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Seminar on Stochastic Analysis, Random Fields and Applications VI

Centro Stefano Franscini, Ascona, May 2008

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Preface

This volume contains the Proceedings of the *Sixth Seminar on Stochastic Analysis, Random Fields and Applications*, which took place at the Centro Stefano Franscini (Monte Verità) in Ascona (Ticino), Switzerland, from May 19 to 23, 2008. *All papers in this volume have been refereed.*

The previous five editions of this conference occurred in 1993, 1996, 1999, 2002 and 2005. This Seminar is a periodically occurring event that attempts to present a partial state of the art in stochastic analysis and certain related fields, both theoretical and applied. The theoretical topics of the conference included infinite-dimensional diffusions and multi-parameter random fields; among the applied topics, significant attention was given to fluid mechanics and mathematical finance, but also to financial issues related to energy management and to the impact of climate variations. In view of the timeliness and importance of this last subject, the meeting was honored by the presence and opening address of On. Marco Borradori, president of the State Council of Ticino (the executive branch of the government of the Italian-speaking canton of Switzerland), who was also in charge of the *Department of Territorio* and whose responsibilities include energy issues.

As was to be expected, an important area of investigation by the Seminar speakers is infinite-dimensional stochastic calculus, which includes fundamental questions such as pathwise uniqueness and uniqueness in law for stochastic partial differential equations, including not only wave and heat equations but also Navier-Stokes and many other equations; in relation to such equations, large deviations estimates, ergodicity results, and perturbations by fractal noise were discussed. Related subjects included infinite-dimensional backward stochastic differential equations, local times of random fields, and, of course, Malliavin calculus.

Malliavin calculus remains an important investigation technique, both with respect to existence, smoothness and estimates of densities of the laws of continuous or jump processes and random fields, and as a technique for stochastic integration with respect to non-semimartingale processes (or random fields). New promising applications appear however, in probabilistic potential theory and in statistics, for instance via generalizations of the classical *Stein's method.*

Multi-parameter processes and infinite-dimensional processes remain an important tool in mathematical finance: they appear naturally in the study of the term structure of interest rates and of other financial assets whose price depends on the present time t and some additional parameter such as a delivery time T

(such assets are also present in commodities and energy markets). Mathematical finance and stochastic analysis remain intimately connected: new stochastic volatility models are being considered, involving both continuous and jump diffusions; risk measures, hedging in incomplete markets, portfolio management with transaction costs, together with the formulation and study of general semimartingale (and even non-semimartingale) models, require extensions of the classical tools of stochastic analysis as well as the creation of new tools; new numerical techniques, which can be deterministic or probabilistic, are also required. In this last topic, substantial efforts have been devoted to simulating solutions of backward stochastic differential equations.

A phenomenon which has been the subject of much recent investigation is the impact of *microstructure noise*. Statistical and econometric tools are being implemented in order to model and analyze such noises using perturbations by classical Lévy or continuous diffusions. Other researchers analyze the robustness of Black-Scholes and related formulas under non log-normal assumptions while conserving the quadratic variation properties of the underlying. Quadratic variation becomes an important *approximatively observed* process related to the price process of a financial asset, and has motivated theoreticians and practitioners to introduce path-dependent options such as variance swaps, which are closely related to this quantity.

Applications of finite- and infinite-dimensional stochastic analysis arise in *climatology*, a science which has been the subject of several interdisciplinary research projects. One afternoon during the conference was devoted to climate and energy; this session was open to the general public. In addition to the address of On. Marco Borradori mentioned above, three presentations were aimed toward a wider audience:

- Prof. René Carmona (Princeton University) spoke on *The European Union emissions trading scheme from a mathematician's perspective*;
- Prof. Arturo Romer (Università della Svizzera Italiana) spoke (in French) on *Energie et environnement. Quel avenir?*
- Prof. Peter Imkeller (Humboldt-Universität Berlin) lectured on *Mathematical challenges of managing energy and weather risk*.

Significant financial support for this meeting was provided by the Fonds National Suisse pour la Recherche Scientifique (Berne), the Centro Stefano Franscini (ETH-Zürich), and the Ecole Polytechnique Fédérale de Lausanne (EPFL). We take this opportunity to thank these institutions.

May 2010

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Stochastic Analysis and Random Fields

The Trace Formula for the Heat Semigroup with Polynomial Potential

Sergio Albeverio and Sonia Mazzucchi

Abstract. We consider the heat semigroup $e^{-\frac{t}{\hbar}H}$, $t > 0$, on $\mathbb{R}^d$ with generator H corresponding to a potential growing polynomially at infinity. Its trace for positive times is represented as an analytically continued infinite-dimensional oscillatory integral. The asymptotics in the small parameter $\hbar$ is exhibited by using Laplace's method in infinite dimensions in the case of a degenerate phase (this corresponds to the limit from quantum mechanics to classical mechanics, in a situation where the Euclidean action functional has a degenerate critical point).

Mathematics Subject Classification (2000). 35K05, 11F72, 28C20, 35C15, 35C20.

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1. Introduction

The study of the asymptotic behavior in the limit $\lambda \downarrow 0$ of infinite-dimensional integrals of the form

$$\int_{\mathcal{B}} e^{\frac{F(\lambda x)}{\lambda^2}} G(\lambda x) d\mu(x) \tag{1.1}$$

(where λ is a real positive parameter, μ a Gaussian measure on a Banach space $\mathcal{B}$, F, G Borel measurable functionals on $\mathcal{B}$) by means of an infinite-dimensional version of the Laplace method is a classical topic of investigation. The first results were obtained by Schilder [31] for the asymptotics of classical Wiener integrals, where $\mathcal{B}$ is the space of continuous functions with the sup norm and μ is the Wiener measure. Schilder's main theorems were generalized by Pincus [26] to the case of more general Gaussian functional integrals, by Kallianpur and Oodaira [21] in an abstract Wiener space setting, and by Ben Arous [14] to the case of path space measures associated to stochastic differential equations. These results

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were successfully applied to the study of the asymptotics of the solution of some partial differential equation, see, e.g., [1, 22]. For some recent results see, e.g., [2, 10, 13, 17, 18, 25, 28, 30].

According to the Laplace method for integrals of the form (1.1), in the case where one is dealing with an abstract Wiener space $(i, \mathcal{H}, \mathcal{B}, \mu)$, the asymptotics should be determined by the maximum of the phase function $F(x) - \|x\|^2/2$, where $\| \, \|$ is the norm in the Hilbert space $\mathcal{H}$, i.e., the reproducing kernel Hilbert space of the Banach space $\mathcal{B}$. The simplest case is the one where there is a unique non degenerate maximum [31].

In this paper we are interested in the study the trace of the heat semigroup $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, and its asymptotics when $\hbar \downarrow 0$, in the case where H is the essentially self-adjoint operator on $C_0^\infty \subset L^2(\mathbb{R}^d)$ given on the functions $\phi \in C_0^\infty$ by

$$H\phi(x) = \Big(- \frac{\hbar^2}{2}\Delta_x + V(x)\Big)\phi(x), \qquad x \in \mathbb{R}^d, \tag{1.2}$$

where $\hbar > 0$ and V is a polynomially growing potential of the form $V(x) = |x|^{2N}$. H can be interpreted as a Schrödinger Hamiltonian, (in which case $\hbar$ is the reduced Planck constant) and consequently $e^{-\frac{t}{\hbar}H}$, $t > 0$, as a Schrödinger semigroup.

In recent years a particular interest has been devoted to the study of the trace of the heat semigroup and of the Schrödinger group $e^{-\frac{it}{\hbar}H}$, $t \in \mathbb{R}$, (related to the heat semigroup by analytic continuation in the "time variable" t) and their asymptotics in the "semiclassical limit $\hbar \downarrow 0$" (see also [15] for a related problem). In particular one is interested in the proof of a trace formula of Gutzwiller's type, relating the asymptotics of the trace of the Schrödinger group and the spectrum of the quantum mechanical energy operator H with the classical periodic orbits of the system. Gutzwiller's heuristic trace formula, which is a basis of the theory of quantum chaotic systems, is the quantum mechanical analogue of Selberg's trace formula, relating the spectrum of the Laplace-Beltrami operator on manifolds with constant negative curvature with the periodic geodesics.

In the case where the potential V is the sum of an harmonic oscillator part and a bounded perturbation V_0 which can be written as the Fourier transform of a complex bounded variation measure on $\mathbb{R}^d$, some rigorous results on the asymptotics of the trace of the Schrödinger group and the heat semigroup have been obtained in [3, 4] by means of an infinite-dimensional version of the stationary phase method for infinite-dimensional oscillatory integrals (see [9] for a review of this topic).

In this paper we extend some of the results of [4] concerning the heat semigroup to the case where the potential has a polynomial growth at infinity, by proving an infinite-dimensional integral representation of the trace of $e^{-\frac{t}{\hbar}H}$, $t > 0$, and by studying its asymptotics when $\hbar \to 0$. This corresponds to exhibiting the detailed behavior of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ "near the classical limit". The difficulties present in the case we handle are twofold. First of all the polynomial growth of the potential $V(x)$ does not allow a direct application of the classical results on the asymptotic expansion for infinite-dimensional integrals [9, 31]. Moreover the maximum of the phase function is degenerate. To handle the degeneracy we prove a functional integral representation for $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ (formula (4.3) below). Such a representation is particularly flexible to handle and allows to reduce the study of the degeneracy to the study of the asymptotics of a finite-dimensional integral.

In Sections 2 and 3 we recall the definitions and the main results on abstract Wiener spaces, as well as on infinite-dimensional (oscillatory) integrals and the relations between them. In Section 4 we prove an infinite-dimensional integral representation for the trace of the heat semigroup $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, with H given by (1.2). In Section 5 we study the detailed behavior of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, for $\hbar \downarrow 0$.

2. Asymptotics of integrals on abstract Wiener spaces

In this section we recall some classical results on the Laplace method on abstract Wiener spaces.

Let $\mathcal{H}$ be a real separable infinite-dimensional Hilbert space, with inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$. Let ν be the finitely additive cylinder measure on $\mathcal{H}$, defined by its characteristic functional $\hat{\nu}(x) = e^{-\frac{\hbar}{2}\|x\|^2}$. Let $|\ |$ be a "measurable" norm on $\mathcal{H}$ in the sense of L. Gross [19, 23], that is $|\ |$ is such that for every $\epsilon > 0$ there exist a finite-dimensional projection $P_\epsilon : \mathcal{H} \to \mathcal{H}$, such that for all $P \perp P_\epsilon$ one has

$$\nu(\{x \in \mathcal{H}|\ |P(x)| > \epsilon\}) < \epsilon,$$

where P and P_ϵ are called orthogonal ($P \perp P_\epsilon$) if their ranges are orthogonal in $(\mathcal{H}, \langle\ ,\ \rangle)$. One can easily verify that $|\ |$ is weaker than $\|\ \|$. Denoted by $\mathcal{B}$ the completion of $\mathcal{H}$ in the $|\ |$-norm and by i the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$, one can prove that $\mu \equiv \nu \circ i^{-1}$ is a countably additive Gaussian measure on the Borel subsets of $\mathcal{B}$. The triple $(i, \mathcal{H}, \mathcal{B})$ is called an *abstract Wiener space* [19, 23]. Given $y \in \mathcal{B}^*$ one can easily verify that the restriction of y to $\mathcal{H}$ is continuous on $\mathcal{H}$, so that one can identify $\mathcal{B}^*$ as a subset of $\mathcal{H}$. Moreover $\mathcal{B}^*$ is dense in $\mathcal{H}$ and we have the dense continuous inclusions $\mathcal{B}^* \subset \mathcal{H} \subset \mathcal{B}$. Each element $y \in \mathcal{B}^*$ can be regarded as a random variable $n(y)$ on $(\mathcal{B}, \mu)$. A direct computation shows that $n(y)$ is normally distributed, with covariance $|y|^2$. More generally, given $y_1, y_2 \in \mathcal{B}^*$, one has

$$\int_B n(y_1)n(y_2)d\mu = \langle y_1, y_2\rangle.$$

The latter result allows the extension to the map $n : \mathcal{H} \to L^2(\mathcal{B}, \mu)$, because $\mathcal{B}^*$ is dense in $\mathcal{H}$. Given an orthogonal projection P in $\mathcal{H}$, with

$$P(x) = \sum_{i=1}^{n} \langle e_i, x\rangle e_i$$

for some orthonormal $e_1, \ldots, e_n \in \mathcal{H}$, the stochastic extension $\tilde{P}$ of P on $\mathcal{B}$ is well defined by

$$\tilde{P}(\,\cdot\,) = \sum_{i=1}^{n} n(e_i)(\,\cdot\,)e_i.$$

Given a function $f : \mathcal{H} \to \mathcal{B}_1$, where $(\mathcal{B}_1, \| \|_{\mathcal{B}_1})$ is another real separable Banach space, the stochastic extension $\tilde{f}$ of f to $\mathcal{B}$ exists if the functions $f \circ \tilde{P} : \mathcal{B} \to \mathcal{B}_1$ converge to $\tilde{f}$ in probability with respect to μ as P converges strongly to the identity in $\mathcal{H}$. If $g : \mathcal{B} \to \mathcal{B}_1$ is continuous and $f := g|_{\mathcal{H}}$, then one can prove [19] that the stochastic extension of f is well defined and it is equal to g μ-a.e.

Let us denote the norm of the embedding $i : (\mathcal{H}, \| \|) \to (\mathcal{B}, | |)$ by $c > 0$. The following holds [21, 31]:

Theorem 2.1. *Let the functions F, G in* (1.1) *satisfy the following assumptions:*

1. $\exists L_1 \in \mathbb{R}$, $\exists L_2 \in (0, 1/2c^2)$ *such that* $\forall x \in \mathcal{B}$ $F(x) \leq L_1 + L_2|x|^2$.
2. $\exists K_1, K_2 > 0$ *such that for μ-a.e.* $x \in \mathcal{B}$ $|G(x)| \leq K_1 e^{K_2|x|^2}$.
3. $\exists \gamma \in \mathcal{H}$ *such that* $F(\gamma) - \|\gamma\|^2/2 > F(x) - \|x\|^2/2$, $\forall x \in \mathcal{H} \setminus \{\gamma\}$.
4. *F is uniformly continuous on every bounded subset of $\mathcal{B}$.*
5. *G is continuous at γ (with γ as in* (3)*).*

Then

$$\lim_{\lambda \downarrow 0} \frac{\int_{\mathcal{B}} e^{\frac{F(\lambda x)}{\lambda^2}} G(\lambda x) d\mu(x)}{\int_{\mathcal{B}} e^{\frac{F(\lambda x)}{\lambda^2}} d\mu(x)} = G(\gamma).$$

Remark 2.2. Condition 1 means that the allowed phase functions can have at most quadratic growth at infinity.

Condition 3 means that the phase function $x \mapsto F(x) - \|x\|^2/2$ possess a maximum which is achieved in one and only one point $\gamma \in \mathcal{H}$.

Remark 2.3. Theorem 2.1 can be extended to the case where the phase function $E(x) = F(x) - \|x\|^2/2$ has a discrete set of non degenerate local maxima, i.e., if the function F is two times Fréchet differentiable in a neighborhood of any maximum γ of the phase function E and the kernel of the second Fréchet derivative of E at γ is trivial. This is so because one can use a decomposition of the unit to "localize", see, e.g., [12, 13, 27, 28].

3. Infinite-dimensional integrals

In the present section we recall some results on analytic continuation of infinite-dimensional oscillatory integrals and their relations with abstract Wiener spaces. Let $(\mathcal{H}, \langle \ , \ \rangle, \| \|)$ be a real separable infinite-dimensional Hilbert space, s a complex number such that $\mathrm{Re}(s) \geq 0$, $g : \mathcal{H} \to \mathbb{C}$ a Borel function. The infinite-dimensional integral

$$I(s) = \widetilde{\int_{\mathcal{H}}} e^{-\frac{s}{2}\|x\|^2} g(x) dx$$

is defined in the following way [4, 16]:

Definition 3.1. A Borel measurable function $g : \mathcal{H} \to \mathbb{C}$ is called $\mathcal{F}^s$ integrable if for each sequence $\{P_n\}_{n \in \mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to I$ strongly as $n \to \infty$ (I being the identity operator in $\mathcal{H}$), the finite-dimensional approximations of the oscillatory integral of f, with parameter s,

$$\mathcal{F}^s_{P_n}(g) = \frac{\int_{P_n\mathcal{H}} e^{-\frac{s}{2}\|P_n x\|^2} g(P_n x) d(P_n x)}{\int_{P_n\mathcal{H}} e^{-\frac{s}{2}\|P_n x\|^2} d(P_n x)} \tag{3.1}$$

are well defined and the limit $\lim_{n\to\infty} \mathcal{F}^s_{P_n}(g)$ exists and is independent of the sequence $\{P_n\}$.

In this case the limit is called the infinite-dimensional oscillatory integral of g with parameter s and is denoted by

$$\widetilde{\int_{\mathcal{H}}} e^{-\frac{s}{2}\|x\|^2} g(x) dx.$$

Strictly speaking $I(s)$ has an oscillatory behavior only for s being a purely imaginary number. In this case, if $g \circ P_n$ is not summable on $P_n\mathcal{H}$, the finite-dimensional approximations in equation (3.1) have to be suitably defined as limits of regularized integrals (see [6, 9, 16, 20]). For the applications we have in mind we are interested in the case where s is real positive, $s = 1/\hbar$, $\hbar > 0$.

Let us recall some well-known results on infinite-dimensional oscillatory integrals.

Theorem 3.2 (Fubini theorem). *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ decompose into the direct sum of two closed and mutually orthogonal subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then*

$$\widetilde{\int_{\mathcal{H}}} e^{-\frac{s}{2}\|x\|^2} g(x) dx = \widetilde{\int_{\mathcal{H}_1}} e^{-\frac{s}{2}\|x_1\|^2} \Big(\widetilde{\int_{\mathcal{H}_2}} e^{-\frac{s}{2}\|x_2\|^2} g(x_1 + x_2) dx_2 \Big) dx_1.$$

Let $\mathcal{H}$ be a Hilbert space with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$. Let also $\|\cdot\|$ be an equivalent norm on $\mathcal{H}$ with scalar product denoted by $\langle\cdot,\cdot\rangle$. Let us denote the new Hilbert space by $\tilde{\mathcal{H}}$. Let us assume moreover that

$$\langle x_1, x_2\rangle = (x_1, x_2) + (x_1, Lx_2), \qquad x_1, x_2 \in \tilde{\mathcal{H}}$$

$$\|x\|^2 = |x|^2 + (x, Lx), \qquad x \in \tilde{\mathcal{H}}$$

where L is a self-adjoint trace class operator on $\mathcal{H}$. The following two theorems hold (see [4, 5]):

Theorem 3.3. *Let $f : \mathcal{H} \to \mathbb{C}$ be a Borel function. f is integrable on $\mathcal{H}$ (in the sense of Definition* 3.1*) if and only if f is integrable on $\tilde{\mathcal{H}}$ and in this case*

$$\widetilde{\int_{\tilde{\mathcal{H}}}} e^{-\frac{s}{2}|x|^2} f(x) dx = \det(I + L)^{1/2} \widetilde{\int_{\mathcal{H}}} e^{-\frac{s}{2}|x|^2} f(x) dx. \tag{3.2}$$

In the case where $s \in \mathbb{R}$, $s > 0$ and the Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle, \|\ \|)$ is an element of an abstract Wiener space $(i, \mathcal{H}, \mathcal{B})$, it is possible to prove the following interesting relation between the infinite-dimensional oscillatory integral on $\mathcal{H}$ with parameter $s = 1$ and the Gaussian integral on the Banach space $\mathcal{B}$:

Theorem 3.4. *Let $g : \mathcal{B} \to \mathbb{C}$ be a continuous bounded function. Let $f : \mathcal{H} \to \mathbb{C}$ be the restriction of g to the Hilbert space $\mathcal{H}$. Then the stochastic extension of f is well defined, it is equal to g μ-a.e. and*

$$\widetilde{\int_{\mathcal{H}}} e^{-\frac{1}{2}\|x\|^2} f(x)dx = \int_{\mathcal{B}} \tilde{g}(x)d\mu(x).$$

Remark 3.5. A corresponding result holds for the case of infinite-dimensional oscillatory integrals with parameter $s > 1$.

4. The trace of the heat semigroup

In the present section we prove an infinite-dimensional integral representation for the trace of the heat semigroup $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, in the case where H is the quantum mechanical Hamiltonian given on the vectors $\phi \in S(\mathbb{R}^d)$ by

$$H\phi(x) = -\frac{\hbar^2}{2}\Delta_x\phi(x) + V(x)\phi(x), \tag{4.1}$$

where $V(x) = \lambda|x|^{2N}$, $\lambda > 0$, or, more generally, $V(x) = \lambda A_{2N}(x, x, \dots, x)$, where $A_{2N} : \times_{i=1}^{2N}\mathbb{R}^d \to \mathbb{R}$ is a completely symmetric, strictly positive $2N$-order covariant tensor on $\mathbb{R}^d$. Below we shall write explicit formulae for the case $V(x) = \lambda|x|^{2N}$, but all formulae can be easily adapted to the case $V(x) = \lambda A_{2N}(x, x, \dots, x)$.

It is well known that H is an essentially self adjoint operator on $C_0^\infty(\mathbb{R}^d)$ (see [29], Theorem X.28). H is a positive operator and is the generator of an analytic semigroup, denoted by $e^{-\frac{t}{\hbar}H}$, $t \geq 0$, moreover its trace (see, e.g., [32, 33]) is given, for $t > 0$ by:

$$\begin{aligned}\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] &= \int_{\mathbb{R}^d} \frac{dx}{(2\pi t)^{d/2}} \int_{C_{[0,t]}} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\alpha(s)+\sqrt{\hbar}x)ds} d\mu(\alpha) \\ &= \int_{\mathbb{R}^d} \frac{dx}{(2\pi t)^{d/2}} \int_{C_{[0,t]}} e^{-\lambda\hbar^{N-1}\int_0^t |\alpha(s)+x|^{2N}ds} d\mu(\alpha)\end{aligned} \tag{4.2}$$

where $C_{[0,t]}$ is the space of continuous paths $\alpha : [0, t] \to \mathbb{R}^d$ such that $\alpha(0) = \alpha(t)$ and μ is the Brownian Bridge probability measure on it.

Let us introduce the Hilbert space $Y_{0,t}$,

$$Y_{0,t} := \{\gamma \in H^1(0, t; \mathbb{R}^d) : \ \gamma(0) = \gamma(t) = 0\}$$

with norm

$$|\gamma|^2 = \int_0^t \dot{\gamma}(s)^2 ds.$$

$(i, Y_{0,t}, C_{[0,t]})$ is an abstract Wiener space.

Let us introduce the Hilbert spaces $Y_{p,t}$ and $\mathcal{H}_{p,t}$, given by

$$Y_{p,t} := \{\gamma \in H^1(0, t; \mathbb{R}^d) : \ \gamma(0) = \gamma(t) = 0\},$$
$$\mathcal{H}_{p,t} := \{\gamma \in H^1(0, t; \mathbb{R}^d) : \ \gamma(0) = \gamma(t)\},$$

both with norm

$$\|\gamma\|^2 = \int_0^t \dot{\gamma}(s)^2 ds + \int_0^t \gamma(s)^2 ds.$$

The following holds.

Theorem 4.1. *The function* $f : \mathcal{H}_{p,t} \to \mathbb{R}$ *given by*

$$f(\gamma) := e^{\frac{1}{2}\int_0^t \gamma(s)^2 ds - \lambda \hbar^{N-1} \int_0^t \gamma(s)^{2N} ds}, \qquad \gamma \in \mathcal{H}_{p,t}$$

is $\mathcal{F}^1$*-integrable on* $\mathcal{H}_{p,t}$ *in the sense of Definition* 3.1*. Moreover the trace of the heat semigroup* $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$*,* $t > 0$ *for* H *as in equation* (4.1) *is given by*

$$\begin{aligned} \mathrm{Tr}[e^{-\frac{t}{\hbar}H}] &= (2\cosh t - 2)^{-d/2} \widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\|\gamma\|^2} f(\gamma) d\gamma \\ &= (2\cosh t - 2)^{-d/2} \widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\int_0^t \dot{\gamma}(s)^2 ds - \lambda \hbar^{N-1} \int_0^t \gamma(s)^{2N} ds} d\gamma. \end{aligned} \tag{4.3}$$

Proof. The proof of (4.3) is divided into 3 steps.

1*st Step:* First of all, by Theorem 3.4, the integral in (4.2) on $C_{[0,t]}$ with respect to the Brownian bridge measure can be written in terms on an infinite-dimensional integral on the Hilbert space $Y_{0,t}$:

$$\int_{C_{[0,t]}} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\alpha(s)+\sqrt{\hbar}x) ds} d\mu(\alpha) = \widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x) ds} d\gamma,$$

so that

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = \int_{\mathbb{R}^d} \frac{dx}{(2\pi t)^{d/2}} \widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x) ds} d\gamma. \tag{4.4}$$

2*nd Step:* By the transformation formula relating infinite-dimensional integrals on Hilbert spaces with varying norms (Theorem 3.3), we get a relation between the integral on $Y_{0,t}$ and the integral on $Y_{p,t}$. Indeed

$$\|\gamma\|^2 = |\gamma|^2 + (\gamma, L\gamma),$$

where L is the unique self-adjoint trace class operator on $Y_{0,t}$ defined by the quadratic form

$$(\gamma_1, L\gamma_2) = \int_0^t \gamma_1(s)\gamma_2(s) ds.$$

Indeed (see [4] for details) $\eta = L\gamma$, $\gamma \in Y_{0,t}$ if and only if

$$\begin{cases} \ddot{\eta}(s) + \gamma(s) = 0, & s \in [0,t] \\ \dot{\eta}(0) = 0 \\ \dot{\eta}(t) = 0 \end{cases}$$

and $\det(I+L)=\left(\frac{\sinh t}{t}\right)^d$. By inserting this into equation (3.2) we obtain:

$$\widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma$$

$$= \left(\frac{t}{\sinh t}\right)^{d/2} \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma$$

and by equation (4.4)

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = \int_{\mathbb{R}^d} \frac{dx}{(2\pi\sinh t)^{d/2}} \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma.$$

3rd Step: The final step is a transformation of variable formula for integrals on the Hilbert space $\mathcal{H}_{p,t}$. $Y_{p,t}$ can be regarded as a subspace of $\mathcal{H}_{p,t}$ and any vector $\gamma\in\mathcal{H}_{p,t}$ can be written as a sum of a vector $\eta\in Y_{p,t}$ and a constant in the following way:

$$\gamma(s)=\eta(s)+x, \qquad s\in[0,t], \gamma\in\mathcal{H}_{p,t}, \eta\in Y_{p,t}, x=\gamma(0).$$

We have to compute a constant C_t such that for integrable functions f

$$\widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\|\gamma\|^2} f(\gamma)d\gamma = C_t\int_{\mathbb{R}^d} dx \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+x\|^2} f(\eta+x)d\eta.$$

By the Fubini theorem 3.2

$$\widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\|\gamma\|^2} f(\gamma)d\gamma = \int_{Y_{p,t}^{\perp}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+\xi\|^2} f(\eta+\xi)d\eta\Big)d\xi \tag{4.5}$$

where $Y_{p,t}^{\perp}$ is the space orthogonal to $Y_{p,t}$ in $\mathcal{H}_{p,t}$. One can easily verify that $Y_{p,t}^{\perp}$ is d-dimensional and it is generated by the vectors $\{v_i\}_{i=1,\dots,d}$, with $v_i(s)=\hat{e}_i\left(\frac{e^s(1-e^{-t})+e^{-s}(e^t-1)}{2\sqrt{2}\sqrt{\sinh t(\cosh t-1)}}\right)$, $s\in[0,t]$, $\hat{e}_i$ being the ith vector of the canonical basis in $\mathbb{R}^d$. The right-hand side of (4.5) is equal to

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+\sum_i y_i v_i\|^2} f(\eta+\sum_i y_i v_i)d\eta\Big)dy,$$

where $\xi(s)=\sum_i y_i v_i(s)$, $i=1,\dots,d$. By writing the finite-dimensional approximation of $\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+\sum_i y_i v_i\|^2} f(\eta+\sum_i y_i v_1)d\eta$, by the formula for the change of variables in finite-dimensional integrals and by noticing that

$$\langle u_j, v_i\rangle_{\mathcal{H}_{p,t}} = \delta_i^j \frac{\sqrt{2\cosh t-2}}{\sqrt{\sinh t}},$$

where $u_j \in \mathcal{H}_{p,t}$ is the vector given by $u_j(s) = \hat{e}_j$, $s \in [0,t]$, we get

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta + \sum_i y_i v_i\|^2} f(\eta + \sum_i y_i v_i) d\eta \Big) dy$$
$$= \Big(\frac{\sqrt{2\cosh t - 2}}{\sqrt{\sinh t}} \Big)^d \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta + \sum_i x_i u_i\|^2} f(\eta + \sum_i x_i u_i) d\eta \Big) dx$$

so that the constant C_t is equal to $\left(\frac{\sqrt{2\cosh t - 2}}{\sqrt{2\pi \sinh t}} \right)^d$.

By combining these results we get equation (4.3). □

Remark 4.2. In [4, 7] the equality (4.3) is proved for the case where V is a quadratic function plus a bounded perturbation (which is Fourier transform of a complex measure) by means of a different technique (a Fubini theorem for infinite-dimensional oscillatory integrals with respect to non-degenerate quadratic forms), that cannot be applied in the case of our Hamiltonian with potential V having polynomial growth. Indeed the quadratic part of the phase function appearing in the integral on the right-hand side of (4.3) can be written as

$$\int_0^t \dot{\gamma}^2(s) ds = \langle \gamma, T\gamma \rangle,$$

with $T : \mathcal{H}_{p,t} \to \mathcal{H}_{p,t}$ a self-adjoint operator. One can verify (see the next section) that T is not invertible and $\det T = 0$. This fact forbids the application of the Fubini theorem as stated in [4, 7] and a direct application of the methods of [4, 7].

5. The detailed behavior of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ for $\hbar \downarrow 0$

The present section is devoted to the study of the asymptotic behavior of the integral

$$I(\hbar) := \widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\int_0^t \dot{\gamma}(s)^2 ds - \frac{\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\gamma(s)|^{2N} ds} d\gamma \tag{5.1}$$

in the limit $\hbar \downarrow 0$ (for $t > 0$, $\lambda > 0$).

Integral (5.1) can be written as $\widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{\hbar}\Phi(\gamma)} d\gamma$, where the phase function $\Phi : \mathcal{H}_{p,t} \to \mathbb{R}$ is given by

$$\Phi(\gamma) = \frac{1}{2}\int_0^t \dot{\gamma}(s)^2 ds + \lambda \int_0^t |\gamma(s)|^{2N} ds.$$

According to the inspiration coming from the finite-dimensional Laplace method, the asymptotic behavior of $I(\hbar)$ should be determined by the stationary points of the phase functional Φ, i.e., the points such that

$$\Phi'(\gamma)(\phi) = 0, \qquad \forall \phi \in \mathcal{H}_{p,t},$$

Φ' being the Fréchet derivative. For $\gamma, \phi \in \mathcal{H}_{p,t}$ we have

$$\Phi'(\gamma)(\phi) = \int_0^t \dot{\gamma}(s)\dot{\phi}(s)ds + 2N\lambda \int_0^t |\gamma(s)|^{2N-2}\gamma(s)\phi(s)ds. \tag{5.2}$$

A function $\gamma \in \mathcal{H}_{p,t}$ is a stationary point of Φ iff γ is a solution of the following boundary value problem:

$$\begin{cases} \ddot{\gamma}(s) - 2N\lambda|\gamma(s)|^{2N-2}\gamma(s) = 0, & s \in [0,t] \\ \gamma(0) = \gamma(t) \\ \dot{\gamma}(0) = \dot{\gamma}(t) \end{cases} \tag{5.3}$$

that is a solution of the equation $\ddot{\eta} = \nabla V(\eta)$ with period t. (We remark that (5.3) is the equation of motion of a classical particle moving in a potential $-V$.) Indeed if γ satisfies (5.3), then by the regularity of the solutions of elliptic equations, we have $\gamma \in H^2([0,t],\mathbb{R}^d)$, and integrating by parts in formula (5.2), it is easy to see that $\Phi'(\gamma) = 0$. Conversely, if $\Phi'(\gamma) = 0$, then for any $\phi \in C_0^\infty([0,t],\mathbb{R}^d)$, one has:

$$-\int_0^t \gamma(s)\ddot{\phi}(s)ds + 2N\lambda \int_0^t |\gamma(s)|^{2N-2}\gamma(s)\phi(s)ds = 0$$

and γ is a weak solution of

$$\ddot{\gamma}(s) - 2N\lambda|\gamma(s)|^{2N-2}\gamma(s) = 0. \tag{5.4}$$

As $\gamma \in H^1([0,t],\mathbb{R}^d)$, the regularity theory implies that $\gamma \in H^2([0,t],\mathbb{R}^d)$ and equation (5.4) is satisfied in the strong sense. By taking $\phi \in C^1([0,t],\mathbb{R}^d)$, with $\phi(0) \neq 0$, and integrating by parts (exploiting the regularity of γ) one obtains that

$$(\dot{\gamma}(t) - \dot{\gamma}(0))\phi(0) = 0.$$

As it is easily seen, the trivial path 0 (i.e., $\eta(s) = 0$, $\forall s \in [0,t]$) is a solution of (5.3) and the function $\gamma \mapsto \Phi(\gamma)$ has a minimum which is achieved only in 0 (i.e., the minimum is the constant path $\gamma(s) = 0$ for all $s \in [0,t]$).

Analogously as we computed for (5.2) we obtain

$$\langle \Phi''(\gamma)(\phi), \psi \rangle = \int_0^t \dot{\phi}(s)\dot{\psi}(s)ds + 2N(2N-1)\lambda \int_0^t |\gamma(s)|^{2N-2}\phi(s)\psi(s)ds,$$

in particular

$$\langle \Phi''(0)(\phi), \psi \rangle = \int_0^t \dot{\phi}(s)\dot{\psi}(s)ds. \tag{5.5}$$

Let

$$\langle \Phi''(0)(\phi), \psi \rangle = \langle \phi, (I+L)\psi \rangle,$$

where L is the unique self-adjoint operator on $\mathcal{H}_{p,t}$ defined by the quadratic form

$$\langle \phi, L\psi \rangle = -\int_0^t \phi(s)\psi(s)ds.$$

We easily see that L for any $\psi \in \mathcal{H}_{p,t}$ is given by:

$$L\psi(s) = \int_0^s \sinh(s-u)\psi(u)du - \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t \sinh(s-u)\psi(u)du + \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t \sinh(t+s-u)\psi(u)du.$$

The kernel of $I+L$ is given by the solution of the equation

$$\psi(s) + \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t (\sinh(t+s-u) - \sinh(s-u))\psi(u)du + \int_0^s \sinh(s-u)\psi(u)du = 0 \quad (5.6)$$

with the periodic condition $\psi(0)=\psi(t)$. By differentiating (5.6) twice, it is easy to see that if ψ satisfies (5.6) then

$$\ddot{\psi}(s) = 0, \qquad \forall s \in [0,t],$$

so that the only solutions of (5.6) satisfying the periodic condition $\psi(0)=\psi(t)$ are the constant paths. From (5.5) the kernel of $\Phi''(0)$ is the d-dimensional subspace:

$$Ker[\Phi''(0)] = \{\gamma \in \mathcal{H}_{p,t} : \ \gamma(s) = x \ \forall s \in [0,t], \ x \in \mathbb{R}^d\}.$$

As the stationary point $\eta \equiv 0$ of the phase functional is degenerate, the classical theorem for asymptotic expansions of Gaussian integrals on abstract Wiener spaces (see Theorem 2.1) cannot be directly applied to the integral occurring in (5.1) and we have to study the asymptotic behavior of $I(\hbar)$ for $\hbar \downarrow 0$ by using a different method.

Let us decompose the Hilbert space $\mathcal{H}_{p,t}$ into the direct sum $\mathcal{H}_{p,t} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 = Ker[\Phi''(0)]$ and $\mathcal{H}_2 = Ker[\Phi''(0)]^\perp$. In particular

$$\mathcal{H}_2 = \left\{\gamma \in \mathcal{H}_{p,t} \ : \ \int_0^t \gamma(s)ds = 0\right\}.$$

By Theorem 3.2

$$\widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\int_0^t \dot{\gamma}(s)^2 ds - \frac{\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\gamma(s)|^{2N} ds} d\gamma = \widetilde{\int_{\mathcal{H}_1}}\widetilde{\int_{\mathcal{H}_2}} e^{-\frac{1}{2}\int_0^t \dot{\gamma}_2(s)^2 ds - \frac{\lambda}{\hbar}\int_0^t |\sqrt{\hbar}(\gamma_1(s)+\gamma_2(s)|^{2N} ds} d\gamma_2 d\gamma_1$$

where $\gamma(s) = \gamma_1(s) + \gamma_2(s)$, $\gamma_1(s) = t^{-1}\int_0^t \gamma(s)ds$, $\gamma_2(s) = \gamma(s) - \gamma_1(s)$.

By putting $x := \sqrt{\hbar}\gamma_1$ and expanding the term $|\sqrt{\hbar}\gamma_2(s) + x|^{2N}$ we have

$$I(\hbar) = \left(\frac{2\pi\hbar}{t}\right)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{t\lambda}{\hbar}|x|^{2N}} f(x,\hbar)dx,$$

where

$$f(x,\hbar)=\widetilde{\int_{\mathcal{H}_2}} e^{-(\frac{1}{2}\int_0^t \dot\gamma_2(s)^2ds+\frac{\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\gamma_2(s)+x|^{2N}ds-\frac{\lambda t}{\hbar}|x|^{2N})}d\gamma_2.$$

The asymptotic behavior of $f(x,\hbar)$ as $\hbar\downarrow 0$ can be simply determined by expanding the integrand in powers of $\hbar$. Indeed

$$f(x,\hbar)=\int_{\mathcal{H}_2} e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle}e^{-\frac{\lambda}{\hbar}P_{2N}(x,\sqrt{\hbar}\gamma_2)}d\gamma_2,$$

where $L_x:\mathcal{H}_2\to\mathcal{H}_2$ is the unique bounded self-adjoint operator determined by the quadratic form

$$\begin{aligned}\langle\phi,(I+L_x)\psi\rangle=&\int_0^t\dot\phi(s)\dot\psi(s)ds+2N\lambda|x|^{2N-2}\int_0^t\phi(s)\psi(s)ds\\&+4N(N-1)\lambda|x|^{2N-4}\int_0^t x\phi(s)\,x\psi(s)ds,\qquad\phi,\psi\in\mathcal{H}_2,\end{aligned}$$

and one can easily see that L_x is given by

$$\begin{aligned}L_x\psi(s)=B\int_0^s\sinh(u-s)\psi(u)du&+\frac{B}{(1-e^t)(1-e^{-t})}\int_0^t\sinh(s-u)\psi(u)du\\&-\frac{B}{(1-e^t)(1-e^{-t})}\int_0^t\sinh(t+s-u)\psi(u)du,\end{aligned}$$

where B is the $d\times d$ matrix defined by $B:=A^2(x)-1_{d\times d}$ and

$$A^2(x)_{i,j}=2N\lambda|x|^{2N-2}\delta_i^j+4N(N-1)\lambda|x|^{2N-4}x_ix_j,\qquad i,j=1,\dots,d.$$

Moreover

$$\begin{aligned}P_{2N}(x,\sqrt{\hbar}\gamma_2)=&\int_0^t|\sqrt{\hbar}\gamma_2(s)+x|^{2N}ds-t|x|^{2N}-2N|x|^{2N-2}\int_0^t\sqrt{\hbar}x\gamma_2(s)ds\\&-\hbar N|x|^{2N-2}\int_0^t|\gamma(s)|^2ds-2N(N-1)\hbar|x|^{2N-4}\int_0^t(x\gamma(s))^2ds\\:=&\,\hbar^{3/2}g(x,\hbar,\gamma_2)\end{aligned}\tag{5.7}$$

(we have used the fact that $\int_0^t\gamma_2(s)ds=0$ as $\gamma_2\in\mathcal{H}_2$), and for any x,γ_2 we have

$$\begin{aligned}\lim_{\hbar\downarrow 0}g(x,\hbar,\gamma_2)=&\frac{N!}{(N-3)!3!}8|x|^{2N-6}\int_0^t(x\gamma_2(s))^3ds\\&+2N(N-1)|x|^{2N-4}\int_0^t x\gamma_2(s)|\gamma_2(s)|^2ds.\end{aligned}$$

By expanding $e^{-\lambda\hbar^{1/2}g(x,\hbar,\gamma_2)}$ around $\hbar=0$ we have:

$$\begin{aligned}f(x,\hbar)&=\widetilde{\int_{\mathcal{H}_2}}e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle}e^{-\lambda\hbar^{1/2}g(x,\hbar,\gamma_2)}d\gamma_2\\&=f_1(x,\hbar)-\lambda\hbar^{1/2}f_2(x,\hbar),\end{aligned}\tag{5.8}$$

where

$$f_1(x,\hbar) = \widetilde{\int_{\mathcal{H}_2}} e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle} = \det(I+L_x)^{-1/2}$$

and

$$f_2(x,\hbar) = \widetilde{\int_{\mathcal{H}_2}} g(x,\hbar,\gamma_2)e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle} e^{-u\lambda\hbar^{1/2}g(x,\hbar,\gamma_2)}d\gamma_2 \tag{5.9}$$

with $u \in (0,1)$.

For the calculation of the spectrum $\sigma(L_x)$ of L_x, it is convenient to replace the standard basis of $\mathbb{R}^d$ by an orthonormal basis which diagonalizes the symmetric matrix $A^2(x)$. Denoting its eigenvalues a_i^2, $i = 1,\dots,d$, it is easy to verify that the spectrum of L_x is given by $\sigma(L_x) = \{\lambda_{i,n}, i = 1,\dots,d,\ n = 1,2,\dots\}$, where

$$\lambda_{i,n} = \frac{a_i^2 - 1}{1 + \frac{4\pi^2 n^2}{t^2}}, \qquad i = 1,\dots,d, \quad n = 1,2,\dots$$

are eigenvalues of multiplicity 2. By applying Lidskij's theorem [33] and the Hadamard factorization theorem (see [34], Theorem 8.24) one gets

$$\det(I+L_x) = \begin{cases} \det\left(\frac{\cosh(A(x)t)-1}{A^2(x)(\cosh t-1)}\right), & \text{for } x \neq 0 \\ (2\cosh t - 2)^{-d}, & \text{for } x = 0 \end{cases}$$

The next result follows easily by the integral representation (5.9) of the function f_2.

Lemma 5.1. *$f_2(x,\epsilon)$ is a C^∞ function of both $x \in \mathbb{R}^d$ and $\epsilon := \sqrt{\hbar} \in \mathbb{R}^+$. Moreover for any $x \in \mathbb{R}^d$, $f_2(x,0) = 0$ and $\lim_{\hbar\downarrow 0} \frac{f_2(x,\hbar)-f_2(x,0)}{\hbar^{1/2}} = C$, where C is a positive constant (depending on $x \in \mathbb{R}^d$).*

Proof. First of all we have

$$f_2(x,\hbar) = \widetilde{\int_{\mathcal{H}_2}} e^{\frac{u\lambda t|x|^{2N}}{\hbar}} g(x,\hbar,\gamma_2)e^{-\frac{1}{2}\int_0^t \dot\gamma_2^2(s)ds} e^{-\frac{u\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\gamma_2(s)+x|^{2N}ds}$$
$$e^{-\frac{1-u}{2}\left(2N|x|^{2N-2}\int_0^t|\gamma(s)|^2ds+4N(N-1)|x|^{2N-4}\int_0^t(x\gamma(s))^2ds\right)}d\gamma_2.$$

By expressing the infinite-dimensional integral on the Hilbert space $\mathcal{H}_2$ as an integral on the abstract Wiener space $(i,\mathcal{H}_2,\mathcal{B}_2)$ associated with $\mathcal{H}_2$ one gets:

$$f_2(x,\hbar) = e^{\frac{u\lambda t|x|^{2N}}{\hbar}} \int_{\mathcal{B}_2} \tilde g(x,\hbar,\omega_2)e^{\frac{1}{2}\langle\omega_2,L_0\omega_2\rangle} e^{-\frac{u\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega_2(s)+x|^{2N}ds}$$
$$e^{-\frac{1-u}{2}\left(2N|x|^{2N-2}\int_0^t|\omega_2(s)|^2ds+4N(N-1)|x|^{2N-4}\int_0^t(x\omega_2(s))^2ds\right)}d\mu(\omega_2), \tag{5.10}$$

where the functions

$$\omega_2 \mapsto \tilde g(x,\hbar,\omega_2)$$
$$\omega_2 \mapsto \langle\omega_2, L_0\omega_2\rangle$$
$$\omega_2 \mapsto \int_0^t |\sqrt{\hbar}\omega_2(s)+x|^{2N}ds$$

$$\omega_2 \mapsto 2N|x|^{2N-2}\int_0^t |\omega_2(s)|^2 ds + 4N(N-1)|x|^{2N-4}\int_0^t (x\omega_2(s))^2 ds$$

represent the stochastic extensions to $\mathcal{B}_2$ of the corresponding functions on $\mathcal{H}_2$. The stochastic extensions are well defined because of the regularity of the functions involved (see Section 2). Analogously

$$f_2(x,\hbar) = \int_{\mathcal{B}_2} \tilde{g}(x,\hbar,\omega_2) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} e^{-u\lambda\hbar^{1/2}\tilde{g}(x,\hbar,\omega_2)} d\mu(\omega_2). \tag{5.11}$$

Representation (5.10) shows the absolute convergence of the integrals involved, while representation (5.11) shows the regularity of f_2 as a function of $\epsilon = \sqrt{\hbar}$.

By a direct computation we get

$$f_2(x,0) = \int_{\mathcal{B}_2} \tilde{g}(x,0,\omega_2) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} d\mu(\omega_2),$$

where

$$\tilde{g}(x,0,\omega_2) = \begin{cases} \frac{N!}{(N-3)!3!} 8|x|^{2N-6}\int_0^t (x\omega_2(s))^3 ds \\ +2N(N-1)|x|^{2N-4}\int_0^t x\omega_2(s)|\omega_2(s)|^2 ds, \qquad 2N \geq 6 \\ \\ 4\int_0^t x\omega_2(s)|\omega_2(s)|^2 ds, \qquad 2N = 4 \end{cases} \tag{5.12}$$

and

$$\lim_{\hbar\downarrow 0} \frac{f_2(x,\hbar) - f_2(x,0)}{\hbar^{1/2}} = \int_{\mathcal{B}_2} g_4(\omega_2,x) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} d\mu(\omega_2) < \infty \tag{5.13}$$

with

$$g_4(\omega_2,x) = \begin{cases} \int_0^t |\omega_2(s)|^4 ds, \qquad 2N = 4 \\ 3|x|^2\int_0^t |\omega_2(s)|^4 ds + 12\int_0^t (x\omega_2(s))^2|\omega_2(s)|^2 ds, \qquad 2N = 6 \\ \binom{N}{2}|x|^{2N-4}\int_0^t |\omega_2(s)|^4 ds \\ \quad + 4\binom{N}{2}\binom{N-2}{1}|x|^{2N-6}\int_0^t (x\omega_2(s))^2|\omega_2(s)|^2 ds \\ \quad + 16\binom{N}{4}|x|^{2N-8}\int_0^t (x\omega_2(s))^4 ds, \qquad 2N \geq 8. \end{cases} \tag{5.14}$$

□

By equation (5.8), the integral $I(\hbar)$ can be represented as the sum

$$\widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\int_0^t \dot{\gamma}(s)^2 ds - \frac{\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\gamma(s)|^{2N} ds} d\gamma = I_1(\hbar) + I_2(\hbar),$$

where

$$I_1(\hbar) = (2\pi\hbar)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{t\lambda}{\hbar}|x|^{2N}} f_1(x,\hbar) dx,$$

$$I_2(\hbar) = -\lambda\hbar^{1/2}(2\pi\hbar)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{t\lambda}{\hbar}|x|^{2N}} f_2(x,\hbar) dx.$$

Lemma 5.2. *$I_2(\hbar) = O(\hbar^{\frac{4-d}{2}-\frac{4-d}{2N}})$, as $\hbar \downarrow 0$.*

Proof. By scaling we get

$$\begin{aligned} I_2(\hbar) &= -\lambda \hbar^{1/2}\Big(\frac{2\pi}{t}\Big)^{-d/2} \hbar^{d/2N-d/2} \int_{\mathbb{R}^d} e^{-t\lambda|x|^{2N}} f_2(\hbar^{1/2N}x,\hbar)dx \\ &= -\lambda\Big(\frac{2\pi}{t}\Big)^{-d/2} \hbar^{d/2N-d/2+1/2} \int_{\mathbb{R}^d} e^{-t\lambda(1-u)|x|^{2N}} \int_{\mathcal{B}_2} \tilde{g}(\hbar^{1/2N}x,\hbar,\omega_2) \\ &\quad e^{-\frac{1-u}{2}\left(2N|\hbar^{1/2N}x|^{2N-2}\int_0^t|\omega_2(s)|^2ds+4N(N-1)|\hbar^{1/2N}x|^{2N-4}\int_0^t(\hbar^{1/2N}x\omega_2(s))^2ds\right)} \\ &\quad e^{-\frac{u\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega_2(s)+\hbar^{1/2N}x|^{2N}ds} e^{\frac{1}{2}\langle\omega_2,L_0\omega_2\rangle} d\mu(\omega_2)dx. \end{aligned}$$

By dominated convergence theorem, the definition (5.7) of the function g, and by Lemma 5.1 and equation (5.13) we get:

$$\lim_{\hbar\downarrow 0}\frac{I_2(\hbar)}{\hbar^{\frac{3-d}{2}-\frac{3-d}{2N}}} = -\lambda\Big(\frac{2\pi}{t}\Big)^{-d/2}\int_{\mathbb{R}^d} e^{-t\lambda(1-u)|x|^{2N}} \int_{\mathcal{B}_2}\tilde{g}(x,0,\omega_2)e^{\frac{1}{2}\langle\omega_2,L_0\omega_2\rangle}d\mu(\omega_2)dx = 0,$$

where $g(x,0,\omega_2)$ is equal to (5.12), and

$$\lim_{\hbar\downarrow 0}\frac{I_2(\hbar)}{\hbar^{\frac{4-d}{2}-\frac{4-d}{2N}}} = -\lambda\Big(\frac{2\pi}{t}\Big)^{-d/2}\int_{\mathbb{R}^d} e^{-t\lambda(1-u)|x|^{2N}} \int_{\mathcal{B}_2} g_4(\omega_2,x)e^{\frac{1}{2}\langle\omega_2,L_0\omega_2\rangle}d\mu(\omega_2)dx < \infty,$$

with $g_4(\omega_2,x)$ given by (5.14) □

Lemma 5.3. *$I_1(\hbar) = \hbar^{-d\frac{N-1}{2N}}\left(\frac{\cosh t-1}{2\pi}\right)^{d/2} 2^{d/2}t^{-d(1/2+1/2N)}\lambda^{-d/2N}\int_{\mathbb{R}^d}e^{-|x|^{2N}}dx + O(\hbar^{(2-d)\frac{N-1}{2N}})$ as $\hbar\downarrow 0$.*

Proof.

$$\begin{aligned} I_1(\hbar) &= \Big(\frac{2\pi\hbar}{t}\Big)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{\lambda t}{\hbar}|x|^{2N}}\det(I+L_x)^{-1/2}dx \\ &= \Big(\frac{2\pi\hbar}{t}\Big)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{\lambda t}{\hbar}|x|^{2N}}\det\Big(\frac{\cosh(A(x)t)-1}{A^2(x)(\cosh t-1)}\Big)^{-1/2}dx \\ &= \Big(\frac{\cosh t-1}{2\pi\hbar/t}\Big)^{d/2}\int_{\mathbb{R}^d} e^{-\frac{\lambda t}{\hbar}|x|^{2N}}\det\Big(\frac{\cosh(A(x)t)-1}{A^2(x)}\Big)^{-1/2}dx. \end{aligned}$$

By scaling

$$\begin{aligned} I_1(\hbar) &= C_t\hbar^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}}\det\Big(\frac{\cosh(A(\hbar^{1/2N}x)t)-1}{A^2(\hbar^{1/2N}x)}\Big)^{-1/2}dx \\ &= C_t\hbar^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}}\det\Big(\frac{\cosh(\hbar^{(N-1)/2N}A(x)t)-1}{\hbar^{(N-1)/N}A^2(x)}\Big)^{-1/2}dx \end{aligned}$$

with $C_t = \left(\frac{\cosh t - 1}{2\pi/t}\right)^{d/2}$. Let $a_i^2(x)$, $i = 1, \dots, d$ be the eigenvalues of the matrix $A^2(x)$. Then

$$I_1(\hbar) = C_t \hbar^{\frac{d}{2N} - \frac{d}{2}} \int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} \frac{\hbar^{\frac{d(N-1)}{2N}} \prod_i a_i(x)}{\prod_i \sqrt{\cosh(\hbar^{(N-1)/2N} a_i(x)t) - 1}} dx$$

$$= C_t \hbar^{\frac{d}{2N} - \frac{d}{2}} \int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} \frac{2^{d/2} t^{-d}}{\prod_i \sqrt{1 + \frac{\cosh(\theta_i)}{12} \hbar^{(N-1)/N} a_i^2(x) t^2}} dx$$

$$= C_t \hbar^{\frac{d}{2N} - \frac{d}{2}} 2^{d/2} t^{-d} \int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} \prod_i \Big(1 - \frac{\frac{\cosh(\theta_i)}{24} \hbar^{(N-1)/N} a_i^2(x) t^2}{(1 + \frac{\xi_i \cosh(\theta_i)}{12} \hbar^{(N-1)/N} a_i^2(x) t^2)^{3/2}}\Big) dx$$

with $\theta_i \in (0, \hbar^{(N-1)/2N} a_i(x)t)$ and $\xi_i \in (0,1)$. We have

$$I_1(\hbar) = I_{1,1}(\hbar) + I_{1,2}(\hbar),$$

where the first term is equal to

$$I_{1,1}(\hbar) = \hbar^{-d\frac{N-1}{2N}} \Big(\frac{\cosh t - 1}{2\pi}\Big)^{d/2} 2^{d/2} t^{-d/2} \int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} dx$$

$$= \hbar^{-d\frac{N-1}{2N}} \Big(\frac{\cosh t - 1}{2\pi}\Big)^{d/2} 2^{d/2} t^{-d(1/2+1/2N)} \lambda^{-d/2N} \int_{\mathbb{R}^d} e^{-|x|^{2N}} dx,$$

and the second term is equal to

$$I_{1,2}(\hbar) = \Big(\frac{\cosh t - 1}{2\pi\hbar}\Big)^{d/2} \hbar^{d/2N} 2^{d/2} t^{-d/2} \int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}}$$

$$\Big(\prod_i \Big(1 - \frac{\frac{\cosh(\theta_i \hbar^{(N-1)/2N} a_i(x)t)}{24} \hbar^{(N-1)/N} a_i^2(x) t^2}{(1 + \frac{\xi_i \cosh(\theta_i \hbar^{(N-1)/2N} a_i(x)t)}{12} \hbar^{(N-1)/N} a_i^2(x) t^2)^{3/2}}\Big) - 1\Big) dx$$

and it satisfies the following relation

$$\lim_{\hbar \downarrow 0} \frac{I_{1,2}(\hbar)}{\hbar^{-d\frac{N-1}{2N} + \frac{N-1}{N}}} = -\frac{t^2}{24} \Big(\frac{\cosh t - 1}{2\pi}\Big)^{d/2} 2^{d/2} t^{-d/2} \int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} \sum_i a_i^2(x) dx < \infty.$$

□

By combining Lemma 5.2 and 5.3 we get:

Theorem 5.4. *Let H be the quantum mechanical Hamiltonian given on the vectors $\phi \in C_0^\infty(\mathbb{R}^d)$ by*

$$H\phi(x) = -\frac{\hbar^2}{2} \Delta\phi(x) + V(x)\phi(x),$$

where $V(x) = \lambda|x|^{2N}$, $\lambda > 0$.

Then the trace $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ of the evolution semigroup $e^{-\frac{t}{\hbar}H}$, $t > 0$, in $L^2(\mathbb{R}^d)$, is given by

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = (2\cosh t - 2)^{-d/2} \int_{\mathcal{H}_{p,t}} e^{-\frac{1}{\hbar}\Phi(\gamma)} d\gamma \tag{5.15}$$