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Pseudo-Differential Operators: Analysis, Applications and Computations

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Preface

The ISAAC Group in Pseudo-Differential Operators (IGPDO) met again on July 13–18, 2009 at Imperial College London in England on the occasion of the Seventh Congress of the International Society for Analysis, its Applications and Computations (ISAAC). The special session for IGPDO turned out to be the largest session with over forty speakers filling out the entire schedule completely. Talks presented at the IGPDO session reflected the diversity of topics cutting across disciplines in the Analysis, Applications and Computations of Pseudo-Differential Operators.

This volume contains eighteen peer-reviewed papers related to the talks given at the IGPDO session. Chapters 1–3 feature a chapter on the adaptive wavelet computations of inverses of pseudo-differential operators (Q. Guo and M.W. Wong) and two chapters on the pseudo-differential operators on the unit circle (M. Pirhayati; S. Molahajloo). The latter two chapters pave the way towards the discretization and numerical computations of pseudo-differential operators. Chapters 4–7 are on pseudo-differential operators and boundary value problems on manifolds with singularities, non-smooth domains and Riemannian manifolds (B.-W. Schulze and M.W. Wong; B.-W. Schulze; V.B. Vasilyev; C. Iwasaki). Chapters 8–11 are devoted to concrete partial differential equations that are of interest in physics and geometry (J. Delgado; V. Catană; V.S. Rabinovich; R. DeLeo, T. Gramchev and A. Kirilov). Chapters 12–14 consist of chapters on microlocal analysis, hyperbolic equations and systems (Y. Chiba; W. Ichinose; K. Benmeriem and C. Bouzar). Chapters 15–18 are on topics related to Wigner transforms, Weyl transforms and localization operators (L. Cohen; L. Galleani; P. Boggiatto, E. Carypis and A. Oliaro; E. Cordero and F. Nicola).

In an era of interdisciplinary studies in academia fuelled by research and development for societal and global needs, the role of pseudo-differential operators in the mathematical, physical, biological, atmospherical, geological and medical sciences is vital. Underpinning novel *applications* are deep understanding in the *analysis* and efficient numerical *computations*. It is expected that new developments in Analysis, Applications and Computations of Pseudo-Differential Operators will deepen our understanding of science in general and hence improve the knowledge-based well-being of the world. Future developments of IGPDO are geared in the direction of interdisciplinarity.

Adaptive Wavelet Computations for Inverses of Pseudo-Differential Operators

Qiang Guo and M.W. Wong

Abstract. For invertible pseudo-differential operators T_σ with symbols σ in S^m, $m \in \mathbb{R}$, we use biorthogonal wavelets to develop an adaptive algorithm to compute the Galerkin approximations of the solution u in the Sobolev space $H^{m,2}$ of the equation $T_\sigma u = f$ on $\mathbb{R}$ for every f in $L^2(\mathbb{R})$.

Mathematics Subject Classification (2000). 47G30, 65M60, 65T10.

Keywords. Multiresolution analysis, scaling functions, wavelets, biorthogonal wavelets, vanishing moments, Sobolev spaces, pseudo-differential operators, Galerkin approximations, adaptive algorithms.

1. Introduction

Wavelet methods are relatively recent developments with applications in pure and applied mathematics [1, 5]. Due to the localization properties that wavelets display both in space and frequency, the wavelet multiresolution analysis allows us to obtain an efficient sparse representation of a function, especially when the function exhibits singular behavior and large wavelet coefficients are near the singularity. Wavelet methods can distinguish smooth and singular regions automatically and hence lead to adaptive techniques based on multilevel methods. Reliable and efficient a posteriori error estimators have been derived for adaptive wavelet Galerkin schemes for elliptic partial differential equations, which are based on stable multiscale biorthogonal wavelet bases in, e.g., [3]. The developed adaptive refinement strategy guarantees an improvement for the approximate solution after the refinement step. We extend the adaptive strategy developed in [3] to compute inverses of pseudo-differential operators.

The paper is organized as follows. In Section 2, we first recall the wavelet multiresolution analysis and describe the properties of biorthogonal wavelets. Then

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the basics of pseudo-differential operators required for this paper are recalled. Residual estimates are given in Section 3 and a posteriori error estimates are derived in Section 4. Finally, an adaptive algorithm for the computations of inverses of nonsymmetric pseudo-differential operators is presented in Section 5.

2. Wavelets and pseudo-differential operators

A multiresolution analysis (MRA) is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}_0}$ of $L^2(\mathbb{R})$ such that

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z},$$
$$\cap_{j\in\mathbb{Z}} V_j = \{0\},$$
$$\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}),$$
$$f \in V_j \Leftrightarrow D_2 f \in V_{j+1}, \quad j \in \mathbb{Z},$$

and

$$f \in V_0 \Leftrightarrow T_{-k} f \in V_0, \quad k \in \mathbb{Z},$$

where D_2 and T_{-k} are the dilation and the translation given, respectively, by

$$(D_2 g)(x) = g(2x), \quad x \in \mathbb{R},$$

and

$$(T_{-k} g)(x) = g(x-k), \quad x \in \mathbb{R},$$

for all measurable functions g on $\mathbb{R}$.

Let $\varphi \in L^2(\mathbb{R})$. Then we consider the translations and dilations $\varphi_{j,k}$ of φ given by

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k), \quad x \in \mathbb{R},$$

for $j,k \in \mathbb{Z}$. If for each fixed $j \in \mathbb{Z}$, the sequence $\{\varphi_{j,k} : k \in \mathbb{Z}\}$ is an orthonormal sequence for V_j such that the sequence is uniformly stable in the sense that

$$\left\| \sum_{k\in\mathbb{Z}} c_{j,k}\varphi_{j,k} \right\|_2 \sim \left(\sum_{k\in\mathbb{Z}} |c_{j,k}|^2 \right)^{1/2}$$

uniformly with respect to j in $\mathbb{Z}$, i.e., there exist positive constants C and C' such that

$$C\left(\sum_{k\in\mathbb{Z}} |c_{j,k}|^2 \right)^{1/2} \le \left\| \sum_{k\in\mathbb{Z}} c_{j,k}\varphi_{j,k} \right\|_2^2 \le C'\left(\sum_{k\in\mathbb{Z}} |c_{j,k}|^2 \right)^{1/2}, \quad j \in \mathbb{Z},$$

then we call φ a scaling function of the MRA.

For $j \in \mathbb{Z}$, we denote the orthogonal complement of V_{j-1} in V_j by W_j. The *raison d' être* for W_{j-1} is that an element in W_j contains the details needed to pass from an approximation at level $j-1$ to an approximation at level j. Let $\psi \in W_0$. Then the translations and dilations $\psi_{j,k}$ of ψ are defined by

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad x \in \mathbb{R},$$

for all $j,k \in \mathbb{Z}$. If for each j in $\mathbb{Z}$, the set $\{\psi_{j,k} : k \in \mathbb{Z}\}$ forms an orthonormal basis for W_j, then we call ψ a mother wavelet and $\psi_{j,k}$, $j,k \in \mathbb{Z}$, the wavelets for the MRA. It is well known that Daubechies [4, 5] has constructed for $L^2(\mathbb{R})$ orthonormal bases consisting of compactly supported wavelets that can be represented by polynomials of a fixed degree. The support of the Daubechies scaling function is $[0, 2N-1]$, where N is a positive integer. The length of the support increases linearly with the regularity. The corresponding mother wavelet then has compact support given by $[1-N, N]$ and has N vanishing moments in the sense that

$$\int_{-\infty}^{\infty} x^k \psi(x)\, dx = 0, \quad k = 0, 1, 2, \ldots, N-1.$$

If we denote for convenience W_0 by V_0, then for all positive integers n, every element v_n in V_n given by

$$v_n = \sum_{k\in\mathbb{Z}} c_{n,k}\varphi_{n,k},$$

where each $c_{n,k}$ is a complex number, has an alternative multiscale representation given by the wavelets. More precisely,

$$v_n = \sum_{j=0}^{n}\sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k},$$

where each $d_{j,k}$ is a complex number. Equivalently, we can write

$$V_n = \oplus_{j=0}^{n} W_j.$$

Now, we start with two biorthogonal MRAs of $L^2(\mathbb{R})$. This means that $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ are MRAs of $L^2(\mathbb{R})$ such that the primal MRA $\{V_j\}_{j\in\mathbb{Z}}$ and the dual MRA $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ can be equipped with, respectively, Riesz bases $\Phi_j = \{\varphi_{j,k} : k \in \mathbb{Z}\}$ and $\tilde{\Phi}_j = \{\tilde{\varphi}_{j,k} : k \in \mathbb{Z}\}$ with the property of biorthogonality to the effect that

$$(\varphi_{j,k}, \tilde{\varphi}_{j,k'}) = \delta_{k,k'}, \quad k, k' \in \mathbb{Z},$$

where $(\,,\,)$ is the inner product in $L^2(\mathbb{R})$. Each of the primal scaling function φ and the dual scaling function $\tilde{\varphi}$ is assumed to have compact support such that the measure of $\varphi_{j,k}$ and that of $\tilde{\varphi}_{j,k}$ are $\sim 2^{-j}$ for all $j,k \in \mathbb{Z}$. These biorthogonal bases also define projections $P_j : L^2(\mathbb{R}) \to V_j$ and $\tilde{P}_j : L^2(\mathbb{R}) \to \tilde{V}_j$, which are uniformly stable in $L^2(\mathbb{R})$. They are given by

$$P_j v = \sum_{k\in\mathbb{Z}} (v, \tilde{\varphi}_{j,k})\varphi_{j,k}$$

and

$$\tilde{P}_j v = \sum_{k\in\mathbb{Z}} (v, \varphi_{j,k})\tilde{\varphi}_{j,k}$$

for all v in $L^2(\mathbb{R})$ and $j = 0, 1, 2, \ldots$. The nestedness of the MRA spaces gives us the properties that

$$P_j P_{j+1} = P_j$$

and

$$\tilde{P}_j \tilde{P}_{j+1} = \tilde{P}_j$$

for all $j \in \mathbb{Z}$. Hence for $j \in \mathbb{Z}$, the operators Q_j and $\tilde{Q}_j$ given by

$$Q_j = P_{j+1} - P_j$$

and

$$\tilde{Q}_j = \tilde{P}_{j+1} - \tilde{P}_j$$

are also projections.

For $j \in \mathbb{Z}$, the wavelet spaces W_j and $\tilde{W}_j$ are given by

$$W_j = V_{j+1} \cap \tilde{V}_j^{\perp}$$

and

$$\tilde{W}_j = \tilde{V}_{j+1} \cap V_j^{\perp},$$

which are, respectively, the range $R(Q_j)$ of Q_j and the range $R(\tilde{Q}_j)$ of $\tilde{Q}_j$. The wavelet spaces $\{W_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{W}_j\}_{j\in\mathbb{Z}}$ induce two multiscale decompositions of $L^2(\mathbb{R})$ via

$$v = P_1 v + \sum_{j=1}^{\infty} Q_j v = \sum_{j=0}^{\infty} Q_j v, \quad v \in L^2(\mathbb{R}),$$

where $Q_0 = P_1$ and

$$\tilde{v} = \tilde{P}_1 + \sum_{j=1}^{\infty} \tilde{Q}_j v, \quad v \in L^2(\mathbb{R}).$$

Furthermore, we assume that for $j \in \mathbb{Z}$, the wavelet spaces W_j and $\tilde{W}_j$ are equipped with compactly supported biorthogonal Riesz bases denoted, respectively, by

$$\Psi_j = \{\psi_{j,k} : k \in \mathbb{Z}\}$$

and

$$\tilde{\Psi}_j = \{\tilde{\psi}_{j,k} : k \in \mathbb{Z}\}.$$

For all nonnegative integers n, we can introduce the canonical truncated projections Q_n and Q'_n by

$$Q_n v = \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}} (v, \tilde{\psi}_{j,k}) \psi_{j,k}$$

and

$$Q'_n v = \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}} (v, \psi_{j,k}) \tilde{\psi}_{j,k}$$

for all functions v in $L^2(\mathbb{R})$.

For $s \in \mathbb{R}$, let $H^{s,2}$ be the L^2-Sobolev space of order s defined to be the set of all tempered distributions u on $\mathbb{R}$ such that

$$\sigma_{-s}\hat{u} \in L^2(\mathbb{R}),$$

where

$$\sigma_s(\xi) = (1+\xi^2)^{-s/2}, \quad \xi \in \mathbb{R},$$

$\hat{u}$ is the Fourier transform of u and the Fourier transform $\hat{f}$ of a function f in $L^1(\mathbb{R})$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx, \quad \xi \in \mathbb{R}.$$

The norm $\| \, \|_{s,2}$ in $H^{s,2}$ is given by

$$\|u\|_{s,2}^2 = \int_{-\infty}^{\infty} (1+\xi^2)^s |\hat{u}(\xi)|^2 d\xi,$$

Henceforth, we let $\lambda = (j,k)$, where j is the level of resolution and k is the location. We let J be the index set given by

$$J = \{\lambda = (j,k) : j = 0,1,2,\ldots,\, k \in \mathbb{Z}\},$$

and for $\lambda = (j,k)$ in J, we define $|\lambda|$ by

$$|\lambda| = j.$$

Then we have the following result.

Theorem 2.1. *Suppose that* $\Psi = \{\psi_\lambda : \lambda \in J\}$ *and* $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in J\}$ *are biorthogonal collections in* $L^2(\mathbb{R})$ *such that the associated sequence* $\{Q_n\}_{n=0}^{\infty}$ *of projections defined by*

$$Q_n v = \sum_{j=0}^{n} \sum_{k \in \mathbb{Z}} (v, \tilde{\psi}_{j,k}) \psi_{j,k}, \quad v \in L^2(\mathbb{R}),$$

is uniformly bounded in the sense that there exists a positive constant C *such that*

$$\|Q_n v\|_{s,2} \le C \|v\|_{s,2}, \quad n = 0,1,2,\ldots.$$

Then for all $v \in H^{s,2}$, *we have*

$$\|v\|_{s,2} \sim \left(\sum_{\lambda \in J} 2^{2|\lambda| s} |(v, \tilde{\psi}_\lambda)|^2 \right)^{1/2}, \quad s \in (-\gamma', \gamma),$$

where

$$\gamma = \sup\{s \in \mathbb{R} : \varphi \in H^{s,2}\}$$

and

$$\gamma' = \sup\{s \in \mathbb{R} : \psi \in H^{s,2}\}.$$

It is worth pointing out that γ and γ' are, respectively, less than or equal to the vanishing moments of φ and $\tilde{\varphi}$.

For every real number m, we define S^m to be the set of all functions σ in $C^\infty(\mathbb{R} \times \mathbb{R})$ such that for all nonnegative integers α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta} (1+|\xi|)^{m-\beta}, \quad x,\xi \in \mathbb{R}.$$

Then we call σ a symbol of order m. Let $\sigma \in S^m$. Then we define the pseudo-differential operator T_σ on the Schwartz space $\mathcal{S}$ on $\mathbb{R}$ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ix\xi} \sigma(x,\xi) \hat{\varphi}(\xi)\, d\xi, \quad x \in \mathbb{R},$$

for all functions φ in $\mathcal{S}$. The following result is well known.

Theorem 2.2. *T_σ can be extended to a bounded linear operator from $H^{s,2}$ into $H^{s-m,2}$.*

A proof can be found in, for instance, the book [7].

Let $\sigma \in S^m$. Suppose that there exist positive constants C and R such that

$$|\sigma(x,\xi)| \geq C(1+|\xi|)^m, \quad |\xi| > R.$$

Then we say that the symbol σ is elliptic or the pseudo-differential operator T_σ is elliptic.

The following result on spectral invariance [6] is well known. See also Theorem 4.9 in [2] in this connection.

Theorem 2.3. *Let $\sigma \in S^m$ be such that the pseudo-differential operator $T_\sigma : H^{m/2,2} \to H^{-m/2,2}$ is invertible. Then σ is elliptic and T_σ^{-1} is an elliptic pseudo-differential operator with symbol in S^{-m}.*

The following estimate is useful to us.

Theorem 2.4. *Let $\sigma \in S^m$ be such that the pseudo-differential operator $T_\sigma : H^{m/2,2} \to H^{-m/2,2}$ is invertible. Then there exist positive constants C_1 and C_2 such that*

$$C_1 \|T_\sigma u\|_{-m/2,2} \leq \|u\|_{m/2,2} \leq C_2 \|T_\sigma u\|_{-m/2,2}, \quad u \in H^{m/2,2}.$$

Proof. The "first" inequality follows from Theorem 2.2. By Theorems 2.2 and 2.3, there exists a positive constant C such that

$$\|u\|_{m/2,2} = \|T_\sigma^{-1} T_\sigma u\|_{m/2,2} \leq C \|T_\sigma u\|_{-m/2,2}, \quad u \in H^{m/2,2}.$$

This completes the proof. □

The aim of this paper is to use adaptive wavelets to compute numerically the inverse of an invertible pseudo-differential operator $T_\sigma : H^{m,2} \to L^2(\mathbb{R})$, where $\sigma \in S^m$ and $m = \min(\gamma, \gamma')$. This amounts to solving the pseudo-differential equation

$$T_\sigma u = f \tag{2.1}$$

on $\mathbb{R}$ for $u \in H^{m,2}$ for all functions f in $L^2(\mathbb{R})$. To do this, we transform the equation (2.1) to the equation

$$T_\sigma^* T_\sigma u = T_\sigma^* f \tag{2.2}$$

on $\mathbb{R}$, where T_σ^* denotes the formal adjoint of T_σ. Now, $T_\sigma^* T_\sigma$ is a pseudo-differential operator T_τ of order $2m$ and $T_\sigma^* f \in H^{-m,2}$. Furthermore, T_τ is symmetric and there exist positive constants A and B such that

$$A\|u\|_{m,2}^2 \leq (T_\tau u, u) \leq B\|u\|_{m,2}^2, \quad u \in H^{m,2}. \tag{2.3}$$

The "second" inequality follows from Theorem 2.2. In fact, there exists a positive constant B such that

$$(T_\tau u, u) \leq |(T_\tau u, u)| \leq \|T_\tau u\|_{-m,2}\|u\|_{m,2} \leq B\|u\|_{m,2}^2, \quad u \in H^{m,2}.$$

On the other hand, we get from Theorems 2.2 and 2.3 a positive constant C such that

$$\|u\|_{m,2}^2 = \|T_\sigma^{-1} T_\sigma u\|_{m,2}^2 \leq C\|T_\sigma u\|_{m,2}^2, \quad u \in H^{m,2}.$$

With slight abuse of notation, the problem (2.1) is then the same as solving for u in $H^{m,2}$ to the equation

$$T_\sigma u = f$$

on $\mathbb{R}$ for every f in $H^{-m,2}$, where T_σ is a symmetric pseudo-differential operator of order $2m$ such that there exist positive constants A' and B' for which

$$A'\|u\|_{m,2} \leq \|u\|_{T_\sigma} \leq B'\|u\|_{m,2}, \quad u \in H^{m,2},$$

where

$$\|u\|_{T_\sigma}^2 = (T_\sigma u, u).$$

Remark 2.5. The existence of a positive constant A' for which

$$\|u\|_{T_\sigma}^2 \geq A'\|u\|_{m,2}^2, \quad u \in H^{m,2},$$

is a condition related to Gårding's inequality on the symbol σ. See, e.g., the paper [8] in this connection.

Adaptive wavelet methods in finding solutions to differential and integral equations can be found in [1, 3].

3. Residual estimates

The problem of computing the inverse of $T_\sigma : H^{m,2} \to H^{-m,2}$ numerically is equivalent to the computation of subspaces V_Λ of the form

$$V_\Lambda = \overline{\text{span}\{\psi_\lambda : \lambda \in \Lambda\}}$$

that are adapted to the unique solution u in $H^{m,2}$ of the pseudo-differential equation

$$T_\sigma u = f \tag{3.1}$$

on $\mathbb{R}$ for every function f in $H^{-m,2}$. To do this, we use the weak formulation of (3.1) to the effect of finding a solution u_Λ in V_Λ such that

$$(T_\sigma u_\Lambda, v) = (f, v), \quad v \in V_\Lambda. \tag{3.2}$$

More precisely, for every tolerance eps, we seek a subset Λ of J such that the Galerkin approximation u_Λ in V_Λ defined by (3.2) satisfies the estimate

$$\|u - u_\Lambda\|_{m,2} \leq \text{eps}.$$

This is to be achieved by successively upgrading Λ based on appropriate *a posteriori* estimates of a current Galerkin approximation u_Λ. To this end, we define the residual term r_Λ by

$$r_\Lambda = T_\sigma u_\Lambda - f,$$

which is the same as

$$r_\Lambda = T_\sigma(u_\Lambda - u).$$

So, by Theorem 2.4, we can find positive constants C_1 and C_2 such that

$$C_1\|r_\Lambda\|_{-m,2} \leq \|u - u_\Lambda\|_{m,2} \leq C_2\|r_\Lambda\|_{-m,2}$$

for all subsets Λ of J. Thus, we can find positive constants C_3 and C_4 such that

$$C_3\left(\sum_{\lambda\in J\setminus\Lambda} 2^{-2m|\lambda|}|(r_\Lambda, \psi_\lambda)|^2\right)^{1/2} \leq \|r_\Lambda\|_{-m,2} \leq C_4\left(\sum_{J\setminus\Lambda} 2^{-2m|\lambda|}|(r_\Lambda, \psi_\lambda)|^2\right)^{1/2}.$$

Now, for $\lambda \in J \setminus \Lambda$, define δ_λ by

$$\delta_\lambda = 2^{-m|\lambda|}|(r_\Lambda, \psi_\lambda)|.$$

Since $u_\Lambda \in V_\Lambda$, it follows that

$$u_\Lambda = \sum_{\lambda'\in\Lambda} u_{\lambda'}\psi_{\lambda'},$$

where

$$u_{\lambda'} = (u_\Lambda, \tilde{\psi}_{\lambda'}).$$

So, for $\lambda \in J \setminus \Lambda$,

$$\delta_\lambda = 2^{-m|\lambda|}\left|f_\lambda - \sum_{\lambda'\in\Lambda}(T_\sigma\psi_{\lambda'}, \psi_\lambda)u'_\lambda\right|.$$

Let μ be the Hölder exponent of $\partial^\gamma\varphi$. Then for all positive numbers ε and δ with $\delta < \mu - \frac{1}{2}$, we can choose positive numbers ε_1 and ε_2 such that

$$\varepsilon_1^{2(\tilde{r}+1)} + 2^{-\delta/\varepsilon_2} \leq \varepsilon,$$

where $\tilde{r}$ is the vanishing moment of $\tilde{\varphi}$.

For all λ in J and for every positive number ε, we define the tolerance set $J_{\lambda,\varepsilon}$ by

$$J_{\lambda,\varepsilon} = \{\lambda' \in J : ||\lambda| - |\lambda'|| \leq \varepsilon_2^{-1},\ 2^{\min(|\lambda|,|\lambda'|)}d(\text{supp}(\psi_\lambda), \text{supp}(\psi_{\lambda'})) \leq \varepsilon_1^{-1}\}.$$

Then we have the following lemma, which is Lemma 4.2 in [3].

Lemma 3.1. *For $\lambda \in J \setminus \Lambda$, let e_λ be defined by*

$$e_\lambda = \sum_{\lambda' \in \Lambda \setminus J_{\lambda,\varepsilon}} 2^{-m|\lambda|}(T_\sigma \psi_{\lambda'}, \psi_\lambda) u_{\lambda'}.$$

Then there exists a positive constant C_5 such that

$$\left(\sum_{\lambda \in J \setminus \Lambda} |e_\lambda|^2 \right)^{1/2} \leq C_5 \varepsilon \|Q'_\Lambda f\|_{-m,2},$$

where

$$Q'_\Lambda f = \sum_{\lambda \in \Lambda} (f, \psi_\lambda) \tilde{\psi}_\lambda.$$

We note that for $\lambda \in J \setminus \Lambda$,

$$\begin{aligned} \delta_\lambda &= 2^{-m|\lambda|} \left| f_\lambda - \left(\sum_{\lambda' \in \Lambda \cap J_{\lambda,\varepsilon}} + \sum_{\lambda' \in \Lambda \setminus J_{\lambda,\varepsilon}} \right) (T_\sigma \psi_{\lambda'}, \psi_\lambda) u_\lambda \right| \\ &\leq |d_\lambda| + |e_\lambda|, \end{aligned}$$

where

$$d_\lambda = 2^{-m|\lambda} \left| f_\lambda - \sum_{\lambda' \in \Lambda \cap J_{\lambda,\varepsilon}} (T_\sigma \psi_{\lambda'}, \psi_\lambda) u_{\lambda'} \right|.$$

Let $N_{\Lambda,\varepsilon}$ be the set of all indices in the complement of Λ with influence set intersecting Λ. More precisely,

$$N_{\Lambda,\varepsilon} = \{\lambda \in J \setminus \Lambda : J_{\lambda,\varepsilon} \cap \Lambda \neq \phi\}.$$

It can be shown that

$$N_{\Lambda,\varepsilon} = \cup_{\lambda' \in \Lambda} J_{\lambda',\varepsilon}$$

and $N_{\Lambda,\varepsilon}$ has at most a finite number of elements. Hence

$$\lambda' \in J \setminus (\Lambda \cup N_{\Lambda,\varepsilon}) \Rightarrow J_{\lambda',\varepsilon} \cap \Lambda = \phi.$$

Since

$$f \in H^{-m,2} \Leftrightarrow \sum_{\lambda \in J} 2^{-2m|\lambda|} |f_\lambda|^2 < \infty,$$

it follows that $\sum_{\lambda \in J \setminus (N_{\Lambda,\varepsilon} \cup \Lambda)} 2^{-2m|\lambda|} |f_\lambda|^2$ can be made arbitrarily small by choosing Λ appropriately. Indeed,

$$\begin{aligned} \sum_{\lambda \in J \setminus (N_{\Lambda,\varepsilon} \cup \Lambda)} 2^{-2m|\lambda|} |f_\lambda|^2 &= \sum_{\lambda \in J} 2^{2m|\lambda|} |f_\lambda|^2 - \sum_{\lambda \in (N_{\Lambda,\varepsilon} \cup \Lambda)} 2^{-2m|\lambda|} |f_\lambda|^2 \\ &= \|f - Q'_{\Lambda \cup N_{\Lambda,\varepsilon}} f\|^2_{-m,2} \\ &\sim \inf_{v \in \tilde{V}_{\Lambda \cup N_{\Lambda,\varepsilon}}} \|f - v\|^2_{-m,2} \\ &\leq \inf_{v \in \tilde{V}_\Lambda} \|f - v\|^2_{-m,2}. \end{aligned}$$

We can now lay out the basic assumptions to the effect that there are positive constants C_6 and C_7 such that

$$C_5\|Q'_\Lambda f\|_{-m,2} \leq C_6\|f\|_{-m,2}$$

and

$$\left(\sum_{\lambda\in J\setminus\Lambda} 2^{-2m|\lambda|}|f_\lambda|^2\right)^{1/2} \leq C_7 \inf_{v\in\tilde{V}_\Lambda} \|f-v\|_{-m,2}$$

for all subsets Λ of J.

4. A posteriori error bounds

For $\lambda \in J\setminus\Lambda$, we define a_λ by

$$a_\lambda = 2^{-m|\lambda|}\left|\sum_{\lambda'\in\Lambda\cap J_{\lambda,\varepsilon}} (T_\sigma\psi_{\lambda'},\psi_\lambda)u_{\lambda'}\right|.$$

Theorem 4.1. *Under the hypotheses of* Lemma 3.1*, we have*

$$\|u-u_\Lambda\|_{m,2} \leq C_2C_4\left\{\left(\sum_{\lambda\in N_{\Lambda,\varepsilon}} a_\lambda^2\right)^{1/2} + C_6\varepsilon\|f\|_{-m,2} + C_7\inf_{v\in\tilde{V}_\Lambda}\|f-v\|_{-m,2}\right\}$$

and

$$\left(\sum_{\lambda\in N_{\Lambda,\varepsilon}} a_\lambda^2\right)^{1/2} \leq \frac{1}{C_1C_3}\|u-u_\Lambda\|_{m,2} + C_6\varepsilon\|f\|_{-m,2} + C_7\inf_{v\in\tilde{V}_\Lambda}\|f-v\|_{-m,2}.$$

Theorem 4.2. *Suppose that $\Lambda\subset\tilde{\Lambda}\subset J$. Then*

$$\left(\sum_{\lambda\in\tilde{\Lambda}\cap N_{\Lambda,\varepsilon}} a_\lambda^2\right)^{1/2} \leq \frac{1}{C_1C_3}\|u_{\tilde{\Lambda}}-u_\Lambda\|_{m,2} + C_6\varepsilon\|f\|_{-m,2} + C_7\inf_{v\in\tilde{V}_\Lambda}\|f-v\|_{-m,2}.$$

Proof. Let $\lambda\in\tilde{\Lambda}$. Then

$$(T_\sigma u_\Lambda,\psi_\lambda) = (T_\sigma(u_\Lambda-u_{\tilde{\Lambda}}),\psi_\lambda) + f_\lambda.$$

So,

$$d_\lambda(\Lambda,\varepsilon) \leq 2^{-m|\lambda|}|(T_\sigma(u_\Lambda-u_{\tilde{\Lambda}}),\psi_\lambda)| + |e_\lambda|.$$

Moreover,

$$\sum_{\lambda\in\tilde{\Lambda}\setminus\Lambda} 2^{-2m|\lambda|}|(T_\sigma(u_\Lambda-u_{\tilde{\Lambda}}),\psi_\lambda)|^2 \leq \frac{1}{C_3^2}\|T_\sigma(u_\Lambda-u_{\tilde{\Lambda}})\|_{-m,2}^2 \leq \frac{1}{C_1^2C_3^2}\|u_\Lambda-u_{\tilde{\Lambda}}\|_{m,2}^2$$

So, by Lemma 3.1,

$$\left(\sum_{\lambda\in\tilde{\Lambda}\setminus\Lambda} d_\lambda(\Lambda,\varepsilon)^2\right)^{1/2} \leq \frac{1}{C_1C_3}\|u_{\tilde{\Lambda}}-u_\Lambda\|_{m,2}+C_5\varepsilon\|Q'_\Lambda f\|_{-m,2}.$$

Hence

$$|a_\lambda(\Lambda,\varepsilon)|\leq|d_\lambda(\Lambda,\varepsilon)|+2^{-m|\lambda|}|f_\lambda|,$$

and the proof is complete. □

5. An adaptive algorithm

We prove in this section that for a set $\tilde{\Lambda}$ containing Λ, the solution in $V_{\tilde{\Lambda}}$ approximates the actual solution better than the one in V_Λ. To do this, we recall our assumptions spelled out at the end of Section 2 that the pseudo-differential operator T_σ is *symmetric* and there exist positive constants C_8 and C_9 such that

$$C_8\|u\|_{m,2}\leq\|u\|_{T_\sigma}\leq C_9\|u\|_{m,2},\quad u\in H^{m,2},$$

where

$$\|u\|^2_{T_\sigma}=(T_\sigma u,u).$$

Theorem 5.1. *Let* eps *be a given tolerance. For $\theta^*\in(0,1)$, we define the number C_e by*

$$C_e=\left(\frac{1}{C_1C_3}+\frac{1-\theta^*}{2C_2C_4}\right).$$

Let μ^* be a positive number such that

$$\mu^*C_e\leq\frac{1-\theta^*}{2(2-\theta^*)C_2C_4}.$$

Let ε be the positive number defined by

$$\varepsilon=\frac{\mu^*\mathrm{eps}}{2C_6\|f\|_{-m,2}}.$$

Suppose that Λ is a subset of J such that

$$C_7\inf_{v\in\tilde{V}_\Lambda}\|f-v\|_{-m,2}\leq\frac{1}{2}\mu^*\mathrm{eps}.$$

Then for all subsets $\tilde{\Lambda}$ of J such that $\Lambda\subset\tilde{\Lambda}$ and

$$\left(\sum_{\lambda\in\tilde{\Lambda}\cap N_{\Lambda,\varepsilon}} a_\lambda^2\right)^{1/2}\geq(1-\theta^*)\left(\sum_{\lambda\in N_{\Lambda,\varepsilon}} a_\lambda^2\right)^{1/2},$$

there exists a number κ in $(0,1)$ such that

$$\|u-u_{\tilde{\Lambda}}\|_{T_\sigma}\leq\kappa\|u-u_\Lambda\|_{T_\sigma}.$$

Proof. We begin with the assumption that

$$\|u-u_\Lambda\|_{m,2} \geq \frac{\mathrm{eps}}{C_e}.$$

By Theorems 4.1 and 4.2,

$$\begin{aligned}
\|u_{\tilde\Lambda}-u_\Lambda\|_{m,2} &\geq C_1C_3\left\{\left(\sum_{\lambda\in\tilde\Lambda\cap N_{\Lambda,\varepsilon}} a_\lambda^2\right)^{1/2} - C_6\varepsilon\|f\|_{-m,2} - C_7\inf_{v\in\tilde V_\Lambda}\|f-v\|_{-m,2}\right\}\\
&\geq C_1C_3\{(1-\theta^*)((C_2C_4)^{-1}\|u-u_\Lambda\|_{m,2} - C_6\varepsilon\|f\|_{-m,2}\\
&\quad - C_7\inf_{v\in\tilde V_\Lambda}\|f-v\|_{-m,2}) - C_6\varepsilon\|f\|_{-m,2} - C_7\inf_{v\in\tilde V_\Lambda}\|f-v\|_{-m,2}\}\\
&\geq C_1C_3((1-\theta^*)(C_2C_4)^{-1}\|u-u_\Lambda\|_{m,2} - (2-\theta^*)C_6\varepsilon\|f\|_{-m,2}\\
&\quad - (2-\theta^*)C_7\inf_{v\in\tilde V_\Lambda}\|f-v\|_{-m,2}).
\end{aligned}$$

So,

$$\|u_{\tilde\Lambda}-u_\Lambda\|_{m,2} \geq C_1C_3\left(\frac{1-\theta^*}{C_2C_4}\|u-u_\Lambda\|_{m,2} - (2-\theta^*)\mu^*\mathrm{eps}\right),$$

and consequently

$$\begin{aligned}
\|u_{\tilde\Lambda}-u_\Lambda\|_{m,2} &\geq C_1C_3\left(\frac{1-\theta^*}{C_2C_4} - (2-\theta^*)\mu^*C_e\right)\|u-u_\Lambda\|_{m,2}\\
&\geq \frac{C_1C_3(1-\theta^*)}{2C_2C_4}\|u-u_\Lambda\|_{m,2}.
\end{aligned}$$

Now,

$$\begin{aligned}
\|u_{\tilde\Lambda}-u_\Lambda\|_{T_\sigma}^2 &= (T_\sigma u_{\tilde\Lambda} - T_\sigma u_\Lambda, u_{\tilde\Lambda}-u_\Lambda)\\
&= \|u_{\tilde\Lambda}\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - (T_\sigma u_{\tilde\Lambda}, u_\Lambda) - (T_\sigma u_\Lambda, u_{\tilde\Lambda})\\
&= \|u_{\tilde\Lambda}\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - (f, u_\Lambda) - (u_\Lambda, f)\\
&= \|u_{\tilde\Lambda}\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - (T_\sigma u_\Lambda, u_\Lambda) - (u_\Lambda, T_\sigma u_\Lambda)\\
&= \|u_{\tilde\Lambda}\|_{T_\sigma}^2 - \|u_\Lambda\|_{T_\sigma}^2.
\end{aligned} \tag{5.1}$$

Also,

$$\begin{aligned}
\|u-u_{\tilde\Lambda}\|_{T_\sigma}^2 &= (T_\sigma u - T_\sigma u_{\tilde\Lambda}, u-u_{\tilde\Lambda})\\
&= \|u\|_{T_\sigma}^2 + \|u_{\tilde\Lambda}\|_{T_\sigma}^2 - (T_\sigma u, u_{\tilde\Lambda}) - (T_\sigma u_{\tilde\Lambda}, u)\\
&= \|u\|_{T_\sigma}^2 + \|u_{\tilde\Lambda}\|_{T_\sigma}^2 - (f, u_{\tilde\Lambda}) - (u_{\tilde\Lambda}, f)\\
&= \|u_{\tilde\Lambda}\|_{T_\sigma}^2 + \|u\|_{T_\sigma}^2 - (T_\sigma u_{\tilde\Lambda}, u_{\tilde\Lambda}) - (u_{\tilde\Lambda}, T_\sigma u_{\tilde\Lambda})\\
&= \|u\|_{T_\sigma}^2 - \|u_{\tilde\Lambda}\|_{T_\sigma}^2.
\end{aligned} \tag{5.2}$$

Furthermore,

$$\begin{aligned}\|u-u_\Lambda\|_{T_\sigma}^2 &= (T_\sigma u - T_\sigma u_\Lambda, u-u_\Lambda)\\ &= \|u\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - (T_\sigma u, u_\Lambda) - (T_\sigma u_\Lambda, u)\\ &= \|u\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - (f, u_\Lambda) - (u_\Lambda, f)\\ &= \|u\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - (T_\sigma u_\Lambda, u_\Lambda) - (u_\Lambda, T_\sigma u_\Lambda)\\ &= \|u\|_{T_\sigma}^2 - \|u_\Lambda\|_{T_\sigma}^2. \end{aligned} \tag{5.3}$$

Therefore by (5.1)–(5.3),

$$\|u_{\tilde\Lambda} - u_\Lambda\|_{T_\sigma}^2 = \|u-u_\Lambda\|_{T_\sigma}^2 - \|u-u_{\tilde\Lambda}\|_{T_\sigma}^2,$$

or equivalently

$$\|u-u_{\tilde\Lambda}\|_{T_\sigma}^2 + \|u_{\tilde\Lambda} - u_\Lambda\|_{T_\sigma}^2 = \|u-u_\Lambda\|_{T_\sigma}^2.$$

Now,

$$\begin{aligned}\|u_{\tilde\Lambda} - u_\Lambda\|_{T_\sigma} &\geq C_8\|u_{\tilde\Lambda} - u_\Lambda\|_{m,2}\\ &\geq \frac{C_1C_3C_8(1-\theta^*)}{2C_2C_4}\|u-u_\Lambda\|_{m,2}\\ &\geq \frac{C_1C_3C_8(1-\theta^*)}{2C_2C_4C_9}\|u-u_\Lambda\|_{T_\sigma}. \end{aligned} \tag{5.4}$$

Hence

$$\begin{aligned}\|u-u_{\tilde\Lambda}\|_{T_\sigma}^2 &= \|u-u_\Lambda\|_{T_\sigma}^2 - \|u_{\tilde\Lambda} - u_\Lambda\|_{T_\sigma}^2\\ &\leq \|u-u_\Lambda\|_{T_\sigma}^2 - \left(\frac{C_1C_3C_8(1-\theta^*)}{2C_2C_4C_9}\right)^2 \|u-u_\Lambda\|_{T_\sigma}^2\\ &= \kappa^2\|u-u_\Lambda\|_{T_\sigma}^2, \end{aligned}$$

where

$$\kappa = \sqrt{1-\left(\frac{C_1C_3C_8(1-\theta^*)}{2C_2C_4C_9}\right)^2}. \qquad \square$$

We can now give an adaptive algorithm as promised.

An adaptive algorithm Given $\theta^* \in (0,1)$ and the desired accuracy eps, we proceed as follows:

- Step 1: Compute $\varepsilon = \frac{\mu^* \text{eps}}{2C_6\|f\|_{-m,2}}$.
- Step2: Determine an index set $\Lambda \subset J$ such that

$$C_7 \inf_{v\in\tilde V_\Lambda} \|f-v\|_{-m,2} < \frac{1}{2}\mu^*\text{eps}.$$

- Step 3: Compute the Galerkin solution u_Λ with respect to V_Λ.

- Step 4: Compute

$$\eta_{\Lambda,\varepsilon} = \left(\sum_{\lambda \in N_{\Lambda,\varepsilon}} a_\lambda^2 \right)^{1/2}.$$

If $\eta_{\Lambda,\varepsilon} <$ eps, then we stop and accept u_Λ as a solution. Otherwise, go to the next step.

- Step 5: Determine an index set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset J$ and

$$\left(\sum_{\lambda \in \tilde{\Lambda} \cap N_{\Lambda,\varepsilon}} a_\lambda^2 \right)^{1/2} \geq (1 - \theta^*) \eta_{\Lambda,\varepsilon},$$

and go to Step 3 with Λ replaced by $\tilde{\Lambda}$.

References

[1] A. Cohen, *Numerical Analysis of Wavelet Methods*, North-Holland, 2003.

[2] A. Dasgupta, Ellipticity of Fredholm pseudo-differential operators on $L^p(\mathbb{R}^n)$, in *New Developments of Pseudo-Differential Operators*, Operator Theory: Advances and Applications **189**, Birkhäuser, 2009, 107–116.

[3] S. Dahlke, W. Dahmen, R. Hochmuth and R. Schneider, Stable multiscale bases and local error estimation for elliptic problems, *Appl. Numerical Math.* **23** (1997), 21–47.

[4] I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909–996.

[5] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.

[6] E. Schrohe, Boundedness and spectral invariance for standard pseudodifferential operators on anisotropically weighted L^p-Sobolev spaces, *Integral Equations Operator Theory* **13** (1990), 271–284.

[7] M.W. Wong, *An Introduction to Pseudo-Differential Operators,* Second Edition, World Scientific, 1999.

[8] M.W. Wong, Weak and strong solutions for pseudo-differential operators, in *Advances in Analysis*, World Scientific, 2005, 275–284.

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Spectral Theory of Pseudo-Differential Operators on $\mathbb{S}^1$

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Abstract. For a bounded pseudo-differential operator with the dense domain $C^\infty(\mathbb{S}^1)$ on $L^p(\mathbb{S}^1)$, the minimal and maximal operator are introduced. An analogue of Agmon-Douglis-Nirenberg [1] is proved and then is used to prove the uniqueness of the closed extension of an elliptic pseudo-differential operator of symbol of positive order. We show the Fredholmness of the minimal operator. The essential spectra of pseudo-differential operators on $\mathbb{S}^1$ are described.

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1. Introduction

In this paper the focus is on pseudo-differential operators on the unit circle $\mathbb{S}^1$ centered at the origin. For $-\infty < m < \infty$, let $S^m(\mathbb{S}^1 \times \mathbb{Z})$ be the set all functions σ in $C^\infty(\mathbb{S}^1 \times \mathbb{Z})$ such that for all nonnegative integers α and β there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(\partial_\theta^\alpha \partial_n^\beta \sigma)(\theta, n)| \le C_{\alpha,\beta}(1 + |n|)^{m-\beta}, \quad \theta \in [-\pi, \pi],\ n \in \mathbb{Z}.$$

Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$. Then we define the pseudo-differential operator T_σ on $L^1(\mathbb{S}^1)$ by

$$(T_\sigma f)(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} \sigma(\theta, n)(\mathcal{F}_{\mathbb{S}^1} f)(n), \quad \theta \in [-\pi, \pi],$$

where

$$(\mathcal{F}_{\mathbb{S}^1} f)(n) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta, \quad n \in \mathbb{Z}.$$

Basic properties of pseudo-differential operators with symbols in $S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$, can be found in [2, 3, 4, 6, 10, 9]. The basic calculi for the

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product and the formal adjoint of pseudo-differential operators with symbols in $S^m(\mathbb{S}^1 \times \mathbb{Z})$ can be found in [9].

A symbol σ in $S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$, is said to be elliptic if there exist positive constants C and R such that

$$|\sigma(\theta, n)| \geq C(1 + |n|)^m, \quad |n| \geq R, \quad \theta \in [-\pi, \pi].$$

The following theorem gives a parametrix for an elliptic pseudo-differential operator with symbol in $S^m(\mathbb{S}^1 \times \mathbb{Z})$, $\infty < m < -\infty$, see [9].

Theorem 1.1. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$ be elliptic. Then there exists a symbol $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$ such that*

$$T_\sigma T_\tau = I + K \quad \text{and} \quad T_\tau T_\sigma = I + R,$$

where K and R are infinitely smoothing in the sense that they are pseudo-differential operators with symbols in $\cap_{m \in \mathbb{R}} S^m(\mathbb{S}^1 \times \mathbb{Z})$.

Similar results for the symbol class $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ of the pseudo-differential operators on $\mathbb{R}^n$ have been studied for example in [15].

In Section 2, we recall L^p-Sobolev spaces $H^{s,p}$, $-\infty < s < \infty$, $1 \leq p \leq \infty$, and we give some of the results in [7]. Then in Section 3, we consider bounded pseudo-differential operators T_σ on $L^p(\mathbb{S}^1)$, $1 < p < \infty$ with dense domain $C^\infty(\mathbb{S}^1)$. The smallest and largest closed extension of T_σ are provided. The analogue of Agmon-Douglis-Nirenberg [1], is given to prove that for an elliptic symbol σ of positive order m, the corresponding pseudo-differential operator has a unique closed extension with domain $H^{m,p}$ on $L^p(\mathbb{S}^1)$. In Section 4, we focus on Fredholmness of pseudo-differential operator and its essential spectrum. Results on the Fredholmness of pseudo-differential operators on $\mathbb{R}^n$ can be found in [16, 13]. By using Theorem 2.9 in [7], we see that the minimal operator of an elliptic pseudo-differential operator of positive order is Fredholm. The essential spectra of the pseudo-differential operator and the minimal (maximal) operator are then provided. Similar results for the SG Pseudo-differential operator on $\mathbb{R}^n$ are given in [5, 8].

2. L^p-Sobolev spaces

For $-\infty < s < \infty$, let J_s be the pseudo-differential operator with symbol σ_s given by

$$\sigma_s(n) = (1 + |n|^2)^{-s/2}, \quad n \in \mathbb{Z}.$$

J_s is called the Bessel potential of order s.

Now, for $-\infty < s < \infty$ and $1 \leq p \leq \infty$, we define the L^p-Sobolev space $H^{s,p}$ to be the set of all tempered distributions u for which $J_{-s}u$ is a function in $L^p(\mathbb{S}^1)$. Then $H^{s,p}$ is a Banach space in which the norm $\|\cdot\|_{s,p}$ is given by

$$\|u\|_{s,p} = \|J_{-s}u\|_{L^p(\mathbb{S}^1)}, \quad u \in H^{s,p}.$$

It is easy to show that for $-\infty < s, t < \infty$, J_t is an isometry of $H^{s,p}$ onto $H^{s+t,p}$.

The following theorem is known as Sobolev embedding theorem.

Theorem 2.1. *Let $1 < p < \infty$ and $s \le t$. Then $H^{t,p} \subseteq H^{s,p}$ and*

$$\|u\|_{s,p} \le \|u\|_{t,p}, \quad u \in H^{t,p}.$$

Proposition 2.2. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$. Then $T_\sigma : H^{s,p} \to H^{s-m,p}$ is a bounded linear operator for $1 < p < \infty$.*

Proposition 2.3. *Let $s < t$. Then the inclusion operator $i : H^{t,p} \hookrightarrow H^{s,p}$ is compact for $1 \le p \le \infty$.*

The results above can be found in [7].

3. Minimal and maximal operators

Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m \in \mathbb{R}$. Then the formal adjoint of T_σ, denoted T_σ^* is a linear operator on $C^\infty(\mathbb{S}^1)$ such that

$$(T_\sigma \varphi, \psi) = (\varphi, T_\sigma^* \psi), \quad \varphi, \psi \in C^\infty(\mathbb{S}^1).$$

It can be proved that the formal adjoint of T_σ is a pseudo-differential operator of symbol of order $-m$ (see [10]). The following proposition guarantee that the minimal operator of T_σ exists.

Proposition 3.1. *Let $S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$. Then $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is closable with dense domain $C^\infty(\mathbb{S}^1)$ for $1 < p < \infty$.*

Proof. Let $\{\varphi_k\}_{k=1}^\infty$ be a sequence in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to 0$ and $T_\sigma \varphi_k \to f$ for some f in $L^p(\mathbb{S}^1)$ as $k \to \infty$. We only need to show that $f = 0$. We have

$$(T_\sigma \varphi_k, \psi) = (\varphi_k, T_\sigma^* \psi), \quad \psi \in C^\infty(\mathbb{S}^1),\ k = 1, 2, \ldots.$$

Let $k \to \infty$, then $(f, \psi) = 0$ for all $\psi \in C^\infty(\mathbb{S}^1)$. By the density of $C^\infty(\mathbb{S}^1)$ in $L^p(\mathbb{S}^1)$, it follows that $f = 0$. $\square$

Consider $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ with domain $C^\infty(\mathbb{S}^1)$. Then by Proposition 3.1, T_σ has a closed extension. Let $T_{\sigma,0}$ be the minimal operator of T_σ which is the smallest closed extension of T_σ. Then the domain $\mathcal{D}(T_{\sigma,0})$ of $T_{\sigma,0}$ consists of all functions $u \in L^p(\mathbb{S}^1)$ for which there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to u$ in $L^p(\mathbb{S}^1)$ and $T_\sigma \varphi_k \to f$ for some $f \in L^p(\mathbb{S}^1)$ in $L^p(\mathbb{S}^1)$ as $k \to \infty$. It can be shown that f does not depend on the choice of $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ and $T_{\sigma,0} u = f$.

We define the linear operator $T_{\sigma,1}$ on $L^p(\mathbb{S}^1)$ with domain $\mathcal{D}(T_{\sigma,1})$ by the following. Let f and u be in $L^p(\mathbb{S}^1)$. Then we say that $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$ if and only if

$$(u, T_\sigma^* \varphi) = (f, \varphi), \quad \varphi \in C^\infty(\mathbb{S}^1).$$

It can be proved that $T_{\sigma,1}$ is a closed linear operator from $L^p(\mathbb{S}^1)$ into $L^p(\mathbb{S}^1)$ with domain $\mathcal{D}(T_{\sigma,1})$ containing $C^\infty(\mathbb{S}^1)$. In fact, $C^\infty(\mathbb{S}^1)$ is contained in the domain $\mathcal{D}(T_{\sigma,1}^t)$ of the true adjoint $T_{\sigma,1}^t$ of $T_{\sigma,1}$. Furthermore, $T_{\sigma,1}(u) = T_\sigma(u)$ for all u in $\mathcal{D}(T_{\sigma,1})$.

It is easy to see that $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$. In fact $T_{\sigma,1}$ is the largest closed extension of T_σ in the sense that if B is any closed extension of T_σ such that $C^\infty(\mathbb{S}^1) \subseteq \mathcal{D}(B^t)$, then $T_{\sigma,1}$ is an extension of B. $T_{\sigma,1}$ is called the maximal operator of T_σ. The following theorem is an analogue of Agmon-Douglis-Nirenberg in [1].

Proposition 3.2. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then there exist positive constants C and $D > 0$ such that*

$$C\|u\|_{m,p} \leq \|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)} \leq D\|u\|_{m,p}, \quad u \in H^{m,p}.$$

Proof. By the boundedness of T_σ in Proposition 2.2 and the boundedness of the inclusion operator in Theorem 2.1, there exists a positive constant D such that for all $u \in H^{m,p}$,

$$\|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)} \leq D\|u\|_{m,p}, \quad u \in H^{m,p}.$$

Since $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ is elliptic, by Theorem 1.1, there exists a symbol $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$ such that

$$u = T_\tau T_\sigma u - Ru, \quad u \in H^{m,p},$$

where R is an infinitely smoothing operator in the sense that R is a pseudo-differential operator with symbol in $\cap_{m\in\mathbb{R}} S^m(\mathbb{S}^1 \times \mathbb{Z})$. By using Proposition 2.2 again, $T_\sigma u \in L^p(\mathbb{S}^1)$. Therefore, $T_\tau T_\sigma u \in H^{m,p}$, for all $u \in H^{m,p}$, Moreover there exists a positive constant C such that

$$\|u\|_{m,p} \leq C(\|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)}), \quad u \in H^{m,p}. \qquad \square$$

We have the following result which we use in the next theorem.

Lemma 3.3. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then $C^\infty(\mathbb{S}^1)$ is dense in $H^{s,p}$.*

Proof. Let $u \in H^{s,p}$. Then $J_{-s}u \in L^p(\mathbb{S}^1)$. Since $C^\infty(\mathbb{S}^1)$ is dense in $L^p(\mathbb{S}^1)$, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to J_{-s}u$ in $L^p(\mathbb{S}^1)$ as $k \to \infty$. Let $\psi_k = J_s\varphi_k$, $k = 1, 2, \ldots$. Then $\psi_k \in C^\infty(\mathbb{S}^1)$, $k = 1, 2, \ldots$, and

$$\begin{aligned} \|\psi_k - u\|_{s,p} &= \|J_{-s}\psi_k - J_{-s}u\|_{L^p(\mathbb{S}^1)} \\ &= \|\varphi_k - J_{-s}u\|_{L^p(\mathbb{S}^1)} \to 0, \end{aligned}$$

as $k \to \infty$, which completes the proof. $\square$

The following theorem gives the domain of the minimal operator of an elliptic pseudo-differential operator with symbol of positive order.

Theorem 3.4. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$, be elliptic. Then $\mathcal{D}(T_{\sigma,0}) = H^{m,p}$.*

Proof. Let $u \in H^{m,p}$. Then by using the density of $C^\infty(\mathbb{S}^1)$ in $H^{m,p}$, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to u$ in $H^{m,p}$ and therefore in $L^p(\mathbb{S}^1)$ as $k \to \infty$. By Proposition 3.2, φ_k and $T_\sigma\varphi_k$ are Cauchy sequences in $L^p(\mathbb{S}^1)$. Therefore $\varphi_k \to u$ and $T_\sigma\varphi_k \to f$ for some f in $L^p(\mathbb{S}^1)$ as $k \to \infty$. This implies that $u \in \mathcal{D}(T_{\sigma,0})$ and $T_{\sigma,0}u = f$. Now assume that $u \in \mathcal{D}(T_{\sigma,0})$. Then there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to u$ in $L^p(\mathbb{S}^1)$ and $T_\sigma\varphi_k \to f$, for some $f \in L^p(\mathbb{S}^1)$ as $k \to \infty$. So, by Proposition 3.2, $\{\varphi_k\}_{k=1}^\infty$ is a Cauchy sequence in $H^{m,p}$. Since $H^{m,p}$ is complete, there exists $v \in H^{m,p}$ such that $\varphi_k \to v$ in $H^{m,p}$ as $k \to \infty$. By Sobolev embedding theorem $\varphi_k \to v$ in $L^p(\mathbb{S}^1)$ which implies that $u = v \in H^{m,p}$. $\square$

The following theorem shows that the closed extension of an elliptic pseudo-differential operator on $L^p(\mathbb{S}^1)$ with symbol $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$, is unique and moreover by Theorem 3.4, its domain is $H^{m,p}$.

Theorem 3.5. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$, be elliptic. Then $T_{\sigma,0} = T_{\sigma,1}$.*

Proof. Since $T_{\sigma,1}$ is a closed extension of $T_{\sigma,0}$, by Theorem 3.4, it is enough to show that $\mathcal{D}(T_{\sigma,1}) \subseteq H^{m,p}$. Let $u \in \mathcal{D}(T_{\sigma,1})$. By ellipticity of σ, there exists $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$ such that

$$u = T_\tau T_\sigma u - Ru,$$

where R is an infinitely smoothing operator. Since $T_\sigma u = T_{\sigma,1} u \in L^p(\mathbb{S}^1)$, by Proposition 2.2, it follows that $u \in H^{m,p}$, which completes the proof. $\square$

4. Fredholm pseudo-differential operators

A closed linear operator A from a complex Banach space X into a complex Banach space Y with dense domain $\mathcal{D}(A)$ is said to be Fredholm if

- the range of A, $R(A)$ is closed subspace of Y and
- the null space of A, $N(A)$ and the null space of the true adjoint of A, $N(A^t)$ are finite dimensional.

The index of a Fredholm operator A is defined by

$$i(A) = \dim N(A) - \dim N(A^t)$$

By Atkinson's theorem, a closed linear operator $A : X \to Y$ with dense domain $\mathcal{D}(A)$ is Fredholm if and only if there exists a bounded linear operator $B : Y \to X$ such that $K_1 = AB - I : Y \to Y$ and $K_2 = BA - I : X \to X$ are compact operators.

Let $A : X \to X$ be a closed linear operator with dense domain $\mathcal{D}(A)$ in the complex Banach space X. Then the spectrum of A, $\Sigma(A)$ is defined by

$$\Sigma(A) = \mathbb{C} - \rho(A),$$

where $\rho(A)$ is the resolvent set of A given by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is bijective}\}.$$

The essential spectrum $\Sigma_w(A)$ of A, which has been defined in [14] by Wolf given by

$$\Sigma_w(A) = \mathbb{C} - \Phi_w(A), \text{ where } \Phi_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm}\}.$$

Note that $i(A - \lambda I)$ is constant for all λ in a connected component of $\Phi_w(A)$.

The essential spectrum $\Sigma_s(A)$ of A in sense of Schechter [11] is defined by

$$\Sigma_s(A) = \mathbb{C} - \Phi_s(A), \text{ where } \Phi_s(A) = \{\lambda \in \Phi_w(A) : i(A - \lambda I) = 0\}.$$

For the properties of essential spectra see [12]. The following theorem gives a sufficient condition for $T_\sigma : H^{s,p} \to H^{s-m,p}$ to be a Fredholm operator. The proof can be found in [7].

Theorem 4.1. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$ be elliptic. Then for all $-\infty < s < \infty$ and $1 < p < \infty$, $T_\sigma : H^{s,p} \to H^{s-m,p}$ is a Fredholm operator. In particular if $\sigma \in S^0(\mathbb{S}^1 \times \mathbb{Z})$, then the bounded linear operator $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is Fredholm.*

The following is an immediate corollary of Theorem 3.4 and Theorem 4.1.

Corollary 4.2. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then for $1 < p < \infty$, $T_{\sigma,0}$ is a Fredholm operator on $L^p(\mathbb{S}^1)$ with the domain $H^{m,p}$.*

The following theorem gives the essential spectrum of an elliptic pseudo-differential operator of positive order.

Theorem 4.3. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then*

$$\Sigma_w(T_{\sigma,0}) = \varnothing.$$

Proof. Let $\lambda \in \mathbb{C}$. By Corollary 4.2, we need only to show that $\sigma - \lambda$ is elliptic. The ellipticity of σ, implies that there exist constants $C, R > 0$ such that

$$|\sigma(\theta, n) - \lambda| \geq C(1 + |n|)^m - |\lambda| = (1 + |n|)^m (C - \frac{|\lambda|}{(1 + |n|)^m}),\ \theta \in [-\pi, \pi],$$

whenever $|n| \geq R$. Since $(1 + |n|)^m \to \infty$ as $|n| \to \infty$, there exists $M > 0$ such that

$$|\sigma(\theta, n) - \lambda| \geq \frac{C}{2}(1 + |n|)^m, \quad |n| \geq M,\ \theta \in [-\pi, \pi],$$

which implies that $\sigma - \lambda$ is elliptic. □

Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m \geq 0$. Then the following theorem is a result on the essential spectra of the bounded pseudo-differential operator T_σ with the domain $H^{m,p}$ on $L^p(\mathbb{S}^1)$.

Theorem 4.4. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m \geq 0$. Then for T_σ on $L^p(\mathbb{S}^1)$ with the domain $H^{m,p}$, $1 < p < \infty$, we have*

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_i\},$$

where

$$L_i = \liminf_{|n| \to \infty} \{(\inf_{\theta \in [-\pi,\pi]} |\sigma(\theta, n)|)(1 + |n|)^{-m}\}.$$

Proof. Let $\lambda \in \mathbb{C}$ be such that $|\lambda| < L_i$. Then there exists $\epsilon > 0$ such that

$$|\lambda| + \epsilon < L_i.$$

Since $m \geq 0$, it follows that $|\lambda| < (L_i - \epsilon)(1 + |n|)^m$. On the other hand, there exists a positive constant R such that

$$\inf_{|n| \geq R} \{(\inf_{\theta \in [-\pi,\pi]} |\sigma(n, \theta)|)(1 + |n|)^{-m}\} > L_i - \frac{\epsilon}{2}.$$

So, for $|n| \geq R$,

$$\begin{aligned}|\sigma(\theta, n) - \lambda| &\geq |\sigma(\theta, n)| - |\lambda| \\ &> (L_i - \frac{\epsilon}{2} - L_i + \epsilon)(1 + |n|)^m \\ &= \frac{\epsilon}{2}(1 + |n|)^m, \quad \theta \in [-\pi, \pi].\end{aligned}$$

Therefore, $\sigma - \lambda$ is elliptic and hence $T_\sigma - \lambda I : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ with domain $H^{m,p}$ is Fredholm. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| < L_i\} \subseteq \Phi_w(T_\sigma),$$

which implies that

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_i\}. \qquad \square$$

We have the following theorem on the essential spectrum of a pseudo-differential operator of order 0 from $L^p(\mathbb{S}^1)$ into $L^p(\mathbb{S}^1)$.

Theorem 4.5. *Let $\sigma \in S^0(\mathbb{S}^1 \times \mathbb{Z})$. Then for $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$, $1 < p < \infty$, we have*

$$\Sigma_s(T_\sigma) \subseteq \{\lambda : |\lambda| \leq L_s\},$$

where

$$L_s = \limsup_{|n| \to \infty} \{\sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)|\}.$$

Proof. Let $\lambda \in \mathbb{C}$ such that $|\lambda| > L_s$. Then there exists $\epsilon > 0$ such that

$$|\lambda| - \epsilon > L_s,$$

and there exists a positive number R such that

$$\sup_{|n| \geq R} \{\sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)|\} < L_s + \frac{\epsilon}{2}.$$

For all $|n| \geq R$,

$$\begin{aligned}|\sigma(\theta, n) - \lambda| &\geq |\lambda| - |\sigma(\theta, n)| \\ &> L_s + \epsilon - L_s - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2}, \quad \theta \in [-\pi, \pi].\end{aligned}$$

Hence $\sigma - \lambda$ is elliptic and by Theorem 4.1, $T_\sigma - \lambda I : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is Fredholm. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| > L_s\} \subseteq \Phi_w(T_\sigma),$$

which is the same as

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq L_s\}.$$

Since $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$ is a connected component of $\Phi_w(T_\sigma)$, it follows that $i(T_\sigma - \lambda I)$ is a constant for all λ in $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$. On the other hand,

$$\rho(T_\sigma) \cap \{\lambda \in \mathbb{C} : |\lambda| > L_s\} \neq \varnothing.$$