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Helmut Hofer
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Symplectic Invariants and Hamiltonian Dynamics

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We dedicate this book to our friend

Andreas Floer

1956–1991

Preface

The discoveries of the past decade have opened new perspectives for the old field of Hamiltonian systems and led to the creation of a new field: symplectic topology. Surprising rigidity phenomena demonstrate that the nature of symplectic mappings is very different from that of volume preserving mappings which raised new questions, many of them still unanswered. On the other hand, due to the analysis of an old variational principle in classical mechanics, global periodic phenomena in Hamiltonian systems have been established. As it turns out, these seemingly different phenomena are mysteriously related. One of the links is a class of symplectic invariants, called symplectic capacities. These invariants are the main theme of this book which grew out of lectures given by the authors at Rutgers University, the RUB Bochum and at the ETH Zürich (1991) and also at the Borel Seminar in Bern 1992. Since the lectures did not require any previous knowledge, only a few and rather elementary topics were selected and proved in detail. Moreover, our selection has been prompted by a single principle: the action principle of mechanics. The action functional for loops in the phase space, given by

$$F(\gamma) \;=\; \int_{\gamma} pdq - \int_{0}^{1} H\Big(t, \gamma(t)\Big)dt \;,$$

differs from the old Hamiltonian principle in the configuration space defined by a Lagrangian. The critical points of F are those loops γ which solve the Hamiltonian equations associated with the Hamiltonian H and hence are the periodic orbits. This variational principle is sometimes called the least action principle. However, there is no minimum for F. Indeed, the action principle is very degenerate. All its critical points are saddle points of infinite Morse index, and at first sight, the principle appears quite useless for existence proofs. But surprisingly it is very effective. This will be demonstrated using several variational techniques starting from minimax arguments due to P. Rabinowitz and ending with A. Floer's homology. The book includes the following subjects:

The introductory chapter presents in a rather unsystematic way some background material. We give the definitions of symplectic manifolds and symplectic mappings and briefly recall the Hamiltonian formalism. For convenience, Cartan's calculus is used. The classification of 2-dimensional symplectic manifolds by the Euler-characteristic and the total volume is proved. Some questions dealt with later on in detail are raised and discussed in special examples. We illustrate the so-called direct method of the calculus of variations in order to establish a periodic orbit on a convex energy surface of a Hamiltonian system in $\mathbb{R}^{2n}$. The Birkhoff invariants are introduced in order to describe without proofs the intricate orbit structure of a Hamiltonian system near an equilibrium point or near a periodic solution. These local results are quite in contrast to the global questions dealt with in the following chapters.

In a systematic way the symplectic invariants, called symplectic capacities, are introduced axiomatically in Chapter 2. Considering the family of all symplectic manifolds of fixed dimension $2n$, a capacity c is a map associating with every symplectic manifold (M, ω) a positive number $c(M, \omega)$ or ∞ satisfying these axioms: a monotonicity axiom for symplectic embeddings, a conformality axiom for the symplectic structure, and a normalization axiom which rules out the volume in higher dimensions. For subsets of $\mathbb{R}^{2n}$, the capacity extends a familiar linear symplectic invariant for positive quadratic forms to nonlinear symplectic mappings. If M and N are symplectically diffeomorphic then $c(M, \omega) = c(N, \tau)$. In view of its monotonicity property a capacity represents, in particular, an obstruction to certain symplectic embeddings and it will be used in order to explain rigidity phenomena for symplectic embeddings, discovered by Ya. Eliashberg and M. Gromov. In particular, Gromov's squeezing theorem is deduced using capacities as well as Eliashberg's C^0-stability of symplectic diffeomorphisms. We introduce a notion of a symplectic homeomorphism, a concept which raises many questions. There are many different capacity functions. For example, the size of the largest ball in $\mathbb{R}^{2n}$ which can be symplectically embedded into a symplectic manifold (M, ω) leads to a special capacity called the Gromov width. It is the smallest capacity function originally introduced by M. Gromov. There are many other "embedding" capacities.

Chapter 3 is devoted entirely to a very detailed construction of a distinguished symplectic capacity c_0. It is dynamically defined by means of Hamiltonian systems. It measures the minimal C^0-oscillation of a Hamiltonian function $H : M \to \mathbb{R}$ which allows to conclude the existence of a fast periodic solution of the corresponding Hamiltonian vector field X_H on M. In the special case of a connected 2-dimensional symplectic manifold, the capacity c_0 agrees with the total area. The existence proof is based on the above action principle which is introduced from scratch in its proper functional analytic framework. The interesting aspect of this principle is that it is bounded neither from below nor from above so that standard variational techniques do not apply directly. Techniques going back to P. Rabinowitz permit us to establish effectively distinguished saddle points of the functional representing special periodic solutions of the system. In the special case of a convex, bounded and smooth domain $U \subset \mathbb{R}^{2n}$, the capacity is represented by a distinguished closed characteristic of its boundary ∂U: it has minimal reduced action equal to $c_0(U)$. But, in general, it is rather difficult to compute the invariant c_0. Some of the recent computations based on more advanced techniques of first order elliptic systems and Fredholm theory are presented without proofs. With the construction of the capacity c_0, the proofs of the rigidity phenomena described in Chapter 2 are complete. Due to its special properties this invariant turns out to be useful also for the dynamics of Hamiltonian systems.

In Chapter 4 the dynamical capacity c_0 is applied to an old question of the qualitative theory of Hamiltonian systems originating in celestial mechanics: does a compact energy surface carry a periodic orbit? We shall demonstrate that many well-known global existence results previously obtained by technically intricate proofs emerge immediately from this invariant. The phenomenon is simply this: if a compact hypersurface in a symplectic manifold possesses a neighborhood of finite capacity c_0, then there are always uncountably many closed characteristics nearby. If one poses, in addition, symplectically invariant restrictions, such as of "contact type", then the hypersurface itself carries a closed characteristic. We shall prove, in particular, the seminal solution of the Weinstein conjecture in $\mathbb{R}^{2n}$ due to C. Viterbo. A nonstandard symplectic torus shows that, in contrast to the Gromov width mentioned above, not every compact symplectic manifold is of finite capacity c_0. Our special example is related to M. Herman's celebrated counterexample to the closing lemma which answers a longstanding open question in dynamical systems. M. Herman's "non-closing-lemma" is proved at the end of the chapter.

In Chapter 5 we study the subgroup $\mathcal{D}$ of symplectic diffeomorphisms of $\mathbb{R}^{2n}$ which are generated by time dependent Hamiltonian vector fields of compact support. The distance from the identity map or the energy $E(\varphi)$ of such a symplectic diffeomorphism φ will be measured by means of the oscillation of its generating Hamiltonian function. This will lead to a surprising bi-invariant metric on $\mathcal{D}$ called the Hofer metric and defined by $d(\varphi, \psi) = E(\varphi^{-1} \circ \psi)$. The definition does not involve derivatives of the Hamiltonian and is of C^0-nature. The verification of the metric property requiring that $d(\varphi, \psi) = 0$ if and only if $\varphi = \psi$ is the difficult aspect. It is based on more refined minimax arguments for the action functional valid simultaneously for a large class of Hamiltonians. We shall investigate the relations of this distinguished metric to the dynamical symplectic invariant c_0 introduced in Chapter 3 and also to another symplectic invariant which is defined for subsets of $\mathbb{R}^{2n}$ and called the displacement energy. The displacement energy of a subset U measures the minimal energy $E(\varphi)$ needed in order to dislocate a given set U from itself in the sense that $U \cap \varphi(U) = \emptyset$. The bi-invariant metric will also be compared with the standard sup-metric. Geodesic arcs associated with the metric will be defined and described in detail. A special example of a geodesic arc is the flow generated by an autonomous Hamiltonian. An important role in our approach is played by the action spectrum of a Hamiltonian mapping $\varphi \in \mathcal{D}$, which turns out to be a nowhere dense subset of the real numbers. Our minimax principle singles out a nontrivial continuous section of the action spectrum bundle over $\mathcal{D}$ called the γ-invariant. This invariant is the main technical tool in this chapter. It allows the characterization of the geodesics and is used also in the existence proof of infinitely many nontrivial periodic points for compactly supported Hamiltonian mappings.

The subject of Chapter 6 is the fixed point theory for Hamiltonian mappings on compact symplectic manifolds (M, ω). It differs from topological fixed point theories. A Hamiltonian map is a special symplectic map: it is homotopic to the identity and the homotopy is generated by the flow of a time dependent Hamiltonian vector field. Prompted by H. Poincaré's last geometric theorem, V.I. Arnold conjectured in the sixties that such a Hamiltonian map possesses at least as many fixed points as a real-valued function on M possesses critical points. Reformulated in terms of dynamical systems, the conjecture asks for a Ljusternik-Schnirelman theory respectively for a Morse theory of forced oscillations solving a time periodic Hamiltonian system on M. We shall first prove the conjecture for the special case of the standard torus T^{2n}. The proof is again based on the action principle. But this time the aim is to find all its critical points. Our strategy is inspired by C. Conley's topological approach to dynamical systems: we shall study the topology of the set of all bounded solutions of the regularized gradient equation belonging to the action functional defined on the set of contractible loops on the manifold M. This way the study of the gradient flow in the infinite dimensional loop space is reduced to the study of a gradient like continuous flow of a compact metric space, whose rest points are the desired critical points. Their number is then estimated by Ljusternik-Schnirelman theory presented in 6.3. A reinterpretation will then lead us to the proof of the Arnold conjecture for the larger class of symplectic manifolds satisfying $[\omega]|\pi_2(M) = 0$. In this general case there is no natural regularization and we are forced to investigate in 6.4 the set of bounded solutions of the non regularized gradient system which now are smooth solutions of a special system of first order elliptic partial differential equations of Cauchy Riemann type. These solutions are related to M. Gromov's pseudoholomorphic curves in M. The compactness of the solution set will be based on an analytical technique which is sometimes called bubbling off analysis. Following this procedure, we shall arrive at the high point of these developments: A. Floer's new approach to Morse theory and Floer homology. We shall merely outline Floer's beautiful ideas in 6.5. A combination of Floer's approach with the construction of the dynamical capacity c_0 results in a symplectic homology theory which is not yet in its final form and which will be sketched without proofs in the last section. The technical requirements of these theories are quite advanced and beyond the scope of this book. Floer's ideas and further related developments will be presented in detail in a sequel. Chapter 6 illustrates, in particular, that old problems emerging from celestial mechanics still lead to powerful new techniques useful also in other branches of mathematics. We should point out that the Arnold conjecture for a general symplectic manifold is still open in the dimensions ≥ 8.

The Appendix contains some technical topics presented for the convenience of the reader. In A.1 we show that a symplectic diffeomorphism can be locally represented in terms of a single function, the so-called generating function. This classical fact is used in Chapter 5. Appendix A.2 illustrates the generating functions in the construction of action-angle coordinates for integrable systems occurring in Chapter 4. A special Sobolev embedding theorem required in the analysis of the action functional (Chapter 5) is proved in A.3. We derive some basic estimates for the Cauchy-Riemann operator on the sphere (A.4), elliptic estimates near the boundary (A.5) and prove the generalized Carleman similarity principle (A.6); all these results for special partial differential equations are important in Chapter 6. While the analytical tools required in the first five chapters are introduced in detail, we make use of topological tools without explanations: we use the Brouwer mapping degree (Chapter 2), the Leray-Schauder degree (Chapter 3), the Smale degree (mod2) and (co-) homology theories (Chapter 6). References concerning these topological topics are given in A.7 and A.8 where we explain the Brouwer degree and the continuity property of the Alexander-Spanier cohomology. This continuity property is important to us for the proof of the Arnold conjecture in the general case.

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E.T.H. Zürich, March 28, 1994.

Chapter 1
Introduction

We shall introduce the concepts of symplectic manifolds, symplectic mappings and Hamiltonian vector fields. It is not the intention to give a systematic treatment of the Hamiltonian formalism, because it is already presented in many books. Rather we shall ask some questions related to these concepts which recently lead to new phenomena and interesting open problems. The question: "What can be done with a symplectic mapping?" leads, for example, to new symplectic invariants different from the volume and discussed in detail in subsequent chapters. We shall illustrate that a seemingly very different and old problem originating in celestial mechanics is related to these invariants. Namely, prompted by the Poincaré recurrence theorem, we ask whether a compact energy surface of a Hamiltonian vector field possesses a periodic orbit. For the very special case of a convex hypersurface in $\mathbb{R}^{2n}$, historically one of the landmarks in this qualitative problem of Hamiltonian systems, we shall give an existence proof in order to illustrate the so-called direct method of the calculus of variation. This classical method is in contrast to the more recent methods introduced in the following chapters in order to establish global periodic solutions. At the end of the introduction we shall illustrate without proofs the rich and intricate orbit structure to be expected near a given periodic orbit. The considerations are based on the local, nonlinear Birkhoff-invariants presented in detail.

1.1 Symplectic vector spaces

Definition. A symplectic vector space (V, ω) is a finite dimensional real vector space V equipped with a distinguished bilinear form ω which is antisymmetric and nondegenerate, i.e.,

$$\omega(u,v) \;=\; -\omega(v,u)\;, \quad u,v \in V \tag{1.1}$$

and, for every $u \neq 0 \in V$, there is a $v \in V$ satisfying $\omega(u,v) \neq 0$. This nondegeneracy is equivalent to the requirement that the map

$$V \to V^*\;, \quad v \mapsto \omega(v,\cdot) \tag{1.2}$$

is a linear isomorphism of V onto its dual vector space V^*. An example is the so-called standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ with

$$\omega_0(u,v) \;=\; \langle Ju, v\rangle \quad \text{for all} \quad u,v \in \mathbb{R}^{2n}, \tag{1.3}$$

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where the bracket denotes the Euclidean inner product in $\mathbb{R}^{2n}$, and where the $2n \times 2n$ matrix J is defined by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.4}$$

with respect to the splitting $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Clearly $\det J \neq 0$ and since $J^T = -J$ the form ω_0 is nondegenerate and antisymmetric. We note that

$$J^T = J^{-1} = -J. \tag{1.5}$$

In particular $J^2 = -1$ and

$$\omega_0(u, Jv) = \langle u, v \rangle.$$

Therefore, J is a complex structure on $\mathbb{R}^{2n}$ compatible with the Euclidean inner product. Recall that a complex structure on a real vector space V is a linear transformation $J : V \to V$ satisfying $J^2 = -1$. It makes V into an n-dimensional complex vector space by defining

$$(\alpha + i\beta)v = \alpha v + \beta J v$$

for $\alpha, \beta \in \mathbb{R}$ and $v \in V$. In the example $(\mathbb{R}^{2n}, \omega_0)$ we may identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way by mapping $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ onto $x + iy \in \mathbb{C}^n$. The linear map J corresponds to the multiplication by $-i$ in $\mathbb{C}^n$.

In the following we shall call v orthogonal to u and write $v \perp u$ if $\omega(v, u) = 0$. If E is a linear subspace of V, we define its orthogonal complement by

$$E^\perp = \left\{ u \in V \,\middle|\, \omega(v, u) = 0 \quad \text{for all} \quad v \in E \right\}. \tag{1.6}$$

$E^\perp$ is a linear subspace and in view of the nondegeneracy of the bilinear form ω, we have

$$\dim E + \dim E^\perp = \dim V. \tag{1.7}$$

Indeed, choosing a basis $e_1, \ldots, e_d$ in E, the subspace $E^\perp$ is the kernel of the linearly independent functionals $\omega(e_j, \cdot)$ on V such that $\dim E^\perp = \dim V - \dim E$ as claimed. Since $u \perp v$ is equivalent to $v \perp u$ we see that

$$\left(E^\perp\right)^\perp = E. \tag{1.8}$$

The concept of orthogonality in symplectic geometry differs sharply from that in Euclidean geometry: E and $E^\perp$ need not be complementary subspaces. For example, every vector $v \in V$ is orthogonal to itself since $\omega(v, v) = -\omega(v, v)$. Hence if $\dim E = 1$ we have $E \subset E^\perp$.

We can, of course, restrict the bilinear form ω to a linear subspace $E \subset V$. This restricted form will obviously be antisymmetric but, in general, fails to be nondegenerate. It is nondegenerate on E if and only if

$$E \cap E^{\perp} = \{0\}, \tag{1.9}$$

which follows immediately from the definitions. In view of (1.7) the statement (1.9) holds precisely if E and $E^{\perp}$ are complementary, i.e.,

$$E \oplus E^{\perp} = V.$$

We see that (E, ω) is a symplectic vector space if (1.9) is satisfied and we call E a symplectic subspace. Because of the symmetry of (1.9) in E and $E^{\perp}$, we conclude that E is symplectic if and only if $E^{\perp}$ is symplectic.

The following proposition shows that every symplectic space looks like the standard space $(\mathbb{R}^{2n}, \omega_0)$.

Proposition 1. The dimension of a symplectic vector space (V, ω) is even. If $\dim V = 2n$ there exists a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V satisfying, for $i, j = 1, 2, \ldots n$,

$$\begin{aligned} \omega(e_i, e_j) &= 0 \\ \omega(f_i, f_j) &= 0 \\ \omega(f_i, e_j) &= \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} . \end{aligned}$$

Such a basis is called a symplectic (or canonical) basis of V. Representing $u, v \in V$ in this basis by

$$\begin{aligned} u &= \sum_{j=1}^{n} \Big(x_j \, e_j + x_{n+j} \, f_j \Big) \\ v &= \sum_{j=1}^{n} \Big(y_j \, e_j + y_{n+j} \, f_j \Big) \end{aligned}$$

one computes readily that

$$\omega(u, v) = \langle Jx, y \rangle , \quad x, y \in \mathbb{R}^{2n} ,$$

where the matrix J is defined by (1.4). The subspaces $V_j = \text{span}\ \{e_j, f_j\}$ are symplectic and orthogonal to each other if $i \neq j$, so that the vector space V is the orthogonal sum

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n \tag{1.10}$$

of 2-dimensional symplectic subspaces. With respect to this splitting the bilinear form ω is, in symplectic coordinates, represented by the matrix

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

Proof of Proposition 1. Choose any vector $e_1 \neq 0$ in V. Since ω is nondegenerate we find $u \in V$ satisfying $\omega(u, e_1) \neq 0$, and we can normalize $f_1 = \alpha u$ such that

$$\omega(f_1, e_1) \;=\; 1.$$

Consequently, f_1 and e_1 are linearly independent since ω is antisymmetric so that $E = \text{ span } \{e_1, f_1\}$ is a 2-dimensional symplectic subspace of V. If $\dim V = 2$ the proof is finished. If $\dim V > 2$ we apply the same argument to the complementary symplectic subspace $E^{\perp}$ of V and thus find the desired basis in finitely many steps. ■

We see that for fixed dimension every symplectic vector space (V, ω) can be put into the same normal form, quite in contrast to the situation of nondegenerate symmetric bilinear forms. The symplectic form ω singles out those linear maps of v which leave the form invariant.

Definition. A linear map $A : V \to V$ of a symplectic vector space (V, ω) is called symplectic (or canonical) if

$$A^*\omega \;=\; \omega.$$

By definition, $A^*\omega$ is the so-called pullback 2-form given by $A^*\omega(u,v) = \omega(Au, Av)$. In the standard space $(\mathbb{R}^{2n}, \omega_0)$ a matrix A is, therefore, symplectic if and only if $\langle JAu, Av\rangle = \langle Ju, v\rangle$ for all $u, v \in \mathbb{R}^{2n}$, or equivalently,

$$A^T\, J\, A \;=\; J. \tag{1.11}$$

In the special case $\mathbb{R}^2$ of two dimensions this is equivalent to the condition that $\det A = 1$. In general we conclude from (1.11) immediately that $(\det A)^2 = 1$. It turns out that

$$\det A \;=\; 1, \tag{1.12}$$

so that symplectic matrices in $\mathbb{R}^{2n}$ are volume-preserving. This requires a proof and it is convenient to use the language of differential forms . Recall that, with the coordinates $z = (z_1, \dots, z_{2n}) \in \mathbb{R}^{2n}$, the bilinear form $dz_i \wedge dz_j$ on $\mathbb{R}^{2n}$ is defined by

$$(dz_i \wedge dz_j)(u, v) \;=\; u_i v_j \;-\; u_j v_i,$$

for $u, v \in \mathbb{R}^{2n}$. Introducing the notation $z = (x, y) \in \mathbb{R}^{2n}$, we can, therefore, represent ω_0 in the form

$$\omega_0 \;=\; \sum_{j=1}^{n} dy_j \wedge dx_j.$$

Then the $2n$-form

$$\Omega \;=\; \omega_0 \wedge \omega_0 \wedge \dots \wedge \omega_0 \;\; (n \text{ times })$$

on $\mathbb{R}^{2n}$ is the volume form

$$\Omega \;=\; c\, dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n$$

with a constant $c \neq 0$. If A is a matrix in $\mathbb{R}^{2n}$ then $A^*\Omega = (\det A)\Omega$ by the definition of a determinant. Assuming that $A^*\omega_0 = \omega_0$ we conclude $A^*\Omega = \Omega$ and, hence, $\det A = 1$ as claimed.

The set of symplectic matrices in $\mathbb{R}^{2n}$, which meet the conditions (1.11), is a group under matrix multiplication. It is one of the classical Lie groups and is denoted by $Sp(n)$.

Proposition 2. If A and $B \in Sp(n)$ then $A^{-1}, AB \in Sp(n)$. Moreover, $A^T \in Sp(n)$ and $J \in Sp(n)$.

Proof. By multiplying $A^T J A = J$ with A^{-1} from the right and with $(A^T)^{-1}$ from the left, we find $J = (A^T)^{-1} J A^{-1}$ so that $A^{-1} \in Sp(n)$. Taking now the inverse on both sides of the latter identity we find $J^{-1} = A J^{-1} A^T$, and since $J^{-1} = -J$ we find $\left(A^T\right)^T J A^T = J$ and $A^T \in Sp(n)$. ■

Note that if a $2n$ by $2n$ matrix is written in block form

$$U \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.13}$$

with respect to the splitting $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, it is symplectic if and only if

$$A^T C\,,\; B^T D \quad \text{are symmetric and} \quad A^T D - C^T B = 1,$$

as is readily verified. For example, a matrix U having $B = 0$ is symplectic if and only if A is nonsingular and U can be written as

$$U \;=\; \begin{pmatrix} A & 0 \\ 0 & \left(A^T\right)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ S & 1 \end{pmatrix},$$

with some symmetric matrix S.

Definition. If (V_1, ω_1) and (V_2, ω_2) are two symplectic vector spaces we call a linear map $A : V_1 \to V_2$ symplectic if

$$A^* \omega_2 = \omega_1,$$

where, by definition, $(A^*\omega_2)(u, v) = \omega_2(Au, Av)$ for all $u, v \in V_1$. Clearly A is injective such that $\dim V_1 \leq \dim V_2$.

Proposition 3. If (V_1, ω_1) and (V_2, ω_2) are two symplectic spaces of the same dimension, then there exists a linear isomorphism $A : V_1 \to V_2$ satisfying $A^*\omega_2 = \omega_1$.

This means that all symplectic vector spaces of the same dimension are, in this sense, equivalent; they are symplectically indistinguishable.

Proof. The proof follows immediately from the normal form in Proposition 1. Choosing symplectic bases (e_j, f_j) in (V_1, ω_1) and $(\hat{e}_j, \hat{f}_j)$ in (V_2, ω_2) we define the linear map $A : V_1 \to V_2$ by

$$A\, e_j = \hat{e}_j \quad \text{and} \quad A\, f_j = \hat{f}_j$$

for $1 \leq j \leq n$. Then clearly $A^*\omega_2 = \omega_1$ by definition of a symplectic basis. ∎

Since the choice of e_1 and $\hat{e}_1$ in the construction of the symplectic bases is at our disposal we conclude from the above proof that the group $Sp(n)$ acts transitively in $\mathbb{R}^{2n}$. Moreover, it also acts transitively on the set of symplectic subspaces of $\mathbb{R}^{2n}$ having the same dimension. This follows because a symplectic basis in a subspace E can always be completed to a basis of V by adding a symplectic basis of its complement $E^{\perp}$, as we did in the proof of Proposition 1.

1.2 Symplectic diffeomorphisms and Hamiltonian vector fields in $(\mathbb{R}^{2n}, \omega_0)$

We now turn to nonlinear maps in the symplectic space $(\mathbb{R}^{2n}, \omega_0)$. A diffeomorphism φ in $\mathbb{R}^{2n}$ is called symplectic if

$$\varphi^* \omega_0 = \omega_0, \tag{1.14}$$

where, by definition, the pullback of any 2-form ω is given by

$$(\varphi^*\omega)_x(a, b) = \omega_{\varphi(x)}(\varphi'(x)a,\ \varphi'(x)b)$$

for $x \in \mathbb{R}^{2n}$ and for all $a, b \in T_x\mathbb{R}^{2n} = \mathbb{R}^{2n}$. Here $\varphi'(x)$ denotes the derivative of φ at the point x represented by the Jacobian matrix. In view of the definition of ω_0, a symplectic diffeomorphism in $(\mathbb{R}^{2n}, \omega_0)$ is, therefore, characterized by the identity

$$\varphi'(x)^T J\, \varphi'(x) = J\,, \quad x \in \mathbb{R}^{2n} \tag{1.15}$$

for the first derivatives of φ. Hence $\varphi'(x)$ is a symplectic matrix and, in particular,

$$\det \varphi'(x) \;=\; 1, \tag{1.16}$$

so that symplectic diffeomorphisms are volume-preserving. However, if $n > 1$ the class of symplectic diffeomorphisms is much more restricted than that of volume-preserving diffeomorphisms. This will become clear below when, taking our lead from Gromov, we look at the question: what can be done with symplectic diffeomorphisms?

A symplectic diffeomorphism φ in $(\mathbb{R}^{2n}, \omega_0)$ not only preserves ω_0 and the associated volume form Ω but also the action of closed curves, as we shall see next. The form ω_0 is an exact form, since

$$\omega_0 \;=\; \sum_{j=1}^{n} dy_j \wedge dx_j \;=\; d\lambda, \tag{1.17}$$

with the 1-form λ defined by

$$\lambda \;=\; \sum_{j=1}^{n} y_j \, dx_j .$$

Therefore, $\lambda - \varphi^*\lambda$ is a closed form provided φ is symplectic. Indeed, $d(\lambda - \varphi^*\lambda) = d\lambda - d(\varphi^*\lambda) = d\lambda - \varphi^* d\lambda = \omega_0 - \varphi^*\omega_0 = 0$. Using the Poincaré lemma one finds a function $F : \mathbb{R}^{2n} \to \mathbb{R}$ satisfying

$$\lambda \,-\, \varphi^*\lambda \;=\; dF. \tag{1.18}$$

If γ is an oriented simply closed curve we can integrate and find in view of (1.18)

$$\int_\gamma \lambda \;=\; \int_\gamma \varphi^*\lambda \;=\; \int_{\varphi(\gamma)} \lambda$$

since the integral of an exact form over a closed curve vanishes. Defining the action $A(\gamma)$ of a closed curve γ by

$$A(\gamma) \;=\; \int_\gamma \lambda \,\in \mathbb{R}, \tag{1.19}$$

we see that

$$A\Big(\varphi(\gamma)\Big) \;=\; A(\gamma) \tag{1.20}$$

provided φ is symplectic; hence φ leaves the action invariant as claimed. Conversely, of course, if a diffeomorphism φ in $\mathbb{R}^{2n}$ satisfies (1.20) for all closed curves in $\mathbb{R}^{2n}$ we conclude that φ is symplectic. Parameterizing γ by $x(t)$, with $0 \leq t \leq 1$ and $x(0) = x(1)$, the action becomes

$$A(\gamma) = \frac{1}{2} \int_0^1 \langle -J\dot{x}, x \rangle \, dt. \tag{1.21}$$

Examples of symplectic diffeomorphisms are generated by the so-called Hamiltonian vector fields which we now recall. To the symplectic form ω_0 and to a smooth function

$$H : \mathbb{R}^{2n} \to \mathbb{R},$$

we can associate a vector field X_H on $\mathbb{R}^{2n}$ by requiring

$$\omega_0\Big(X_H(x), a\Big) = -dH(x)a \tag{1.22}$$

for all $a \in \mathbb{R}^{2n}$ and $x \in \mathbb{R}^{2n}$. Since ω_0 is nondegenerate, the vector $X_H(x)$ is determined uniquely. The condition (1.22) is equivalent to $\langle JX_H(x), a\rangle = -\langle \nabla H(x), a\rangle$ where the gradient of H is, as usual, defined with respect to the Euclidean inner product. Therefore, $JX_H(x) = -\nabla H(x)$ and in view of $J^2 = -1$ we find the representation

$$X_H(x) = J\nabla H(x)\,, \quad x \in \mathbb{R}^{2n}. \tag{1.23}$$

Clearly the Hamiltonian vector fields are very special. They differ in particular sharply from vector fields $X = \nabla H(x)$ of gradient type, since J is antisymmetric.

In the following we denote by φ^t the flow of a vector field X. It is defined by

$$\begin{aligned} \frac{d}{dt}\varphi^t(x) &= X\Big(\varphi^t(x)\Big) \\ \varphi^0(x) &= x\,, \quad x \in \mathbb{R}^{2n}\,. \end{aligned}$$

The curve $x(t) = \varphi^t(x)$ solves the Cauchy initial value problem for the initial condition $x \in \mathbb{R}^{2n}$. Assume now that $X = X_H$ is the Hamiltonian vector field determined by ω_0 and H. Then every flow map φ^t preserves the form ω_0:

$$\left(\varphi^t\right)^* \omega_0 = \omega_0, \tag{1.24}$$

and is, therefore, a symplectic map. This is easily verified and will be proved in the next section in a more abstract setting.

It is useful for the following to recall the transformation formula for vector fields X on $\mathbb{R}^m$. Assume $x(t)$ is a solution of the differential equation

$$\dot{x} = X(x)\,, \quad x \in \mathbb{R}^m.$$

If $u : \mathbb{R}^m \to \mathbb{R}^m$ is a diffeomorphism we can define the curve $y(t)$ by

$$x(t) \;=\; u\Big(y(t)\Big).$$

Differentiating in t we conclude that $y(t)$ solves the equation

$$\dot{y} \;=\; Y(y)\;, \quad y \in \mathbb{R}^m$$

for the transformed vector field Y defined by

$$Y(y) \;=\; du(y)^{-1} \cdot X \circ u(y).$$

In the following, we shall use the notation

$$u^*X :=\; (du)^{-1} \cdot X \circ u. \tag{1.25}$$

We have demonstrated that the two flows φ^t of X and ψ^t of u^*X are conjugated by the diffeomorphism u, i.e.,

$$\varphi^t \circ u \;=\; u \circ \psi^t\;.$$

If we subject a Hamiltonian vector field X_H in $\mathbb{R}^{2n}$ to an arbitrary transformation u its special form will be destroyed. However, a symplectic transformation preserves the class of Hamiltonian vector fields. Indeed, if $u^*\omega_0 = \omega_0$ then

$$u^*X_H \;=\; X_K \quad \text{and} \quad K \;=\; H \circ u. \tag{1.26}$$

This is easily verified: defining the function K as the composition $K = H \circ u$, then by the chain rule, $dK = dH \circ u \cdot du$, and the gradient with respect to the Euclidean scalar product becomes $\nabla K = (du)^T \nabla H \circ u$. By assumption, du is, at every point, a symplectic map and, therefore, also $(du)^T$ so that $du \cdot J \cdot (du)^T = J$. Consequently, in view of the definition (1.23) of a Hamiltonian vector field

$$\begin{aligned} X_K \;&=\; J\nabla K \;=\; J(du)^T\,\nabla H \circ u \\ &=\; (du)^{-1}(J\nabla H) \circ u \\ &=\; u^*X_H, \end{aligned}$$

as we set out to prove.

1.3 Hamiltonian vector fields and symplectic manifolds

In order to introduce Hamiltonian vector fields on a manifold, we first have to extend the symplectic structure

$$\omega_0 \;=\; \sum_{j=1}^{n} dy_j \wedge dx_j \quad \text{on} \quad \mathbb{R}^{2n}$$

to even dimensional manifolds.

Definition. A symplectic structure on an even dimensional manifold M is a 2-form ω on M satisfying

(i) $d\omega = 0$, i.e., ω is a closed form.

(ii) ω is nondegenerate.

The second condition requires that for every tangent space T_xM: if $\omega_x(u,v) = 0$ for all $v \in T_xM$ then $u = 0$. The pair (M, ω) is then called a symplectic manifold. Thus every tangent space T_xM of a symplectic manifold becomes a symplectic vector space with respect to the distinguished antisymmetric and nondegenerate bilinear form ω_x at x. Therefore, M has even dimension.

An example is the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$; indeed, since ω_0 is a constant form we have $d\omega_0 = 0$. Since the symplectic form ω is assumed to be closed, every symplectic manifold looks, locally, like $(\mathbb{R}^{2n}, \omega_0)$; we shall now prove that there are always local coordinates in which the symplectic form is represented by the constant form ω_0.

Theorem 1. (Darboux) Suppose ω is a nondegenerate 2-form on a manifold of $\dim M = 2n$. Then $d\omega = 0$ if and only if at each point $p \in M$ there are coordinates (U, φ) where $\varphi : (x_1, \ldots, x_n, y_1, \ldots, y_n) \to q \in U \subset M$ satisfies $\varphi(0) = p$ and

$$\varphi^*\omega \;=\; \omega_0 \;=\; \sum_{j=1}^{n} dy_j \wedge dx_j .$$

Such coordinates are sometimes called symplectic coordinates. They are clearly not determined uniquely; the most general coordinates of this sort are related to (x, y) by symplectic transformation $u^*\omega_0 = \omega_0$ in $\mathbb{R}^{2n}$, as previously introduced. We see that we can define a symplectic manifold alternatively as follows: it is a manifold of $\dim M = 2n$ for which there are local coordinates φ_j mapping open sets $U_j \subset M$ onto open sets of the fixed symplectic space $(\mathbb{R}^{2n}, \omega_0)$ such that the coordinates changes $\varphi_i \circ \varphi_j^{-1}$ defined on $\varphi_j(U_i \cap U_j)$ are symplectic local diffeomorphisms in $(\mathbb{R}^{2n}, \omega_0)$.

Proof. Choosing any local coordinates, we may assume that ω is a 2-form on $\mathbb{R}^{2n}$ depending on $z \in \mathbb{R}^{2n}$ and that p corresponds to $z = 0$. By a linear change of coordinates we can achieve that the form be in normal form at the origin, i.e.,

$$\omega(0) \;=\; \sum_{j=1}^{n} dy_j \wedge dx_j \quad \text{at} \quad z = 0.$$

This is precisely the same as the statement that any nondegenerate antisymmetric bilinear form can be brought into normal form (Proposition 1). With ω_0 we shall denote the constant form $\Sigma dy_j \wedge dx_j$ on $\mathbb{R}^{2n}$. The aim is to find a local diffeomorphism φ in a neighborhood of 0 leaving the origin fixed and solving

$$\varphi^*\omega \;=\; \omega_0 .$$

We shall solve this equation by a deformation argument. We interpolate ω and ω_0 by a family ω_t of forms defined by

$$\omega_t \;=\; \omega_0 \;+\; t(\omega - \omega_0)\,, \quad 0 \le t \le 1,$$

such that $\omega_t = \omega_0$ for $t = 0$ and $\omega_1 = \omega$, and look for a whole family φ^t of diffeomorphisms satisfying $\varphi^0 = id$ and

$$(\varphi^t)^*\omega_t \;=\; \omega_0\,, \quad 0 \le t \le 1. \tag{1.27}$$

The diffeomorphism φ^t for $t = 1$ will then be the solution to our problem. In order to find φ^t we shall construct a t-dependent vector field X_t generating φ^t as its flow. Differentiating (1.27), such a vector field X_t has to satisfy the identity

$$0 \;=\; \frac{d}{dt}(\varphi^t)^*\,\omega_t \;=\; (\varphi^t)^*\left\{L_{X_t}\omega_t \;+\; \frac{d}{dt}\,\omega_t\right\}. \tag{1.28}$$

Here L_Y denotes the Lie derivative of the vector field Y. Now we use Cartan's identity

$$L_X \;=\; i_X \circ d \;+\; d \circ i_X \tag{1.29}$$

and the assumption that $d\omega_t = 0$ and find

$$0 \;=\; (\varphi^t)^*\Big\{d(i_{X_t}\omega_t) \;+\; \omega - \omega_0\Big\}.$$

Hence, X_t has to satisfy the linear equation

$$d(i_{X_t}\omega_t) \;+\; \omega - \omega_0 \;=\; 0. \tag{1.30}$$

In order to solve this equation we observe that $\omega - \omega_0$ is closed, hence, locally exact by the Poincaré lemma and there is a 1-form λ satisfying

$$\omega - \omega_0 \;=\; d\lambda \quad \text{and} \quad \lambda(0) \;=\; 0.$$

Since $\omega_t(0) = \omega_0$ the 2-forms ω_t are nondegenerate for $0 \le t \le 1$ in an open neighborhood of the origin and hence there is a unique vector field X_t determined by

$$i_{X_t}\omega_t \;=\; \omega_t(X_t, \cdot) \;=\; -\lambda$$

for $0 \le t \le 1$ which then solves the equation (1.30). Since we normalized $\lambda(0) = 0$ we have $X_t(0) = 0$ and there is an open neighborhood of the origin on which the flow φ^t of X_t exists for all $0 \le t \le 1$. It satisfies $\varphi^0 = id$ and $\varphi^t(0) = 0$. We can follow our arguments backwards: by construction this family φ^t of diffeomorphisms satisfies

$$\frac{d}{dt}(\varphi^t)^*\omega_t \;=\; 0\,, \quad 0 \le t \le 1,$$

hence $(\varphi^t)^*\omega_t = (\varphi^0)^*\omega_0 = \omega_0$ for all $0 \le t \le 1$, as we wanted to prove. ■

The method employed in the proof above is the so-called deformation method of J. Moser. Its unusual aspect is that one searches for a differential equation to be solved. J. Moser introduced the method in [160] in order to prove, in particular, that two symplectic structures ω_0 and ω_1 on a compact manifold M are equivalent, in the sense that

$$\varphi^*\omega_1 = \omega_0$$

for a diffeomorphism φ of M, provided the forms can be deformed into each other within the class of symplectic forms having their periods fixed. Incidentally, the classification of symplectic forms up to equivalence is still open.

From Darboux's normal form we conclude that any two symplectic manifolds having the same dimension are locally indistinguishable: symplectic manifolds do not possess any local symplectic invariants other than the dimension. This is in sharp contrast to Riemannian manifolds: two different metrics generally are not locally isometric, e.g., the Gaussian curvature is an invariant. It is our aim later on to construct global symplectic invariants.

Every manifold M carries a Riemannian structure. In contrast, not every even dimensional manifold admits a symplectic structure. For example, spheres S^{2n} do not admit a symplectic structure if $n \geq 2$. Indeed, arguing by contradiction we assume ω is a symplectic structure. Then $\Omega = \omega \wedge \omega \wedge \ldots \wedge \omega$ (n times) is a volume form, since ω is nondegenerate. But $\omega = d\alpha$ for a 1-form α on S^{2n} since the second de Rham cohomology group vanishes: $H^2(S^{2n}) = 0$. Therefore, $\Omega = d\beta$ with $\beta = \omega \wedge \omega \wedge \ldots \wedge \omega \wedge \alpha$ and by Stokes' theorem

$$\int_{S^{2n}} \Omega = \int_{\partial S^{2n}} \beta = 0$$

which is, of course, not possible for a volume form. This argument evidently applies to any compact manifold M without boundary having $H^{2j}(M) = 0$ for some $1 \leq j \leq n-1$.

Next we introduce the analogue of symplectic maps in $(\mathbb{R}^{2n}, \omega_0)$. A differentiable map $f : M_1 \to M_2$ between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is called symplectic if

$$f^*\omega_2 = \omega_1,$$

where, by definition of the pullback of a 2-form ω

$$(f^*\omega)_x(u, v) = \omega_{f(x)}(df(x)u, df(x)v) \text{ for all } u,\ v \in T_xM.$$

Since ω_1 is nondegenerate the tangent map $df(x)$ must be injective at every point and hence $\dim M_1 \leq \dim M_2$. If $\dim M_1 = \dim M_2$ then f is a local diffeomorphism. In the case that f maps a symplectic manifold (M, ω) into itself the condition for f to be symplectic becomes

$$f^*\omega = \omega,$$

i.e., f preserves the symplectic structure. Expressed in the distinguished local symplectic coordinates defined by Darboux's theorem, this condition for f agrees with our previous condition for a map to be symplectic in $(\mathbb{R}^{2n}, \omega_0)$. It is useful to point out that locally such a symplectic map can be presented in terms of a single function on $\mathbb{R}^{2n}$, a so-called generating function and we refer to the Appendix for details.

The symplectic structure, being nondegenerate, defines an isomorphism between vector fields X and 1-forms on M given by $X \mapsto \omega(X, \cdot)$. In particular, if

$$H : M \to \mathbb{R}$$

is a smooth function on M, then dH is a 1-form on M and hence together with ω determines the vector field X_H by

$$\text{(1.31)} \qquad \Big(i_{X_H}\omega\Big)(x) \;=\; \omega\Big(X_H(x), \cdot\Big) \;=\; -dH(x),$$

$x \in M$. This distinguished vector field X_H is called the Hamiltonian vector field belonging to the function H. Since $d\omega = 0$ we deduce from (1.31) using Cartan's formula $L_X = di_X + i_X d$ and $ddH = 0$ that

$$\text{(1.32)} \qquad L_{X_H}\omega \;=\; 0.$$

We conclude that the maps φ^t belonging to the flow of a Hamiltonian vector field X_H leave the symplectic form invariant,

$$\text{(1.33)} \qquad (\varphi^t)^*\omega \;=\; \omega,$$

hence are symplectic. Indeed, the derivative $\frac{d}{dt}(\varphi^t)^*\omega = (\varphi^t)^* L_X\omega = 0$ vanishes in view of (1.32) and since $(\varphi^0)^*\omega = \omega$ the claim follows. The set of Hamiltonian vector fields is invariant under symplectic transformations as we shall verify next. Recall that $u^*X = (du)^{-1}X \circ u$ for a vector field X and a diffeomorphism u, and, equivalently, $\varphi^t \circ u = u \circ \psi^t$ for the associated flows φ^t of X and ψ^t of u^*X.

Proposition 4. If $u : M \to M$ satisfies $u^*\omega = \omega$ then for every function $H : M \to \mathbb{R}$

$$u^*X_H \;=\; X_K \quad \text{and} \quad K = H \circ u.$$

Proof. In view of the definition of a Hamiltonian vector field

$$\begin{aligned} i_{X_{H\circ u}}\omega &= -d(H \circ u) &&= -u^*(dH) \\ &= u^*(i_{X_H}\omega) &&= i_{u^*X_H}(u^*\omega) \\ &= i_{u^*X_H}\omega \end{aligned}$$

and since ω is nondegenerate the vector fields $X_{H\circ u}$ and u^*X_H must be equal. ■

From the symplectic structure we shall deduce an auxiliary structure which will be convenient later on. Recall that an almost complex structure on a manifold M associates smoothly with every $x \in M$ a linear map $J = J_x : T_xM \to T_xM$ satisfying $J^2 = -1$.

Proposition 5. If (M, ω) is a symplectic manifold there exists an almost complex structure J on M and a Riemannian metric $\langle \cdot, \cdot \rangle$ on M satisfying

$$\omega_x(v, Ju) \;=\; \langle v, u \rangle_x \tag{1.34}$$

for $v, u \in T_xM$. From the symmetry of the bilinear form $\langle \cdot, \cdot \rangle$ it follows that

$$\omega_x(Jv, Ju) \;=\; \omega_x(v, u), \tag{1.35}$$

i.e., J is a symplectic map of the symplectic vector space (T_xM, ω_x). Moreover,

$$J^* \;=\; J^{-1} \;=\; -J\;, \tag{1.36}$$

where J^* is the adjoint of J in the inner product space $(T_xM, \langle \cdot, \cdot \rangle_x)$.

Proof. We choose any Riemannian metric g on M. Fixing a point $x \in M$ we shall construct $J = J_x$ in T_xM. All the constructions will depend smoothly on x and, for notational convenience, the dependence on x will not be explicitly mentioned. Since ω is nondegenerate there exists a unique linear isomorphism $A : T_xM \to T_xM$ satisfying

$$\omega(u, v) \;=\; g(Au, v)\,, \quad u, v \in T_xM.$$

Since ω is antisymmetric we infer $g(Au, v) = \omega(u, v) = -\omega(v, u) = -g(Av, u) = -g(v, A^*u) = g(-A^*u, v)$, where A^* is the g-adjoint map of A. Hence

$$A^* \;=\; -A.$$

Consequently, $A^*A = AA^* = -A^2$ is a positive definite g-self-adjoint map and we denote by $Q = \sqrt{-A^2}$ the positive square root of $-A^2$. Set

$$J \;=\; AQ^{-1}.$$

Since A and A^* do commute, A is a normal operator and consequently A and Q^{-1} commute and we compute

$$J^2 \;=\; AQ^{-1}AQ^{-1} \;=\; A^2(-A^2)^{-1} \;=\; -id.$$

Finally,

$$\begin{aligned} \omega(u, Jv) \;&=\; g(Au, Jv) \;=\; g(Au, AQ^{-1}v) \\ &=\; g(A^*Au, Q^{-1}v) \;=\; g(Q^2u, Q^{-1}v) \\ &=\; g(Qu, v). \end{aligned}$$

Since Q is symmetric and positive definite we conclude that

$$\langle u, v \rangle :=\; g(Qu, v)$$

defines a Riemannian metric on M which, in general, is different from g. It is the desired metric. The remainder of the statement is now readily verified making use of the fact that the metric $\langle u, v\rangle = \langle v, u\rangle$ is symmetric. Since the construction depends smoothly on x the proof is completed. ■

This almost complex structure compatible with ω extends the complex structure in $(\mathbb{R}^{2n}, \omega_0)$ considered above. Moreover, if ∇H denotes the gradient of a function H with respect to the Riemannian metric $\langle\cdot,\cdot\rangle$ of Proposition 5, i.e., $\langle \nabla H(x), v\rangle = dH(x)v$ for all $v \in T_xM$, we find for the Hamiltonian vector field X_H the representation

$$X_H(x) \;=\; J\nabla H(x) \in T_xM \tag{1.37}$$

using that $J^2 = -1$. This agrees with the representation of X_H in $(\mathbb{R}^{2n}, \omega_0)$.

We should point out that the almost complex structure is not unique. If we denote by $\mathcal{J}_\omega$ the set of almost complex structures compatible with ω in the sense of (1.34), it can easily be shown that this set is contractible. Indeed, for every $J \in \mathcal{J}_\omega$ there exists, by definition, a unique Riemannian metric g_J satisfying $\omega(u, Jv) = g_J(u, v)$. Starting from any Riemannian metric g, we constructed in the proof of the proposition an almost complex structure $J = J_g$ and a metric g_J such that $J_{g_J} = J$. Hence, fixing any metric g^* on M, we can define the contraction in $\mathcal{J}_\omega$ by

$$(t, J) \mapsto J_{(1-t)g_J + tg^*}$$

for $0 \le t \le 1$ and $J \in \mathcal{J}_\omega$.

In view of Darboux's theorem there are locally no symplectic invariants other than the dimension. On the other hand, the total volume is a trivial example of a global symplectic invariant. Indeed, if $u : (M_1, \omega_1) \to (M_2, \omega_2)$ is a symplectic diffeomorphism of M_1 onto M_2 then it follows from $u^*\omega_2 = \omega_1$ that the associated volume forms $\Omega_1 = \omega_1 \wedge \ldots \wedge \omega_1$ (n times) on M_1 and similarly Ω_2 on M_2 are related by

$$u^*\Omega_2 \;=\; \Omega_1. \tag{1.38}$$

Since the diffeomorphism $u : M_1 \to M_2$ preserves the orientation we have

$$\int_{M_1} u^*\Omega_2 \;=\; \int_{M_2} \Omega_2$$

and in view of (1.38),

$$\int_{M_1} \Omega_1 \;=\; \int_{M_2} \Omega_2,$$

so that the total volumes of Ω_1 and Ω_2 have to agree. Consider now the special case of compact, connected and oriented manifolds of dimension 2, i.e., surfaces. The orientation will be given by a volume form denoted by ω. It evidently is a closed form, since every 3-form on a surface vanishes. Therefore, (M, ω) is a symplectic manifold with the volume form as the symplectic structure.