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Lev Bukovský

The Structure of the Real Line

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Lev Bukovský

The Structure of the Real Line

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Lev Bukovský
Institute of Mathematics
University of P.J. Šafárik
Jesenná 5
040 01 Košice
Slovakia
lev.bukovsky@upjs.sk

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To Zuzana

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Preface

V názvu té knihy musí být slovo *struktura*.

Petr Vopěnka, 1975.

The title of this book must contain the word *structure*.

We must distinguish at least four periods – sometimes overlapping – in the history of investigation of the real numbers.

The first period, devoted to understanding the real numbers system, began as early as about 2 000 B.C. in Mesopotamia, then in Egypt, India, China, Greece (Pythagorians showed that $\sqrt{2}$ is an irrational), through Eudoxus, Euclid, Archimedes, Fibonacci, Euler, Bolzano and many others, and continued till the second half of the nineteenth century. Moris Kline [1972], p. 979 says that "one of the most surprising facts in the history of mathematics is that the logical foundation of the real numbers system was not erected until the late nineteenth century".

The rigorization of analysis forced the beginning of the second period, concentrated in the second half of the nineteenth century, and primarily devoted to an exact definition of the real numbers system. The investigations showed that mathematicians needed, as a framework for such a definition, a rigorous theory of infinity. The Zermelo – Fraenkel set theory **ZFC** was accepted as the best solution. Several independent definitions of reals in the framework of set theory turned out to be equivalent and, as usual in mathematics, that was a strong argument showing that the exact definition of an intuitive notion was established correctly.

Establishing an exact notion of the real numbers system, mathematicians began in the third period to intensively study the system. Essentially there were three possibilities: to study the algebraic structure of the field of reals, to study the subsets of the reals related to the topological and measure theoretic properties of reals induced by the order – in this case I speak about the real line instead of the system of real numbers – and finally, taking into account both the algebraic and topological (or measure theoretical) properties.

The fourth period started by arousing many open unanswered questions of the later one. It turned out that they are closely connected to set theoretical questions which were unanswered in set theory. By the invention of forcing as a method for showing the non-provability of a statement in **ZFC**, mathematicians obtained a strong tool to show that many questions concerning the topological and the measure theoretical structure of the real line are undecidable, or at least non-provable, in the considered set theory. The forcing method concerns essentially questions connected with infinite sets. Since the algebraic structure (to be an algebraic number, to be linearly independent over rationals, to be a prime number, etc.) is usually connected with the properties of finite sets, forcing hardly can contribute to a solution of an algebraic problem.

Focusing on this historical point of view, in 1975, I began to work on the Slovak version of a book collecting the main results of the above mentioned second, third and mainly the fourth period. I had previously presented the design of the book to participants in the Prague Set Theory Seminar. After my presentation Petr Vopěnka spontaneously stated the sentence quoted above. And I immediately accepted this proposal that provided an appropriate emphasis in the title on the book's content.

When the book appeared, in 1979, I was pleasantly surprised by the interest it generated in countries where students understood Slovak (which included the former Czechoslovakia, Poland, and some other particular locations, e.g., Hungary). At the end of 1987 I was further surprised to receive permission[1] to publish an English edition abroad. Immediately I began to work on an English version of my book. After the political events in Czechoslovakia in 1989, I could not refuse the (at least moral) responsibility to contribute to academic politics and my work on the English language version of the book was interrupted. When the monograph by T. Bartoszyński and H. Judah [1995] appeared, I thought my rôle in writing about the structure of the real line was finished. However, at the beginning of the 21st Century, several colleagues, mainly Polish, urged me to prepare a new edition of the Slovak version of "The Structure of the Real Line". In March 2003, I was invited to participate in the Boise Extravaganza in Set Theory at Boise State University in Idaho. After the conference I visited Tomek Bartoszyński in his office and was surprised to find my Slovak book open on his desk. This started my reflection, finishing with the conclusion that a book with the intention of my Slovak edition is not a rival book to Bartoszyński and Judah's book but rather a complementary, maybe, useful monograph. This presented a final – convincing – reason for my decision to prepare a new – significantly revised – edition of "The Structure of the Real Line".

I tried to follow the spirit of the Slovak edition. I shall not discuss the history of the real system. Since I present the main consequences of the Axiom of Determinacy **AD**, that contradicts the Axiom of Choice **AC**, in the basic parts of the book I try to avoid any use of **AC**, if possible, or, at least to replace it by the Weak Axiom of Choice **wAC**, that is a consequence of **AD**. Moreover, for some readers this may be interesting. In Chapter 1, I briefly describe the framework

[1] Let me remind the reader, that I lived in a country where everything was strongly controlled by a political party.

for the mathematical treatment of infinity – the Zermelo-Fraenkel set theory **ZF**. Then I sketch the main topological results in the above mentioned way. A precise definition of the real line is given in Chapter 2. It is shown that the definition leads to a unique object (up to a mathematical identification) and the set theoretical framework implies its existence. Chapters 3 to 8 contain mainly the rather classical theory of the structure of the real line – the study of properties of subsets of reals, which is important from the point of view of topology and measure theory. Chapter 7 is devoted to the phenomena of the measure – category duality. Finally, with overlapping topics, Chapters 7 to 10 deal with reductions of many important problems of the structure of the real line to the sentences of set theory, which the forcing method already proved are undecidable, or at least non-provable in set theory. To make the book self-contained, I wrote an Appendix containing miscellaneous, in my opinion, necessary material. I here recall all notions and basic facts of set theory and algebra that I use in the book. Then I present a short introduction to the metamathematics of set theory. Finally, I present the main results obtained (mainly) by the forcing method in the last fifty years, which I needed to answer important questions about the structure of the real line. I tried to attribute each result to its author. Any new notion or denotation can be found in the very large Index or Index of notations.

In the introduction of each chapter I try to explain its content and mainly, whether a reader should read a particular section immediately or before reading another sections. Especially, I suppose that a reader should start with reading Section 1.1 and then, she/he can begin to read any chapter and then go to the results needed for understanding of the presented material with the help of the indexes. Each section is supplemented by a series of exercises that contains a great deal of supplementary information concerning the topics of the section.

I am deeply satisfied that the book is being published in the series Monografie Matematyczne. For many years in Czechoslovakia we had found it difficult to obtain scientific information. The main source were the (often illegal) translations of English language books and articles in Russian and, in the topics of interest to me, Polish books that were mainly published as Monografie Matematyczne. So I consider it a great honor to contribute to this most prestigious series.

I would like to thank those who contributed to the preparation of the manuscript. Mirko Repický advised me as a TEX specialist. Peter Eliaš helped me with presentation of results related to the thin sets of harmonic analysis. My postgraduate students, former or current, Jozef Haleš, Michal Staš and Jaroslav Šupina, read the manuscript, discovering and correcting many errors, both typographical and factual. They contributed significantly to the correctness of the exercises.

I wish to express my gratitude to the Institute of Mathematics of Pavol Jozef Šafárik University at Košice that created good conditions for me while I was writing the book. My work was supported by grants 1/3002/06 and 1/0032/09 of the Slovak Grant Agency VEGA.

My intention for the book was to survey progress in the study of the real line over the period encompassing the highly productive end of one century and the beginning of another. I hope that this effort will prove to be a source of useful and stimulating information for a wide variety of mathematicians.

Košice, October 1, 2010 Lev Bukovský

Chapter 1

Introduction

> Również problematem o dużej doniosłości jest tego rodzaju ujęcie teorii mnogości, które – odpowiednio zacieśniając pojęcie zbioru – eliminowałoby istnienie zbiorów patologicznych (które jest konsekwencją – jak wspomnieliśmy – przede wszystkim aksjomatu wyboru), a nie uszczuplіłoby przy tym istotnie wartościowych osiągnięć teorii mnogości.
>
> Kazimierz Kuratowski and Andrzej Mostowski [1952].

> Equally, there is a problem with strong consequences in establishing a set theory of the kind that – making adequately precise the notion of a set – should eliminate the existence of pathological sets (which is a consequence – as we have already said – mainly of the axiom of choice) and does not weaken the surely worthy results of set theory.

In spite of the belief that the word around us is essentially finite, mathematics, physics and some other natural sciences cannot exist without a concept of infinity. Investigating the notion of a number, mathematicians almost always met an infinity. Starting with the Pythagoreans, continuing with Newton and Leibniz, then Bolzano and Cauchy, finishing with Dedekind and Cantor. As a consequence a necessity to build an appropriate mathematical theory of infinity and to investigate the numbers in a framework of such a theory arose. It was B. Bolzano who began the study of infinity. Then G. Cantor developed an adequate theory, fruitful with important results. All known attempts to build a theory of infinity convenient for a study of numbers essentially converge to the set theory as initiated by G. Cantor. So we have chosen as a framework for investigating numbers the most common set theory called the Zermelo-Fraenkel axiomatic set theory.

In this chapter we summarize necessary terminology and facts of set theory and of topology that we shall need later. In presenting our set theory we emphasize its axiomatization showing the rôle of some of its axioms in our investigations. The axiom of choice and its weak forms will play an important rôle in our next investigations. For this reason we present some well-known results of the set topology. In particular, we must identify the results which depend on some form of the axiom of choice.

L. Bukovský, *The Structure of the Real Line*, Monografie Matematyczne 71,
DOI 10.1007/978-3-0348-0006-8_1,

1.1 Set Theory

We assume that the reader is familiar with elementary set theory, say to the extent of a basic graduate course. Usually a set theory is developed in the framework of the Zermelo-Fraenkel axiom system, including the axiom of choice. Working mathematicians often do not notice when they have used the axiom of choice, even in an essential way. We shall always make it clear. Moreover, we shall try to indicate that a weaker form of an axiom of choice is sufficient for proving a statement. "**Theorem**" is a statement provable in **ZF**. "**Theorem** [φ]" is a statement provable in **ZF** $+ \varphi$, where φ is an additional axiom to **ZF**. Especially, " **Theorem [AC]**" is a statement provable in **ZFC** and "**Theorem [wAC]**" is a statement provable in **ZFW**. Similarly for Corollaries, Lemmas, and Exercises.

In this section we present necessary facts of set theory. We shall present only proofs of those which we consider as not standard and/or which are not usually included in basic courses.

The basic notions of set theory are that of a **set**, denoted usually by letters $a, b, \dots, x, y, z, A, B, \dots, X, Y, Z$ and others, and that of the **membership relation** $\in$. $x \in A$ is read as x is an **element** of A, or x **belongs** to A, or x is a **member** of A. Actually we assume that any object we shall deal with is a set. When all members of a considered set are subsets of a given set, we use the word **family** instead of set. Another notion is that of the inclusion relation $X \subseteq Y$, which is a short denotation for the formula $(\forall x)\,(x \in X \rightarrow x \in Y)$ meaning that X is a **subset** of Y. Similarly, $\emptyset$ is a constant which denotes the unique set satisfying $(\forall x)\, x \notin \emptyset$.

An **atomic formula** of set theory is a formula of the form $x = y$ or $x \in y$, where x, y are variables (or constants, generally terms). The formulas of set theory are built from atomic formulas in an obvious way by logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\equiv$, and quantifiers $\forall$ and $\exists$.

Zermelo-Fraenkel set theory ZF consists of the following axioms.

1. Axiom of Extensionality. If X and Y have the same elements, then $X = Y$.

2. Axiom of Pairing. For any x, y there exists a set X that contains exactly the elements x and y.

3. Axiom of Union. For any set X there exists a set Y such that $x \in Y$ if and only if $x \in u$ for some $u \in X$.

4. Axiom of Power Set. For any set X there exists a set Y that contains all subsets of X.

5. Axiom Scheme of Separation. For any formula $\varphi(x, x_1, \dots, x_k)$ of set theory the following statement is an axiom: for given sets X and $x_1, \dots, x_k$ there exists a set Y such that Y contains exactly those elements $x \in X$ that have the property $\varphi(x, x_1, \dots, x_k)$.

6. Axiom Scheme of Replacement. For any formula $\varphi(x, y, x_1, \ldots, x_k)$ of set theory such that

$$(\forall x, y, z, x_1, \ldots, x_k)\,((\varphi(x, y, x_1, \ldots, x_k) \wedge \varphi(x, z, x_1, \ldots, x_k)) \rightarrow y = z)$$

holds true, the following statement is an axiom: for given sets X and $x_1, \ldots, x_k$ there exists a set Y such that Y contains all those elements y for which there exists an $x \in X$ such that $\varphi(x, y, x_1, \ldots, x_k)$ holds true.

7. Axiom of Infinity. There exists an infinite set.

8. Axiom of Regularity. Every non-empty set X has an $\in$-minimal element, i.e., there exists an $x \in X$ such that x and X have no common element.

One of the main technical consequences of the Axiom of Regularity is that there exists no set x such that $x \in x$.

According to the Axiom of Extensionality the sets whose existence is guaranteed by axioms 2.–6. are unique. So we can introduce a notation for them. The set X of the Pairing Axiom will be denoted by $\{x, y\}$. The set Y of the Axiom of Union will be denoted by $\bigcup X$. Especially, if $X = \{x, y\}$, then we write $x \cup y = \bigcup X$. The set Y of the Axiom of Power Set will be denoted by $\mathcal{P}(X)$. The set Y of the Axiom Scheme of Separation will be denoted by

$$\{x \in X : \varphi(x, x_1, \ldots, x_k)\}.$$

Finally, the set Y of the Axiom Scheme of Replacement will be denoted by

$$\{y : x \in X \wedge \varphi(x, y, x_1, \ldots, x_k)\}.$$

We need to make precise the meaning of the notion of "an infinite set". The simplest way is to find a property of a set which implies that it is "an infinite set". We can consider a set X as infinite if X is non-empty and for each of its elements contains a new element different in some sense from all "previous" ones. It turns out that the following property of a set X is enough[1] for being infinite:

$$(\exists x)\,(x \in X) \wedge (\forall z \in X)\,(z \cup \{z\} \in X).$$

If there exists at least one set, then there exists the empty set $\emptyset$. We can specify that an infinite set contains the empty set, i.e.,

$$\emptyset \in X \wedge (\forall z \in X)\,(z \cup \{z\} \in X). \tag{1.1}$$

It is well known that using the Axiom of Replacement the existence of an infinite set implies the existence of a set with property (1.1). Moreover, the existence of a

[1] By the Axiom of Regularity $z \notin z$ and therefore $z \neq z \cup \{z\}$.

set with property (1.1) implies the existence of a "minimal" set with this property, i.e., the existence of a set with both properties (1.1) and

$$(\forall Y)((\emptyset \in Y \wedge (\forall z \in Y)\,(z \cup \{z\} \in Y)) \to X \subseteq Y). \tag{1.2}$$

The set with properties (1.1) and (1.2) is uniquely determined and will be denoted by ω. By definition, $\emptyset \in \omega$. We shall write $0 = \emptyset$. If $n \in \omega$ we write $n+1 = n \cup \{n\}$. An element of ω is called a **natural number**. Note that $1 = \{0\}, 2 = \{0, 1\}, \ldots, n+1 = \{0, \ldots, n\}$. For our convenience we formulate the Axiom of Infinity as

There exists the set ω.

However if we replace the former Axiom of Infinity by the later formulation, then we need to assume that there exists the empty set.

The definition of the set ω immediately yields a useful method of proof of sentences about elements of ω – mathematical induction. Actually we can show a metatheorem, or for any particular formula φ of set theory we have a particular theorem. So, let $\varphi(x, x_1, \ldots, x_k)$ be a formula of set theory. Then

Theorem 1.1 (Theorem on Mathematical Induction). *Let $x_1, \ldots, x_k$ be given. If*

(IS1) $\varphi(0, x_1, \ldots, x_k)$,

(IS2) $(\forall n \in \omega)\,(\varphi(n, x_1, \ldots, x_k) \to \varphi(n+1, x_1, \ldots, x_k))$, *then* $\varphi(x, x_1, \ldots, x_k)$ *holds true for any* $x \in \omega$.

Proof. Set

$$Y = \{x \in \omega : \varphi(x, x_1, \ldots, x_k)\}.$$

By (IS1) and (IS2) the set $Y \subseteq \omega$ satisfies the premise of the implication (1.2). Thus $Y = \omega$. $\square$

We assume that the reader is familiar with the theory of ordinals. Let us recall that X is a **transitive set** if $(\forall x)\,(x \in X \to x \subseteq X)$. An **ordinal** is a transitive set well-ordered by the relation $\eta \in \zeta \vee \eta = \zeta$. Thus an ordinal is the set of all smaller ordinals. If ξ, η are different ordinals, then either $\xi \in \eta$ or $\eta \in \xi$. If ξ is an ordinal, then $\xi + 1 = \xi \cup \{\xi\}$ is the least ordinal greater than ξ – the immediate successor of ξ. If an ordinal $\xi \neq \emptyset$ is not an immediate successor of any ordinal, then $\xi = \sup\{\eta : \eta < \xi\} = \bigcup \xi$ is called a **limit ordinal**. ω is the least limit ordinal. The ordinal sum $\xi + \eta$ is defined by transfinite induction. Any ordinal α can be expressed as $\alpha = \lambda + n$, where λ is 0 or a limit ordinal and $n \in \omega$. The fundamental property of ordinals is expressed by

Theorem 1.2. *Every well-ordered set $\langle X, \leq \rangle$ is isomorphic to a unique ordinal.*

The unique ordinal ξ is called the **order type** of the well-ordered set X and we write $\xi = \mathrm{ot}(X) = \mathrm{ot}(X, \leq)$.

The method of mathematical induction can be extended for well-ordered sets. So, let $\varphi(x, x_1, \ldots, x_k)$ be a formula of set theory. Then

Theorem 1.3 (Theorem on Transfinite Induction). *Assume that $\langle X, \leq\rangle$ is a well-ordered set. Let $x_1, \dots, x_k$ be given. If for any $x \in X$,*

$$(\forall y < x)\, \varphi(y, x_1, \dots, x_k) \to \varphi(x, x_1, \dots, x_k)$$

holds true, then $\varphi(x, x_1, \dots, x_k)$ holds true for any $x \in X$.

We shall use without any commentary the following result or its adequate modification:

Theorem 1.4 (Definition by Transfinite Induction). *Let ξ be an ordinal, X being a non-empty set. Let $a \in X$, $f : X \longrightarrow X$ and $g : \bigcup_{\eta<\xi} {}^{\eta}X \longrightarrow X$. Then there exists unique function $F : \xi \longrightarrow X$ such that*

a) $F(0) = a$,
b) $F(\eta + 1) = f(F(\eta))$ *for any* $\eta < \xi$,
c) $F(\eta) = g(F|\eta)$ *for any limit* $\eta < \xi$.

Sometimes we shall speak about a **class of sets**. By a class of sets we understand "a collection" of sets satisfying a given formula, for which we do not have any argument for being a set. E.g., by **V** we denote the **class of all sets**. It is easy to see that **V** is not a set. The expression $x \in \mathbf{V}$ simply means "x is a set". Similarly, we can define the class **On** of all ordinals. The class **On** is not a set and the formula $x \in \mathbf{On}$ simply means that x is an ordinal. Actually, a class is an object of metamathematics (a formula of the language of **ZF**). We shall often speak about the class of all topological spaces or about the class of all Polish spaces, see Section 5.2. However, we must deal with the notion of a class very carefully, e.g., saying "for all classes ... holds true" is not a formula of set theory.

We say that two sets A, B have the **same cardinality**, written $|A| = |B|$, if there exists a one-to-one mapping of A onto B. The relation $|A| = |B|$ is reflexive, symmetric and transitive. We customarily say that $|A|$ is the **cardinality** of the set A. However, we do not know what it is. A cardinality has sense only in an interrelation with some other cardinality. The set A has cardinality **not greater** than the set B, written $|A| \leq |B|$, if there exists a one-to-one mapping of A into B. Finally, the set A has cardinality **smaller** than the set B, written $|A| < |B|$, if $|A| \leq |B|$ and not $|A| = |B|$. The relation $|A| \leq |B|$ is reflexive and transitive. In **ZF** one can prove that it is antisymmetric:

Theorem 1.5 (G. Cantor – F. Bernstein). *For any sets A, B, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

We define another relation between cardinalities of sets as

$$|X| \ll |Y| \equiv (\exists f)\,(f : Y \xrightarrow{\text{onto}} X). \tag{1.3}$$

The relation $\ll$ is reflexive and transitive. Evidently $|X| \leq |Y|$ implies $|X| \ll |Y|$, provided $X \neq \emptyset$. As we shall see in Section 9.4 we have no chance to prove in **ZF** that the relation $\ll$ is antisymmetric. One can easily see that

$$|A| \ll |B| \to |\mathcal{P}(A)| \leq |\mathcal{P}(B)|. \tag{1.4}$$

Generally the relation $|A| \leq |B|$ is not dichotomous, i.e., one cannot prove in **ZF** that

$$|A| \leq |B| \vee |B| \leq |A|$$

for any sets A, B.

The arithmetic operations on cardinalities are defined as follows:

$$\begin{aligned} |A| + |B| &= |A \cup B| \text{ provided that } A \cap B = \emptyset, \\ |A| \cdot |B| &= |A \times B| \text{ for any sets } A,\ B, \\ |A|^{|B|} &= |{}^{B}A| \text{ for any sets } A,\ B, \end{aligned}$$

where ${}^{B}A$ denotes the set of all mappings from B into A. The operations satisfy the obvious laws of arithmetics.

Theorem 1.6 (G. Cantor). *There exists no mapping of X onto $\mathcal{P}(X)$. Therefore $|X| < |\mathcal{P}(X)|$. Moreover, if A, X are sets such that $|A| \ll |X|$, then $\neg(2^{|X|} \ll |A|)$ and $|A| < 2^{|X|}$.*

Proof. The former statement follows from the later one.

So assume that $f : X \xrightarrow{\text{onto}} A$. Since the mapping $F(x) = f^{-1}(\{x\})$ is an injection $F : A \xrightarrow{1-1} \mathcal{P}(X)$, we obtain $|A| \leq 2^{|X|}$.

To obtain a contradiction, we shall suppose that there exists a mapping $h : A \xrightarrow{\text{onto}} \mathcal{P}(X)$. Set $g = f \circ h : X \xrightarrow{\text{onto}} \mathcal{P}(X)$. Let $E = \{x \in X : x \notin g(x)\}$. Then there exists an $e \in X$ such that $g(e) = E$. Thus

$$e \in E \equiv e \notin g(e) = E,$$

which is a contradiction. □

A set A is called **finite** if $|A| < |\omega|$. A set A is called **countable** if $|A| \leq |\omega|$. A set that is not finite is **infinite** and a set that is not countable is **uncountable**. If $n \in \omega$ and $|A| = |n|$ we write $|A| = n$. We shall use without reference the following result.

Theorem 1.7. *A set A is finite if and only if $|A| = n$ for some $n \in \omega$. Thus ω is the set of all finite ordinals.*

An ordinal α is called a **cardinal number** or simply **cardinal** if $|\alpha| \neq |\xi|$ for every $\xi < \alpha$. Thus every finite ordinal is a cardinal, ω is a cardinal. The infinite cardinal numbers can be enumerated by ordinals

$$\omega = \aleph_0 < \aleph_1 < \cdots < \aleph_\xi < \cdots .$$

That is why an infinite cardinal number is also called an **aleph**. Sometimes we write ω_ξ instead $\aleph_\xi$, i.e., $\omega_\xi = \aleph_\xi$. A cardinal $\aleph_\xi$ is a **limit cardinal** or a **successor cardinal** if ξ is a limit or successor ordinal, respectively.

$\aleph_1$ is the least uncountable cardinal and also the least uncountable ordinal. Instead of $|A| = |\aleph_\xi|$ we write simply $|A| = \aleph_\xi$ and in such a case $\aleph_\xi$ is called the cardinality of the set A. Similarly for inequalities. Generally we shall denote a cardinality by Fraktur letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{m}$, $\mathfrak{n}$ etc.[2] When we want to emphasize that the considered cardinality is the cardinality of a well-ordered set, we shall use Greek letters κ, λ, μ etc. The smallest cardinality greater than $\mathfrak{m}$ (if it does exist!) we denote by $\mathfrak{m}^+$.

A set A **can be well ordered** if there exists a well-ordering with the field A. If an infinite set A can be well ordered, then its cardinality is an aleph. Thus instead of saying that A can be well ordered we shall say also that $|A|$ **is an aleph**.

For addition and multiplication of alephs the simple **Hessenberg Theorem** holds true:

$$\aleph_\xi + \aleph_\eta = \aleph_\xi \cdot \aleph_\eta = \aleph_{\max\{\xi,\eta\}}. \tag{1.5}$$

Let $\eta < \xi$ be two ordinals. The ordinal ξ is said to be **cofinal** with η if there exists an increasing function $f : \eta \longrightarrow \xi$ such that $\sup\{f(\zeta) : \zeta < \eta\} = \xi$, i.e., if the set $\{f(\zeta) : \zeta < \eta\}$ is cofinal in ξ. By $\mathrm{cf}(\xi)$ we denote the least ordinal η such that ξ is cofinal with η. An infinite ordinal ξ is called **regular** if $\mathrm{cf}(\xi) = \xi$. Otherwise ξ is **singular**. The ordinal $\mathrm{cf}(\xi)$ is always regular. A regular limit ordinal is always a cardinal. Not vice versa. There exist singular cardinals, e.g., $\aleph_\omega$ for which $\mathrm{cf}(\aleph_\omega) = \omega$.

We shall use the following result.

Theorem 1.8 (A. Tarski). *If $|X| \geq \aleph_0$, then $2^{|X|} + |X| = 2^{|X|}$ and for any set Y such that $|Y| + |X| = 2^{|X|}$ we have $|Y| = 2^{|X|}$.*

The statement of this theorem may be expressed as follows: if $\mathfrak{m} \geq \aleph_0$ is a cardinality, then $2^\mathfrak{m} + \mathfrak{m} = 2^\mathfrak{m}$ and for any cardinality $\mathfrak{n}$ such that $\mathfrak{n} + \mathfrak{m} = 2^\mathfrak{m}$ we have $\mathfrak{n} = 2^\mathfrak{m}$. A proof may be found in Exercise 1.6.

Let us recall that if an axiom of **ZF** assures the existence of a set, then the set is uniquely determined (and we have usually denoted it by some symbol). We shall sometimes need such an axiom, which assures the existence of a set with a property that does not determine the set uniquely.

If $\mathcal{F}$ is a family of non-empty sets, then a function $f : \mathcal{F} \longrightarrow \bigcup \mathcal{F}$ is called a **choice function** or a **selector** for $\mathcal{F}$ if $f(A) \in A$ for every $A \in \mathcal{F}$. The **Axiom of Choice AC** says that, for every family of non-empty sets, there exists a choice function. Evidently **AC** is equivalent to the statement: a Cartesian product of a family of non-empty sets is a non-empty set. The theory **ZF**+**AC** will be denoted by **ZFC**. By results of K. Gödel (11.11) and P.J. Cohen (11.17), the axiom **AC** is undecidable in **ZF**.

Using Theorem 1.7 one can easily prove by mathematical induction that, for every finite family of non-empty sets, there exists a selector. As a consequence we obtain

[2]Note that in Sections 5.3, 5.4 and later on, the letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{m}$, $\mathfrak{p}$ denote particular cardinalities.

Theorem 1.9 (Dirichlet Pigeonhole Principle). *Let* $\mathfrak{m}$ *be a cardinality (i.e.,* $\mathfrak{m} = |B|$ *for some set* B*). Let* $f : X \longrightarrow Y$*,* Y *being finite. If* $|Y| \cdot \mathfrak{m} < |X|$*, then there exists a* $y \in Y$ *such that the cardinality of the inverse image* $f^{-1}(\{y\})$ *is not smaller than or equal to* $\mathfrak{m}$*. Equivalently: if* $\mathcal{A}$ *is a finite partition of the set* X *and* $\mathfrak{m} \cdot |\mathcal{A}| < |X|$*, then there exists a set* $A \in \mathcal{A}$ *such that* $\neg|A| \leq \mathfrak{m}$*.*

The following theorem is a basic result.

Theorem 1.10 (E. Zermelo – M. Zorn). *The following are equivalent:*

a) Axiom of Choice **AC**.
b) **Zermelo's Theorem***: Every set can be well ordered.*
c) **Zorn's Lemma***: If every chain in a poset* $\langle X, \leq \rangle$ *is bounded from above, then for any* $x \in X$ *there exists a maximal element* $a \geq x$*.*[3]

Corollary 1.11. *If* **AC** *holds true, then any set* A *is either finite or there exists an ordinal* ξ *such that* $|A| = \aleph_\xi$*. In other words,* **AC** *implies that any cardinality is a cardinal number.*

Thus if **AC** holds true, then the class of cardinalities is equal to the class of all cardinal numbers and therefore is well ordered, i.e., there exists the smallest cardinality (=cardinal number) with given property (if there exists any such). In what follows we shall use this fact without any commentary.

Theorem 1.12. *If* **AC** *holds true, then any* $\aleph_{\xi+1}$ *is a regular cardinal.*

However, as we shall see later, the Axiom of Choice must be essentially used in a proof of Theorem 1.12.

By Theorem 1.6 and Corollary 1.11, assuming **AC**, for any infinite set X the cardinality of the power set $\mathcal{P}(X)$ is an aleph greater than $|X|$. The assumption that this cardinality is the smallest possible is called the **Generalized Continuum Hypothesis** and is denoted as **GCH**. Thus **GCH** says that $(\forall \xi)\, 2^{\aleph_\xi} = \aleph_{\xi+1}$. The **Continuum Hypothesis CH** says that $2^{\aleph_0} = \aleph_1$. Thus **CH** follows from **GCH**. By results of K. Gödel (11.11) and P.J. Cohen (11.17), both **CH** and **GCH** are undecidable in **ZFC**.

A limit regular cardinal κ is called a **weakly inaccessible cardinal**. If moreover, for any $\lambda < \kappa$ we have $2^\lambda < \kappa$, then κ is called **strongly inaccessible**. Note that if $\aleph_\xi$ is weakly inaccessible, then $\aleph_\xi = \xi$. By Metatheorem 11.3 the existence of a strongly inaccessible cardinal cannot be proved in **ZF**. Neither is the existence of a weakly inaccessible cardinal provable in **ZFC**.

For sake of brevity we denote by **IC** the statement "there exists a strongly inaccessible cardinal".

In our reasoning we do not always need the full **AC**. We formulate some weak forms of the axiom of choice. The **Countable Axiom of Choice AC**$^\omega$ says that for every countable family of non-empty sets there exists a choice function. In several investigations we shall need even weaker forms. The **Weak Axiom of Choice wAC** says that for any countable family of non-empty subsets of a given set

[3] For the notions used in the formulation of Zorn's Lemma, see Section 11.1.

of cardinality $2^{\aleph_0}$ there exists a choice function. Finally, the **Axiom of Dependent Choice DC** says that for any binary relation R on a non-empty set A such that for every $a \in A$ there exists a $b \in A$ such that aRb, for every $a \in A$ there exists a function $f : \omega \longrightarrow A$ satisfying $f(n)Rf(n+1)$ for any $n \in \omega$ and $f(0) = a$. One can easily show that

$$\mathbf{AC} \to \mathbf{DC}, \quad \mathbf{DC} \to \mathbf{AC}^{\omega}, \quad \mathbf{AC}^{\omega} \to \mathbf{wAC}.$$

It is well known that no implication can be reversed. We denote by **ZFW** the theory **ZF**+**wAC**.

We shall often need the following simple result.

Theorem 1.13. *The following are equivalent:*

a) *The Weak Axiom of Choice* **wAC**.
b) *For any countable family of non-empty subsets of* $^{\omega}2$ *there exists a choice function.*
c) *For every* X *such that* $|X| \ll 2^{\aleph_0}$ *and every sequence* $\{A_n\}_{n=0}^{\infty}$ *of non-empty subsets of* X *there exists a selector for* $\{A_n\}_{n=0}^{\infty}$.
d) *For every sequence* $\langle A_n : n \in \omega \rangle$ *of non-empty subsets* $^{\omega}2$ *there exists an infinite* $E \subseteq \omega$ *and a function* $f : E \longrightarrow {}^{\omega}2$ *such that* $f(n) \in A_n$ *for every* $n \in E$.

The proof is easy. The implication a) → b), b) → d), and c) → a) are trivial.

Assume b). We show that c) holds true. If $|X| \ll 2^{\aleph_0}$, then there exists a surjection $f : {}^{\omega}2 \xrightarrow{\text{onto}} X$. Hence $\{f^{-1}(A_n)\}_{n=0}^{\infty}$ is a sequence of non-empty subsets of $^{\omega}2$. By b), there exists a selector $\langle a_n \in f^{-1}(A_n) : n \in \omega \rangle$ for this sequence. Then $\{f(a_n)\}_{n=0}^{\infty}$ is a selector for $\{A_n\}_{n=0}^{\infty}$.

Assume d). To show b), consider the sequence $\langle B_n = \Pi_{k \le n} A_k : n \in \omega \rangle$. If $E \subseteq \omega$ is infinite and $g : E \longrightarrow \bigcup_n {}^{n+1}({}^{\omega}2)$ is such that $g(n) \in B_n$ for every $n \in E$, we define

$$f(n) = g(m)(n), \text{ where } m = \min\{k \in E : n \le k\}. \qquad \square$$

The Axiom of Choice implies that the relation $|A| \le |B|$ is dichotomous, i.e., for any sets A, B, we have $|A| \le |B| \vee |B| \le |A|$. As we have already remarked, one cannot prove this statement if **ZF**. See, e.g., Theorem 9.28. However, **wAC** implies a similar statement at least in the most important case.

Theorem [wAC] 1.14. *If* $A \subset X$, $|X| \ll 2^{\aleph_0}$, *then either* A *is countable or* $|A| > \aleph_0$.

Proof. If A is not finite, then using Theorem 1.7 one can easily show by mathematical induction that the sets $\langle \Psi_n = \{f \in {}^{n}A : f \text{ is an injection}\} : n \in \omega \rangle$ are non-empty. Let $\langle f_n : n \in \omega \rangle$ be a choice function. We define

$$f(n) = \begin{cases} f_1(0) & \text{if } n = 0, \\ f_{n+1}(k) & \text{where } k = \min\{l \in \omega : (\forall i < n)\, f_{n+1}(l) \neq f(i)\} \text{ otherwise.} \end{cases}$$

Evidently $f : \omega \xrightarrow{1-1} A$ and therefore $\aleph_0 \le |A|$. Thus if A is not countable, then $\aleph_0 < |A|$. $\square$

A set $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an **ideal** on X if

1) $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$,
2) if $A \in \mathcal{I}$, $B \subseteq A$, then $B \in \mathcal{I}$,
3) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

For simplicity we usually assume that

4) $\{x\} \in \mathcal{I}$ for every $x \in X$.

Let κ be an uncountable regular cardinal. An ideal $\mathcal{I}$ is said to be **κ-additive** if $\bigcup \mathcal{A} \in \mathcal{I}$ for any set $\mathcal{A} \subseteq \mathcal{I}$, $|\mathcal{A}| < \kappa$. An $\aleph_1$-additive ideal is simply called **σ-additive**. A set $\mathcal{I}_0 \subseteq \mathcal{I}$ is a **base of the ideal** $\mathcal{I}$ if every element A of $\mathcal{I}$ is a subset of some $B \in \mathcal{I}_0$.

The dual notion to the notion of an ideal is the notion of a filter. A set $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a **filter**on X if

1) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$,
2) if $A \in \mathcal{F}$, $A \subseteq B$, then $B \in \mathcal{F}$,
3) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Similarly as above, we usually assume that

4) $X \setminus \{x\} \in \mathcal{F}$ for every $x \in X$.

$\mathcal{F}$ is a filter if and only if the family $\{X \setminus A : A \in \mathcal{F}\}$ is an ideal. A filter $\mathcal{F}$ is called an **ultrafilter** if for every $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. It is easy to see that a filter $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ is maximal with respect to ordering by inclusion. A set $\mathcal{F}_0 \subseteq \mathcal{F}$ is a **base of the filter** $\mathcal{F}$ if for every $A \in \mathcal{F}$ there exists a $B \in \mathcal{F}_0$ such that $B \subseteq A$.

For any $x \in X$ the set $\{A \subseteq X : x \in A\}$ is an ultrafilter on X. An ultrafilter of this form is called **trivial**. A filter $\mathcal{F}$ on X is called a **free filter** if $\mathcal{F}$ does not contain any finite set. Thus, an ultrafilter is free if and only if it is not a trivial ultrafilter. The **Boolean Prime Ideal Theorem BPI** says that every filter on any set can be extended to an ultrafilter. Equivalently, any ideal is contained in a maximal ideal. Similarly as in the case of an ultrafilter, one can show that a maximal ideal $\mathcal{I}$ on a set X is a **prime ideal**, i.e., for any subset $A \subseteq X$ we have either $A \in \mathcal{I}$ or $X \setminus A \in \mathcal{I}$.

Theorem 1.15. **AC** *implies* **BPI**.

The proof is based on an application of the Zermelo Theorem. It is known that the converse implication is not true, see J.D. Halpern and A. Lévy [1971].

An ultrafilter $\mathcal{G}$ on an infinite set X is called **uniform** provided that every element of $\mathcal{G}$ has cardinality $|X|$.

Assume **AC**. Then for any infinite set X, the set

$$\mathcal{F} = \{A \subseteq X : |X \setminus A| < |X|\}$$

is a filter[4]. If $\mathcal{G}$ is an ultrafilter extending $\mathcal{F}$, then every element of $\mathcal{G}$ has cardinality $|X|$. Thus, (assuming **AC**) there exists a uniform ultrafilter on any infinite set[5].

If $A \subseteq \omega$ is infinite, then there is a unique increasing **enumeration** e_A of A. The nth element of A is $\mathrm{e}_A(n)$. Thus the enumeration e_A is uniquely determined by

$$\mathrm{e}_A(0) < \mathrm{e}_A(1) < \cdots < \mathrm{e}_A(n) < \mathrm{e}_A(n+1) < \cdots \tag{1.6}$$

and

$$A = \{\mathrm{e}_A(n) : n \in \omega\}. \tag{1.7}$$

A bijection π from $\omega \times \omega$ onto ω is called a **pairing function**. It is well known that there exists a pairing function, e.g.,

$$\pi(n,m) = \frac{1}{2}(n+m)(n+m+1) + m. \tag{1.8}$$

Note that $\pi(n,m) \geq n, m$ for any n, m. We denote by λ and ρ the **left inverse** and **right inverse** functions of π, respectively, i.e.,

$$\pi(\lambda(n), \rho(n)) = n, \quad \lambda(\pi(n,m)) = n, \quad \rho(\pi(n,m)) = m \tag{1.9}$$

for any $n, m \in \omega$. Note also that $\lambda(n) \leq n$ and $\rho(n) < n$ for any $n > 0$.

Using a pairing function we can "identify" the sets ${}^\omega X$ and ${}^\omega({}^\omega X)$. Actually, to a $\varphi \in {}^\omega({}^\omega X)$ we assign a $\psi \in {}^\omega X$ by setting $\psi(n) = \varphi(\lambda(n))(\rho(n))$. A projection of ${}^\omega X$ onto ${}^\omega X$ considered as the nth factor of the product ${}^\omega({}^\omega X)$ is defined by

$$\mathrm{Proj}_n(\varphi) = \psi, \text{ where } \psi(m) = \varphi(\pi(n,m)). \tag{1.10}$$

Similarly we can identify ${}^\omega X$ with ${}^\omega X \times {}^\omega X$ using the mapping Π_X defined as $\Pi_X(\alpha, \beta) = \gamma$, where

$$\gamma(n) = \begin{cases} \alpha(n/2) & \text{if } n \text{ is even,} \\ \beta((n-1)/2) & \text{if } n \text{ is odd.} \end{cases} \tag{1.11}$$

The left inverse Λ_X and right inverse R_X of Π_X from ${}^\omega X$ into ${}^\omega X$ are defined as

$$\Lambda_X(\alpha) = \{\alpha(2n)\}_{n=0}^{\infty}, \qquad R_X(\alpha) = \{\alpha(2n+1)\}_{n=0}^{\infty}. \tag{1.12}$$

Thus, for any $\alpha, \beta, \gamma \in {}^\omega X$ we have

$$\Lambda_X(\Pi_X(\alpha,\beta)) = \alpha, \quad R_X(\Pi_X(\alpha,\beta)) = \beta, \quad \Pi_X(\Lambda_X(\gamma), R_X(\gamma)) = \gamma. \tag{1.13}$$

If the set X is understood we simply write Π, Λ, R.

We shall use many natural modifications of pairing functions.

Theorem 1.16. *There exists a function $f : \mathcal{P}(\omega) \xrightarrow{\text{onto}} \omega_1$, i.e., $\aleph_1 \ll 2^{\aleph_0}$.*

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[4]Actually, this statement is equivalent to **AC**. See Exercise 1.4.

[5]Again, this assertion is equivalent to **AC**, see Exercise 1.4.

Proof. Let π be a pairing function. If $A \subseteq \omega$, then $\pi^{-1}(A) \subseteq \omega \times \omega$, i.e., $\pi^{-1}(A)$ is a binary relation. We define a function f as

$$f(A) = \begin{cases} \xi & \text{if } \pi^{-1}(A) \text{ is a well-ordering of } \omega \text{ in the ordinal type } \xi, \\ n & \text{if } |\pi^{-1}(A)| = n, \\ 0 & \text{otherwise.} \end{cases}$$

Since for every infinite ordinal $\xi < \omega_1$ there exists a well-ordering of ω in the ordinal type ξ, the function f is a surjection of $\mathcal{P}(\omega)$ onto ω_1. □

Corollary 1.17. *There exists a decomposition of $\mathcal{P}(\omega)$ in ω_1 non-empty pairwise disjoint sets*

$$\mathcal{P}(\omega) = \bigcup_{\xi<\omega_1} f^{-1}(\{\xi\}), \tag{1.14}$$

where $|f^{-1}(\{\xi\})| = \mathfrak{c}$ for every $\xi \geq \omega$.

Proof. If $\langle \omega, R \rangle$ is a well-ordered set of an infinite type ξ and $g : \omega \xrightarrow[\text{onto}]{1-1} \omega$, then $\langle \omega, \{\langle n, m \rangle : \langle g(n), g(m) \rangle \in R\} \rangle$ is a well-ordered set of the type ξ as well. For different g's the sets $\{\langle n, m \rangle : \langle g(n), g(m) \rangle \in R\}$ are different. Hence for any $\xi \geq \omega$ we have $|f^{-1}(\{\xi\})| = \mathfrak{c}$. □

The decomposition of (1.14) is called the **Lebesgue decomposition**. In Section 6.4 we describe a related decomposition of the set of dyadic numbers.

The famous fact, that a union of countably many countable sets is a countable set, is usually proved by using some weak form of the axiom of choice.

Theorem 1.18. *Assuming* $\mathbf{AC}^\omega$*, a countable union of countable sets is a countable set. Assuming* **wAC***, a countable union of countable subsets of a given set of cardinality $\ll 2^{\aleph_0}$ is a countable set.*

The theorem cannot be proved in **ZF**, since S. Feferman and A. Lévy (11.19) have constructed a model of **ZF**, in which the following holds true:

$$\mathcal{P}(\omega) \text{ is a countable union of countable sets.} \tag{1.15}$$

We show that in a proof of Theorem 1.12 we need a form of the axiom of choice even in the case of ω_1.

Theorem 1.19. *If* (1.15) *holds true, then* $\mathrm{cf}(\omega_1) = \omega$.

Proof. Assume that

$$\mathcal{P}(\omega) = \bigcup_{n\in\omega} \mathcal{A}_n, \quad \mathcal{A}_n \text{ is countable for any } n. \tag{1.16}$$

Let $f : \mathcal{P}(\omega) \xrightarrow{\text{onto}} \omega_1$ be the function constructed in the proof of Theorem 1.16. Set $\eta_n = \sup\{f(A) : A \in \mathcal{A}_n\}$. If some $\eta_n = \omega_1$, then we are ready. If not, then every η_n is a countable ordinal. If $\xi < \omega_1$, then there exists a set $A \in \mathcal{P}(\omega)$ such that $f(A) = \xi$. By (1.16) there exists an n such that $A \in \mathcal{A}_n$. Then $\xi \le \eta_n$. Thus $\sup\{\eta_n : n \in \omega\} = \omega_1$. □

Theorem 1.20. *The Weak Axiom of Choice* **wAC** *implies that* ω_1 *is a regular cardinal.*

Proof. Let $A \subseteq \omega_1$ be countable, i.e., $A = \{\xi_n : n \in \omega\}$. We can assume that $\xi_n > \omega$ for each n. We have to show that A is bounded in ω_1.

Let f be the function of Theorem 1.16. We denote $\Phi_n = \{A \subseteq \omega : f(A) = \xi_n\}$. Then by **wAC** there exists a choice function $\langle B_n \in \Phi_n : n \in \omega\rangle$. Thus $\langle \omega, \pi^{-1}(B_n)\rangle$ is a well-ordered set of order type ξ_n. We define a well-ordering R on $\omega \times \omega$ of order type equal to $\xi_0 + \cdots + \xi_n + \cdots$ as follows:

$$\langle n_1, m_1\rangle R \langle n_2, m_2\rangle \equiv (n_1 < n_2 \vee (n_1 = n_2 \wedge \langle m_1, m_2\rangle \in \pi^{-1}(B_{n_1}))).$$

Let η be the order type of $\langle \omega \times \omega, R\rangle$. Since $|\omega \times \omega| = \aleph_0$, we obtain $\eta < \omega_1$. On the other hand every $\langle \omega, \pi^{-1}(B_n)\rangle$ can be naturally embedded into $\langle \omega \times \omega, R\rangle$, therefore $\xi_n \le \eta$. Thus η is an upper bound of A. □

We close with a technical result that will be useful in some investigation. Let us consider the following property $\mathrm{COF}(\xi)$ of an ordinal $\xi \le \omega_1$:

> there exists a function $F : \xi \longrightarrow {}^{\omega}\omega_1$ such that for any limit $\eta < \xi$, $F(\eta)$ is an increasing sequence of ordinals $\{\eta_n\}_{n=0}^{\infty}$ and $\eta = \sup\{\eta_n : n \in \omega\}$.

Of course, the Axiom of Choice implies that $\mathrm{COF}(\xi)$ holds true for any $\xi \le \omega_1$. However, we want to avoid a use of **AC**.

Theorem 1.21. $\mathrm{COF}(\xi)$ *holds true for any* $\xi < \omega_1$.

Proof. If $\xi < \omega_1$, then there is a well-ordering R (we suppose that R is antireflexive) on ω such that $\mathrm{ot}(\omega, R) = \xi$. If $\eta < \xi$ is a limit ordinal, then there exists a natural number k such that η is the order type of the set $\{n \in \omega : nRk\}$. We set by induction

$$\eta_n = \max\{\eta_0, \dots, \eta_{n-1}, \mathrm{ot}(\{m \in \omega : mRn \wedge mRk\})\} + 1.$$

If $\zeta < \eta$, then there exists an $l \in \omega$ such that lRk and ζ is the order type of the set $\{n \in \omega : nRl\}$. Then $\eta_l > \zeta$. Set $F(\eta) = \{\eta_n\}_{n=0}^{\infty}$. □

Exercises

1.1 The Cumulative Hierarchy

The smallest transitive set containing a given set x as a subset is called the **transitive closure** of x.

a) Set $x_0 = x$, $x_{n+1} = \bigcup x_n$ for any n. Show that $\mathbf{TC}(x) = \bigcup_n x_n$ is the transitive closure of x.
Hint: If $y \in x_n$, then $y \subseteq x_{n+1}$.

b) We define the **Cumulative Hierarchy** $\langle V_\xi, \xi \in \mathbf{On}\rangle$ by transfinite induction:

$$\begin{aligned} V_0 &= \emptyset, \\ V_\xi &= \bigcup_{\eta<\xi} V_\eta, \text{ if } \xi \text{ is a limit ordinal}, \\ V_{\xi+1} &= \mathcal{P}(V_\xi). \end{aligned}$$

Show that every V_ξ is a transitive set.

c) Show that $x \in V_\xi$ if and only if $\mathbf{TC}(x) \in V_\xi$.

d) For every set x there exists an ordinal ξ such that $x \in V_\xi$, i.e., $\mathbf{V} = \bigcup_{\xi\in\mathbf{On}} V_\xi$.
Hint: Assume that there exists a transitive set x which does not belong to any V_ξ. By the Axiom of Regularity there exists $\in$-minimal element $y \in x$ which does not belong to any V_ξ. Thus every $z \in y$ belongs to some V_η. By suitable use of an instance of the Scheme of Replacement we obtain $y \subseteq V_\zeta$ for some ζ, a contradiction.

e) The **rank** of a set x is $\mathrm{rank(x)} = \min\{\xi : x \in V_{\xi+1}\}$. Show that $x \in y \to \mathrm{rank(x)} < \mathrm{rank(y)}$.

f) $\mathrm{rank(x)} = \sup\{\mathrm{rank(y)} : y \in x\} + 1$.

g) An uncountable regular cardinal κ is a strongly inaccessible cardinal if and only if $|V_\kappa| = \kappa$.

1.2 Cardinal Arithmetics without AC

a) $\aleph_0 \le |X|$ if and only if there exists a set $Y \subseteq X$, $|X| = |Y|$, $Y \ne X$.
Hint: If $f : X \xrightarrow[\text{onto}]{1-1} Y$, $a \in X \setminus Y$, then $|\{f^{-n}(a) : n \in \omega\}| = \aleph_0$.

b) If a set X is non-empty and $|X| + |X| = |X|$, then $\aleph_0 \le |X|$.

c) If $|X| > 1$ and $|X| \cdot |X| = |X|$, then $\aleph_0 \le |X|$.

d) If the cardinalities $|X|$ and $|Y|$ are incomparable (i.e., neither $|X| \le |Y|$ nor $|Y| \le |X|$), then $|X| < |X| + |Y|$ and $|X| < |X| \cdot |Y|$.

e) If $\aleph_1$ and $2^{\aleph_0}$ are incomparable, then $2^{\aleph_0} < 2^{\aleph_1}$.

f) If $\aleph_0 \le |X|$, then $\aleph_0 + |X| = |X|$.

g) Show that for any set X the following are equivalent:

1) $|X| \ge \aleph_0$;
2) $|X| + 1 = |X|$;
3) $|X| + \aleph_0 = |X|$.

1.3 Hartogs' Function

Hartogs' function $\aleph$ is defined as follows: for any set X the value $\aleph(X)$ is the first ordinal ξ such that there is no injection $f : \xi \xrightarrow{1-1} X$.

a) For every set X there exists an ordinal ξ such that $|\xi| \le |\mathcal{P}(X \times X)|$ and $|\xi| \not\le |X|$.
Hint: Consider the set W of all well-orderings of subsets of X. To each element $R \in W$ assign its ordinal type $h(R)$. Show that $\xi = \{h(R) : R \in W\}$ is the desired ordinal.

b) Hartogs' function $\aleph(X)$ is well defined.

c) $\aleph(X) \not\leq |X|$ for any infinite set X.

d) If the set X is infinite, then $\aleph(X)$ is an aleph.

e) A set X can be well ordered if and only if $\aleph(X) > |X|$.

f) $\aleph_1$ and $2^{\aleph_0}$ are incomparable if and only if $\aleph(\mathcal{P}(\omega)) = \aleph_1$.
Hint: $\aleph(\mathcal{P}(\omega)) \geq \aleph(\omega) = \aleph_1$, *since every infinite* $\xi < \omega_1$ *is the order type of a well-ordering of* ω.

1.4 Addition of cardinals and AC

a) If $|X| + \aleph(X) = |X| \cdot \aleph(X)$, then X can be well ordered.
Hint: Let $|Y| = \aleph(X)$, $X \cap Y = \emptyset$. *Assume that* $X \times Y = A \cup B$, *where* $|A| = |X|$ *and* $|B| = \aleph(X)$. *Since* $\aleph(X) \not\leq |X|$, *for every* $a \in X$ *there exists a* $b \in Y$ *such that* $\langle a, b\rangle \in B$. *Since* Y *can be well ordered, one define an injection of* X *into* Y.

b) $|X| + |Y| \leq |X| \cdot |Y|$ for any infinite X, Y.

c) The following are equivalent:

(1) $(\forall \mathfrak{m}, \mathfrak{n} \text{ infinite})\,(\mathfrak{m} + \mathfrak{n} = \mathfrak{m} \vee \mathfrak{m} + \mathfrak{n} = \mathfrak{n})$.

(2) $(\forall \mathfrak{m}, \mathfrak{n} \text{ infinite})\,(\mathfrak{m} \leq \mathfrak{n} \vee \mathfrak{n} \leq \mathfrak{m})$.

(3) $(\forall \mathfrak{m}, \mathfrak{n} \text{ infinite})\,(\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{m} \vee \mathfrak{m} \cdot \mathfrak{n} = \mathfrak{n})$.

(4) $(\forall \mathfrak{m} \text{ infinite})\,(\mathfrak{m}^2 = \mathfrak{m})$.

(5) $(\forall \mathfrak{m}, \mathfrak{n} \text{ infinite})\,(\mathfrak{m}^2 = \mathfrak{n}^2 \to \mathfrak{m} = \mathfrak{n})$.

(6) **AC**.

Hint: **AC** $\to (1) \to (2)$, **AC** $\to (3) \to (4) \to (5)$. (2) *implies* **AC**, *since* $|X| \leq \aleph(X)$ *for any infinite* X.
Assume (5). *If* X *is infinite, set* $\mathfrak{p} = |X|^{\aleph_0}$, $\mathfrak{m} = \mathfrak{p} + \aleph(\mathfrak{p})$, $\mathfrak{n} = \mathfrak{p} \cdot \aleph(\mathfrak{p})$. *Evidently* $\mathfrak{p} = \mathfrak{p} + 1 = 2 \cdot \mathfrak{p} = \mathfrak{p}^2$. *Similarly one has* $\aleph(\mathfrak{p}) = \aleph(\mathfrak{p}) + 1 = 2 \cdot \aleph(\mathfrak{p}) = (\aleph(\mathfrak{p}))^2$. *Then by simple calculation one obtains* $\mathfrak{m}^2 = \mathfrak{n}^2$. *Thus* $\mathfrak{m} = \mathfrak{n}$ *and by part* a) *any set of cardinality* $\mathfrak{p}$ *can be well ordered. Note that* $|X| \leq \mathfrak{p}$.

d) If for any infinite X the family $\{A \subseteq X : |A| < |X|\}$ is an ideal, then **AC** holds true.
Hint: If **AC** *fails, then there exist disjoint infinite sets* X, Y *such that* $|X| < |X| + |Y|$ *and* $|Y| < |X| + |Y|$.

e) If on every infinite set there exists a uniform ultrafilter, then **AC** holds true.
Hint: Let X, Y *be as in* d). *If* $\mathcal{F}$ *were a uniform ultrafilter on* $X \cup Y$, *then either* $X \in \mathcal{F}$ *or* $Y \in \mathcal{F}$.

1.5 Tarski's Lemma

Assume that M, P, Q are pairwise disjoint sets, $A = M \cup P$, $B = M \cup Q$ and $f : A \xrightarrow[\text{onto}]{1-1} B$.

We set $P_1 = \{x \in P : (\forall n > 0)\, f^n(x) \in M\}$, $Q_1 = \{x \in Q : (\forall n > 0)\, f^{-n}(x) \in M\}$, $P_2 = P \setminus P_1$, $Q_2 = Q \setminus Q_1$.

a) Show that $|P_2| = |Q_2|$.
Hint: Set

$$C_n = \{x \in P : (\forall k < n, k > 0)\, f^k(x) \in M \wedge f^n(x) \notin M\},$$
$$D_n = \{x \in Q : (\forall k < n, k > 0)\, f^{-k}(x) \in M \wedge f^{-n}(x) \notin M\}.$$