a^{\rho\sigma}(x,y)(D^\rho u)(x,y)\overline{(D^\sigma v)(x,y)}dxdy \qquad (2.1)$$

where $a^{\rho\sigma} \in L^\infty(\mathbb{R}^d)$, $a^{\rho\sigma} = \overline{a^{\sigma\rho}}$, and $a^{\rho\sigma}(x, y) = a^{\rho\sigma}(x, y + e_j)$ for any $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$, $j = 1, \ldots, d_2$, where $e_1, \ldots, e_{d_2}$ are the unit vectors[1] in $\mathbb{R}^{d_2}$. Moreover, we assume that the leading coefficients satisfy the ellipticity condition[2]

$$\sum_{|\rho|,|\sigma|=m} a^{\rho\sigma}(x, y)\zeta_\rho\overline{\zeta_\sigma} \geq c \sum_{|\alpha|=m} |\zeta_\alpha|^2 \tag{2.2}$$

for some $c > 0$, all $(x, y) \in \mathbb{R}^d$ and all $\zeta = (\zeta_\alpha)_{|\alpha|=m} \in \mathbb{C}^N$, where $N = \#\{\alpha : |\alpha| = m\}$.

Remark 2.1. This condition, which (for the case of real-valued coefficients) appears, e.g., in [12], is, in general, stronger than the usual strong ellipticity condition

$$\mathrm{Re} \sum_{|\rho|,|\sigma|=m} a^{\rho\sigma}(x, y)\xi^\rho\xi^\sigma \geq c|\xi|^{2m}$$

for all $\xi \in \mathbb{R}^d$ and $(x, y) \in \mathbb{R}^d$. We need this stronger condition, since we want to avoid the assumption of continuity of the leading coefficients $a^{\rho\sigma}$.

Throughout this paper, let $\Omega := \mathbb{R}^{d_1} \times [0, 1]^{d_2}$ denote the periodic cell for our problem. We also introduce a bilinear form acting on Ω. Let

$$B_\Omega[u, v] := \sum_{|\rho|,\ |\sigma|\leq m} \int_\Omega a^{\rho\sigma}(x, y)(D^\rho u)(x, y)\overline{(D^\sigma v)(x, y)}dxdy, \tag{2.3}$$

for $u, v \in H^m(\Omega)$.

Due to condition (2.2) and [1, Theorem 5.2], B_Ω satisfies a Gårding inequality of the form

$$B_\Omega[u, u] \geq c \, |\!|\!|u|\!|\!|^2_{H^m(\Omega)} - C \, \|u\|^2_{L^2(\Omega)} \text{ for all } u \in H^m(\Omega).$$

Since we are studying a spectral problem, we therefore may assume without loss of generality (introducing a shift by C) that B and B_Ω are H^m-elliptic, i.e., there is a $c > 0$ such that $B[u, u] \geq c \, |\!|\!|u|\!|\!|^2_{H^m(\mathbb{R}^d)}$ for all $u \in H^m(\mathbb{R}^d)$ and $B_\Omega[v, v] \geq c \, |\!|\!|v|\!|\!|^2_{H^m(\Omega)}$ for all $v \in H^m(\Omega)$, where $|\!|\!|v|\!|\!|^2_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha v\|^2_{L^2(\Omega)}$. (Note that H^m-ellipticity of B_Ω implies H^m-ellipticity for B due to periodicity of the coefficients.)

This allows us to introduce new scalar products on $H^m(\mathbb{R}^d)$ and $H^m(\Omega)$ given by

$$\langle u, v\rangle_{H^m(\mathbb{R}^d)} := B[u, v] \quad \text{and} \quad \langle u, v\rangle_{H^m(\Omega)} := B_\Omega[u, v]$$

which are equivalent to the usual scalar products in $H^m(\mathbb{R}^d)$ and $H^m(\Omega)$, respectively. By $\|\cdot\|_{H^m(\mathbb{R}^d)}$ and $\|\cdot\|_{H^m(\Omega)}$ we denote the associated norms.

[1] This assumption is made for simplicity; in general, we only require d_2 linearly independent vectors in $\mathbb{R}^{d_2}$.

[2] The authors wish to thank Gerd Grubb and Hans-Christoph Grunau for their related remarks.

Definition 2.2. Let $H^{-m}(\mathbb{R}^d)$ denote the dual space of $H^m(\mathbb{R}^d)$. Let $\phi : H^m(\mathbb{R}^d) \to H^{-m}(\mathbb{R}^d)$ be defined by

$$\langle \phi[u], \varphi\rangle = B[u,\varphi] \quad \text{for all} \;\; u,\varphi \in H^m(\mathbb{R}^d) \tag{2.4}$$

where the $\langle\cdot,\cdot\rangle$-notation indicates the dual pairing, i.e.,

$$\langle w, \varphi\rangle = w[\overline{\varphi}] \text{ for all } w \in H^{-m}(\mathbb{R}^d),\; \varphi \in H^m(\mathbb{R}^d).$$

ϕ is an isometric isomorphism, and hence the scalar product on $H^{-m}(\mathbb{R}^d)$ given by

$$\langle u, v\rangle_{H^{-m}(\mathbb{R}^d)} := \langle \phi^{-1}u, \phi^{-1}v\rangle_{H^m(\mathbb{R}^d)}$$

induces a norm which coincides with the usual operator sup-norm on $H^{-m}(\mathbb{R}^d)$.

Proposition 2.3. *We define an operator* $\mathrm{L} : D(\mathrm{L}) \to H^{-m}(\mathbb{R}^d)$ *by* $D(\mathrm{L}) := H^m(\mathbb{R}^d) \subset H^{-m}(\mathbb{R}^d)$ *and*

$$\mathrm{L}\, u := \phi u.$$

Then L *is self-adjoint.*

Proof. For $u, v \in H^m(\mathbb{R}^d)$,

$$\begin{aligned} \langle \mathrm{L}\, u, v\rangle_{H^{-m}(R^d)} &= \langle \phi^{-1}\,\mathrm{L}\, u, \phi^{-1}v\rangle_{H^m(\mathbb{R}^d)} \\ &= \langle u, \phi^{-1}v\rangle_{H^m(\mathbb{R}^d)} = \overline{\langle \phi^{-1}v, u\rangle_{H^m(\mathbb{R}^d)}} = \overline{B[\phi^{-1}v, u]} \\ &= \overline{\langle v, u\rangle} = \overline{\langle v, u\rangle_{\mathrm{L}^2}} = \langle u, v\rangle_{L^2}; \end{aligned}$$

the last line follows by (2.4). Thus L is symmetric.

Since ϕ is bijective it follows that L is bijective, thus $\mathrm{L}^{-1} : H^{-m}(\mathbb{R}^d) \to H^{-m}(\mathbb{R}^d)$ is defined on the whole space and is also symmetric. Therefore, L^{-1} is self-adjoint. Hence L is self-adjoint. ☐

3. Floquet transform in $H^m(\mathbb{R}^d)$ and $H^{-m}(\mathbb{R}^d)$

In this section, we recall the Floquet transform on $L^2(\mathbb{R}^d)$ and show that its restriction to $H^m(\mathbb{R}^d)$ is an isometric isomorphism between $H^m(\mathbb{R}^d)$ and a suitable Hilbert space $\mathcal{H}$. By a simple duality argument, we extend the Floquet transform to an isometric isomorphism between $H^{-m}(\mathbb{R}^d)$ and $\mathcal{H}'$.

Definition 3.1. For a lattice $R \subset \mathbb{R}^n$ the reciprocal lattice K consists of those points k in $\mathbb{R}^n$ such that

$$e^{ir\cdot k} = 1$$

for all $r \in R$. The first Brillouin zone associated with a lattice R consists of those points in $\mathbb{R}^n$ whose distance to the origin is smaller than or equal to their distance from any other point in the reciprocal lattice.

The Brillouin zone $\mathcal{K} \subset \mathbb{R}^{d_2}$ for the lattice $\mathbb{Z}^{d_2}$, which corresponds to our periodic cell, is $\mathcal{K} := [-\pi, \pi]^{d_2}$.

Definition 3.2. For all $k \in \mathcal{K}$, we now introduce an extension operator $\mathrm{E}_k : L^2(\Omega) \to L^2_{\mathrm{loc}}(\mathbb{R}^d)$ with

$$(\mathrm{E}_k\, u)(x, y + p) := e^{ik\cdot p} u(x, y)$$

for all $(x, y) \in \Omega$, $p \in \mathbb{Z}^{d_2}$.

The Floquet transform

$$\mathrm{U} : L^2(\mathbb{R}^d) \to L^2(\Omega \times \mathcal{K})$$

is given by

$$(\mathrm{U}\, u)(x, y, k) := \frac{1}{(2\pi)^{d_2/2}} \sum_{n\in\mathbb{Z}^{d_2}} e^{ik\cdot n} u(x, y - n) \quad \text{for } (x, y) \in \Omega, k \in \mathcal{K}.$$

We need the following lemma which together with a proof can be found in [10, Theorem 2.2.5].

Lemma 3.3. U *is an isometric isomorphism and*

$$(\mathrm{U}^{-1}\, v)(x, y) = \frac{1}{(2\pi)^{d_2/2}} \int_{\mathcal{K}} (\mathrm{E}_k\, v(\cdot, \cdot, k))(x, y) dk.$$

The following lemma shows that the formula for U has a canonical extension.

Lemma 3.4. *For all* $u \in L^2(\mathbb{R}^d)$, $k \in \mathcal{K}$ *and* $(x, y) \in \mathbb{R}^d$

$$\mathrm{E}_k[\mathrm{U}\, u(\cdot, \cdot, k)](x, y) = \frac{1}{(2\pi)^{\frac{d_2}{2}}} \sum_{n\in\mathbb{Z}^{d_2}} e^{ik\cdot n} u(x, y - n).$$

Proof. It follows from the definition of E_k that, for $p \in \mathbb{Z}^{d_2}$ and $(x, y) \in \Omega$,

$$\begin{aligned} \mathrm{E}_k[\mathrm{U}\, u(\cdot, \cdot, k)](x, y + p) &= e^{ik\cdot p}\, \mathrm{U}\, u(x, y, k) \\ &= \frac{1}{(2\pi)^{\frac{d_2}{2}}} \sum_{n\in\mathbb{Z}^{d_2}} e^{ik\cdot(n+p)} u(x, y + p - (n + p)) \\ &= \frac{1}{(2\pi)^{\frac{d_2}{2}}} \sum_{\tilde{n}\in\mathbb{Z}^{d_2}} e^{ik\cdot\tilde{n}} u(x, y + p - \tilde{n}). \end{aligned}$$

Noting that $(x, y + p)$ runs through $\mathbb{R}^d$ completes the proof. □

Definition 3.5. For all $k \in \mathcal{K}$, let

$$\mathcal{H}_k := \{u \in H^m(\Omega) : \mathrm{E}_k\, u \in H^m_{\mathrm{loc}}(\mathbb{R}^d)\}.$$

Note that being an element of $\mathcal{H}_k$ requires a weak form of semi-periodic boundary conditions on $\partial\Omega$.

We denote by N_k the mapping

$$N_k : \mathcal{H}_0 \to \mathcal{H}_k, \quad (N_k u)(x, y) := e^{ik\cdot y} u(x, y)$$

and extend it to a mapping $\mathcal{H}_0' \to \mathcal{H}_k'$ by

$$\langle N_k u, \varphi\rangle := \langle u, N_k^{-1}\varphi\rangle \text{ for all } u \in \mathcal{H}_0', \varphi \in \mathcal{H}_k.$$

Let

$$\mathcal{H} = \Big\{ u \in L^2(\Omega \times \mathcal{K}) : \forall' k \in \mathcal{K} \quad u(\cdot,\cdot,k) \in \mathcal{H}_k, \\ \left\{ \begin{array}{l} \mathcal{K} \to \mathbb{C} \\ k \mapsto \langle N_k^{-1} u(\cdot,\cdot,k), \varphi \rangle_{H^m(\Omega)} \end{array} \right\} \begin{array}{l} \text{measurable for all } \varphi \in \mathcal{H}_0, \\ \text{and } \|u\|_{\mathcal{H}} < \infty \end{array} \Big\}$$

where the norm $\|\cdot\|_{\mathcal{H}}$ is induced by the scalar product

$$\langle u, v \rangle_{\mathcal{H}} = \int_{\mathcal{K}} \langle u(\cdot,\cdot,k), v(\cdot,\cdot,k) \rangle_{H^m(\Omega)} dk.$$

$\mathcal{H}$ can be viewed as the space of all functions $u(x,y,k) = (N_k v(k))(x,y)$ with $v \in L^2(\mathcal{K}, \mathcal{H}_0)$. $N_k : \mathcal{H}_0 \to \mathcal{H}_k$ is a homeomorphism, with N_k and N_k^{-1} being uniformly bounded with respect to k in the compact set $\mathcal{K}$, which implies in particular that $\mathcal{H}$ is a Hilbert space.

Lemma 3.6. *Let $M \subseteq \mathbb{R}^{d_2}$ be any open bounded set. Then*

a) *the operator $\mathrm{E}_k : L^2(\Omega) \to L^2(\mathbb{R}^{d_1} \times M)$ is bounded,*

b) *the operator $\mathrm{E}_k : \mathcal{H}_k \to H^m(\mathbb{R}^{d_1} \times M)$ is bounded, and $D^\rho(\mathrm{E}_k u) = \mathrm{E}_k(D^\rho u)$, for $u \in \mathcal{H}_k, |\rho| \leq m$.*

c) *for all $k \in \mathcal{K}$, $\mathcal{H}_k \subseteq H^m(\Omega)$ is closed,*

Proof. a) Let $M \subseteq [-l,l]^{d_2}$. Then

$$\int_{\mathbb{R}^{d_1} \times M} |\, \mathrm{E}_k u \,|^2 \, dxdy \leq (2l)^{d_2} \int_{\Omega} |\, \mathrm{E}_k u \,|^2 \, dxdy = (2l)^{d_2} \int_{\Omega} |\, u \,|^2 \, dxdy.$$

b) For all $p \in \mathbb{Z}^{d_2}$, all $\varphi \in C_0^\infty(\Omega + p)$, $u \in \mathcal{H}_k$ and $|\, \rho \,| \leq m$ we have

$$\begin{aligned} \langle D^\rho(\mathrm{E}_k u), \varphi \rangle_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} D^\rho(\mathrm{E}_k u) \overline{\varphi} dxdy \\ &= (-1)^{|\rho|} \int_{\mathbb{R}^d} \mathrm{E}_k u D^\rho \overline{\varphi} dxdy \\ &= (-1)^{|\rho|} \int_{\mathbb{R}^d} e^{ik\cdot p} u(x, y-p) D^\rho \overline{\varphi(x,y)} dxdy \\ &= \int_{\mathbb{R}^d} e^{ik\cdot p} (D^\rho u)(x, y-p) \overline{\varphi(x,y)} dxdy \\ &= \int_{\mathbb{R}^d} \mathrm{E}_k(D^\rho u) \overline{\varphi} dxdy \;=\; \langle \mathrm{E}_k(D^\rho u), \varphi \rangle_{L^2(\mathbb{R}^d)}. \end{aligned}$$

This implies that $D^\rho(\mathrm{E}_k u) = \mathrm{E}_k(D^\rho u)$. Hence, by part a), for all $|\, \rho \,| \leq m$,

$$\|D^\rho(\mathrm{E}_k u)\|_{L^2(\mathbb{R}^{d_1} \times M)} = \|\mathrm{E}_k(D^\rho u)\|_{L^2(\mathbb{R}^{d_1} \times M)} \leq (2l)^{d_2} \|D^\rho u\|_{L^2(\Omega)} .$$

c) Suppose $(u_\mu) \in \mathcal{H}_k^{\mathbb{N}}$ is a sequence with $u_\mu \to u$ in $H^m(\Omega)$ as $\mu \to \infty$. Part b) proves that $(\mathrm{E}_k u_\mu)$ is a Cauchy sequence in $H^m(\mathbb{R}^{d_1} \times M)$ and hence converges to some $w \in H^m(\mathbb{R}^{d_1} \times M)$. On the other hand, $\mathrm{E}_k u_\mu \to \mathrm{E}_k u$ in $L^2(\mathbb{R}^{d_1} \times M)$ by part a). Hence, $\mathrm{E}_k u = w \in H^m(\mathbb{R}^{d_1} \times M)$, which proves $u \in \mathcal{H}_k$. ∎

We are now ready to introduce the Floquet transform on $H^m(\mathbb{R}^d)$.

Theorem 3.7. *Let* V *be given by* $\mathrm{V} := \mathrm{U}|_{H^m(\mathbb{R}^d)}$. *For* $u, v \in H^m(\mathbb{R}^d)$ *we have* $\mathrm{V}\,u, \mathrm{V}\,v \in \mathcal{H}$, *and*

$$\int_{\mathcal{K}} B_\Omega[\mathrm{V}\,u(\cdot,\cdot,k), \mathrm{V}\,v(\cdot,\cdot,k)]dk = B[u,v],$$

that is

$$\langle \mathrm{V}\,u, \mathrm{V}\,v\rangle_{\mathcal{H}} = \langle u, v\rangle_{H^m(\mathbb{R}^d)}, \tag{3.1}$$

i.e., $\mathrm{V} : H^m(\mathbb{R}^d) \to \mathcal{H}$ *is an isometry.*

Proof. Let $u \in H^m(\mathbb{R}^d)$ have compact support. Then

$$\mathrm{E}_k[\mathrm{V}\,u(\cdot,\cdot,k)](x,y) = \frac{1}{(2\pi)^{\frac{d_2}{2}}} \sum_{n\in\mathbb{Z}^{d_2}} e^{ik\cdot n} u(x, y-n) \text{ on } \mathbb{R}^d$$

by Lemma 3.4, and hence $(\mathrm{V}\,u)(\cdot,\cdot,k) \in \mathcal{H}_k$ since the sum is locally finite. Furthermore, for $\varphi \in \mathcal{H}_0$,

$$\langle N_k^{-1}(\mathrm{V}\,u)(\cdot,\cdot,k), \varphi\rangle_{H^m(\Omega)} \frac{1}{(2\pi)^{\frac{d_2}{2}}} \sum_{n\in\mathbb{Z}^{d_2}} e^{ik\cdot n} B_\Omega[N_k^{-1}u(\cdot,\cdot-n),\varphi]$$

is a measurable function of k.

With $v \in H^m(\mathbb{R}^d)$ denoting a second compact support function, we get

$$\begin{aligned}
&\langle (\mathrm{V}\,u)(\cdot,\cdot,k), (\mathrm{V}\,v)(\cdot,\cdot,k)\rangle_{H^m(\Omega)} \\
&\quad = \frac{1}{(2\pi)^{d_2}} B_\Omega\Big[\sum_{n\in\mathbb{Z}^{d_2}} e^{ik\cdot n} u(\cdot,\cdot-n), \sum_{\widetilde{n}\in\mathbb{Z}^{d_2}} e^{ik\cdot\widetilde{n}} v(\cdot,\cdot-\widetilde{n})\Big] \\
&\quad = \frac{1}{(2\pi)^{d_2}} \sum_{n,\widetilde{n}\in\mathbb{Z}^{d_2}} e^{ik\cdot(n-\widetilde{n})} B_\Omega[u(\cdot,\cdot-n), v(\cdot,\cdot-\widetilde{n})].
\end{aligned}$$

Since the sum is finite, this expression is integrable over $\mathcal{K}$ and

$$\int_{\mathcal{K}} \langle (\mathrm{V}\,u)(\cdot,\cdot,k), (\mathrm{V}\,v)(\cdot,\cdot,k)\rangle_{H^m(\Omega)} dk = \sum_{n\in\mathbb{Z}^{d_2}} B_\Omega[u(\cdot,\cdot-n), v(\cdot,\cdot-n)]$$

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^{d_2}} \sum_{|\rho|,|\sigma|\le m} \int_\Omega a^{\rho\sigma}(x,y) D^\rho u(x,y-n)\overline{D^\sigma v(x,y-n)}dxdy \\
&= \sum_{n\in\mathbb{Z}^{d_2}} \sum_{|\rho|,|\sigma|\le m} \int_{\Omega-(0,n)} a^{\rho\sigma}(x,\widetilde{y}+n) D^\rho u(x,\widetilde{y})\overline{D^\sigma v(x,\widetilde{y})}dxd\widetilde{y} \\
&= \sum_{|\rho|,|\sigma|\le m} \int_{\mathbb{R}^d} a^{\rho\sigma}(x,y) D^\rho u(x,y)\overline{D^\sigma v(x,y)}dxdy \\
&= B[u,v] \;=\; \langle u, v\rangle_{H^m(\mathbb{R}^d)},
\end{aligned}$$

which shows that $\mathrm{V}\,u, \mathrm{V}\,v \in \mathcal{H}$, and that (3.1) holds.