Giovanni Peccati
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Wiener Chaos: Moments, Cumulants and Diagrams

A survey with computer implementation

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Preface

The theoretical aspects of Time Series Analysis in the Gaussian context have been well understood since the end of the Second World War, more than 60 years ago. Linear transformations (or filters) preserve the Gaussian property and hence fit well in that framework. Norbert Wiener wanted to extend the theory to the non-linear world by considering non-linear transformations of Gaussian processes, while requesting that the output of the non-linear filters preserve the finite variance-covariance property, which is one the hallmarks of Gaussian processes. Kiyoshi Itô attempted to do the same thing in Japan. This extension is now known as the "Wiener chaos" and the corresponding stochastic integrals as "Wiener-Itô stochastic integrals". The versatility of the non-linear theory, however, turned out to be more limited than what was hoped for. It is not easy, for example, to write down the distributions of the random variables that live in this Wiener chaos. This mathematical challenge led several researchers to develop ad-hoc graphical devices, known as diagram formulae, allowing to derive moments and cumulants of chaotic random variables by means of combinatorial identities. Although these tools are not always easy to manipulate, they have been successfully used in order to develop not only new types of central limit theorems where the limit is Gaussian but also non-central limit theorems where the limit is non-Gaussian.

This is a book about combinatorial structures in the Wiener chaos. The combinatorial structures involved in our analysis are those of lattices of partitions of finite sets, over which we define incidence algebras, Möbius functions and associated inversion formulae. As discussed in the text, this combinatorial standpoint (which is originally due to Rota and Wallstrom [132]) provides an ideal framework in order to systematically deal with diagram formulae. Several applications are described, in particular recent limit theorems for chaotic random variables. An explicit computer implementation into the *Mathematica* language completes the text.

Chaotic random variables play now a crucial role in several areas of theoretical and applied probability. For instance, the fact that every functional of a Gaussian process or a Poisson measure can be written as a series of multiple integrals (the so-called "Wiener-Itô chaotic representation property") is one of the staples of Malliavin calculus and of its many applications, for example, to stochastic partial differential equations or stochastic calculus. In a recent series of papers (that are described and analyzed in the book), essentially written by Nourdin, Nualart, Peccati and Taqqu, it has been shown that the properties of chaotic random variables are the key elements for deriving very general (and strikingly simple) convergence results for non-linear functionals of Gaussian fields, Poisson measures or Lévy processes.

The goal of this monograph is to render this subject more accessible. We do this in a number of ways. We provide many examples to illustrate the theory and we also implement many of the formulas in *Mathematica*, so that the user can get a concrete feeling for the various topics. The theoretical exposition is rigorous. We have tried to fill in many of the steps in the proofs we provide, and when proofs are not given, we include detailed references. The bibliography, for example, is rather extensive, with more than 150 references. Our emphasis is on the combinatorial aspect of the subject because it is through combinatorics that the various objects are related. We start with describing partitions of a integer, of a set, the relations between them, we continue with moments and cumulants and cover a number of graphical descriptions of the various diagram formulae. When considering stochastic integrals, we do not always eliminate diagonals as is usually done, but we consider integrals where an arbitrary subset of diagonals has been eliminated, and we specify the explicit relations between them. The stochastic integrals include not only multiple integrals with respect to a Gaussian measure but also multiple integrals with respect to Poisson measures.

As anticipated, the subject is very much in flux with new limit theorems being developed and further applications, for example, to the Malliavin calculus. Although we do not cover these subjects, we provide an overview of various directions that are being pursued by researchers. This survey provides a basis for understanding the new developments.

The readership we have in mind includes researchers, but also graduate students who are either starting their research or are already working on a doctoral thesis. For a detailed description of the contents, please refer to the introduction and its subsection 1.1 “Overview”. The Contents, which include the title of the sections and subsections, also provide a good overview of the covered topics.

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who developed the *Mathematica* code, and without whom the computer applications would not have existed.

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1

Introduction

1.1 Overview

The aim of this work is to provide a unified treatment of moments and cumulants associated with non-linear functionals of completely random measures. A "completely random measure" (also called an "independently scattered random measure") is a measure φ with values in a space of random variables, such that $\varphi(A)$ and $\varphi(B)$ are independent random variables, whenever A and B are disjoint sets. Examples are Gaussian, Poisson or Gamma random measures. We will specifically focus on multiple stochastic integrals with respect to the random measure φ. These integrals are of the form

$$\int_{\sigma} f(z_1, ..., z_n)\varphi(dz_1)\cdots\varphi(dz_n) \tag{1.1.1}$$

and

$$\int_{\geq\sigma} f(z_1, ..., z_n)\varphi(dz_1)\cdots\varphi(dz_n), \tag{1.1.2}$$

where f is a symmetric function and φ is a completely random measure (for instance, Poisson or Gaussian) on the real line. The integration is not over all of $\mathbb{R}^n$, but over a "diagonal" subset of $\mathbb{R}^n$ defined by a partition σ of the integers $\{1, ..., n\}$ as illustrated below.

We shall mainly adopt a combinatorial point of view. Our main inspiration is a truly remarkable paper by Rota and Wallstrom [132], building (among many others) on earlier works by Itô [40], Meyer [76, 77] and, most importantly, Engel [26] (see also Bitcheler [9], Kussmaul [59], Linde [64], Masani [73], Neveu [79] and Tsilevich and Vershik [159] for related works). In particular, in [132] the authors point out a crucial connection between the machinery of multiple stochastic integration and the structure of the lattice of partitions of a finite set, with specific emphasis on the role played by the associated Möbius function (see e.g. [2], as well as Chapter 2 below). As we will see later on, the connection between multiple stochastic integration and partitions is given by the natural isomorphism between the partitions of the set $\{1, ..., n\}$ and the

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diagonal sets associated with the Cartesian product of n measurable spaces (a diagonal set is just a subset of the Cartesian product consisting of points that have two or more coordinates equal).

For example, going back to (1.1.1), if $n = 3$ and

$$\sigma = \{\{1,2\},\{3\}\}$$

in (1.1.1), then one integrates over $z_1, z_2, z_3 \in \mathbb{R}^3$ with

$$z_1 = z_2, \ z_2 \neq z_3.$$

If

$$\sigma = \{\{1\},\{2\},\{3\}\},$$

then the integration is over

$$z_1 \neq z_2, \ z_1 \neq z_3, \ z_2 \neq z_3$$

(commonly denoted $z_1 \neq z_2 \neq z_3$). In (1.1.2), one integrates over "$\geq \sigma$", that is, over all partitions that are coarser than σ. For example, if $n = 3$ and $\sigma = \{\{1,2\},\{3\}\}$, then "$\geq \sigma$" indicates that one has not only to integrate over $z_1 = z_2$, $z_2 \neq z_3$, but also on the hyperdiagonal $z_1 = z_2 = z_3$ (corresponding to the one-block partition $\{\{1,2,3\}\}$). If $n \geq 2$, the integrals (1.1.1) and (1.1.2) are non-linear functionals of φ, and thus, even if φ is Gaussian, these integrals are non-Gaussian random variables.

As we will see in Chapter 5, a crucial element of our analysis is given by random variables of the type (1.1.1), where the integration is performed over a set *without diagonals*, that is, where all the coordinates z_i are different. Random variables of this type belong to the so-called **Wiener chaos** associated with the random measure φ. Chaotic random variables play now a crucial role in several areas of theoretical and applied probability. For instance, we will see that they enjoy several remarkable connections with orthogonal polynomials, such as Hermite and Charlier polynomials; also, we will show that every square-integrable functional of a Gaussian process or of a Poisson measure can be written as a series of multiple integrals over non-diagonal sets (the so-called "Wiener-Itô chaotic representation property"). This last fact is one of the staples of Malliavin calculus (see [93, 94, 21]) and of its many applications e.g. to stochastic partial differential equations or stochastic calculus. In a recent series of papers (that are described and analyzed in the last chapters of this book), it has been shown that the properties of chaotic random variables are fundamental in deriving convergence results for non-linear functionals of Gaussian fields, Poisson measures or Lévy processes.

The best description of the approach to stochastic integration followed in the present work is still given by the following sentences, taken from [132]:

> The basic difficulty of stochastic integration is the following. We are given a measure φ on a set S, and we wish to extend such a measure to the product set S^n. There is a well-known and established way of carrying out such

> an extension, namely, taking the product measure. While the product measure is adequate in most instances dealing with a scalar valued measure, it turns out to be woefully inadequate when the measure is vector-valued, or, in the case dealt with presently, random-valued. The product measure of a nonatomic scalar measure will vanish on sets supported by lower-dimensional linear subspaces of S^n. This is not the case, however, for random measures. The problem therefore arises of modifying the definition of product measure of a random measure in such a way that the resulting measure will vanish on lower-dimensional subsets of S^n, or diagonal sets, as we call them.

As pointed out in [132], as well as in Chapter 5 below, the combinatorics of partition lattices provide the correct framework in order to define a satisfactory stochastic product measure.

As discussed in detail in Chapter 8, part of the results presented in this work extend to the case of *isonormal Gaussian processes*, that is, centered Gaussian families whose covariance structure is isomorphic to some (real or complex) Hilbert space. Note that isonormal Gaussian processes have gained enormous importance in recent years, for instance in connection with fractional processes (see e.g. the second edition of Nualart's book [94]), or as limit objects (known as *Gaussian Free Fields*) appearing e.g. in the theory of random matrices and random surfaces (see [129] and [139] for some general discussions in this direction).

To make the notions we introduce concrete, we have included an Appendix to this survey. It is devoted to a *Mathematica*[1] implementation of various formulae. We stress, however, that no knowledge of the *Mathematica* language is required. This is because we provide in the Appendix detailed instructions and examples. This unusual addendum may be welcome, in particular, by graduate students and new researchers in this area.

As apparent from the title, in the subsequent chapters a prominent role will be played by moments and cumulants. In particular, the principal aims of our work are the following:

– **Put diagram formulae in a proper algebraic setting.** Diagram formulae are mnemonic devices, allowing to compute moments and cumulants associated with one or more random variables. These tools have been developed and applied in a variety of frameworks: see e.g. [141, 151] for diagram formulae associated with general random variables; see [10, 12, 34, 68] for non-linear functionals of Gaussian fields; see [150] for non-linear functionals of Poisson measures. They can be quite useful in the obtention of Central Limit Theorems (CLTs) by means of the so-called *method of moments and cumulants* (see e.g. [66]). Inspired by the works by McCullagh [74], Rota and Shen [131] and Speed [145], we shall show that all diagram formulae quoted above can be put in a unified framework, based on the

[1] *Mathematica* is a computational software program developed by Wolfram Research, which is used widely in scientific and mathematical fields.

use of partitions of finite sets. Although somewhat implicit in the previously quoted references, this clear algebraic interpretation of diagrams is new. In particular, in Chapter 4 we will show that all diagrams encountered in the probabilistic literature (such as Gaussian, non-flat and connected diagrams) admit a neat translation in the combinatorial language of partition lattices.

– **Illustrate the Engel-Rota-Wallstrom theory.** We shall show that the theory developed in [26] and [132] allows to recover several crucial results of stochastic analysis, such as multiplication formulae for multiple Gaussian and Poisson integrals see [49, 94, 150]. This extends the content of [132], which basically dealt with product measures. See also [28] for other results in this direction.
– **Shed light the combinatorial implications of new CLTs.** In a recent series of papers (see [69, 82, 86, 87, 90, 95, 98, 103, 106, 109, 110, 111]), a new set of tools has been developed, allowing to deduce simple CLTs involving random variables having the form of multiple stochastic integrals. All these results can be seen as simplifications of the method of moments and cumulants. In Chapter 11, we will illustrate these results from a combinatorial standpoint, by providing some neat interpretations in terms of diagrams and graphs. In particular, we will prove that in these limit theorems a fundamental role is played by the so-called *circular diagrams*, that is, connected Gaussian diagrams whose edges only connect subsequent rows.

We will develop the necessary combinatorial tools related to partitions, diagram and graphs from first principles in Chapter 2 and Chapter 4. Chapter 3 provides a self-contained treament of moments and cumulants from a combinatorial point of view. Stochastic integration is introduced in Chapter 5. Chapter 6 and Chapter 7 deal, respectively, with product formulae and diagram formulae. Chapter 8 deals with Gaussian random measures, isonormal Gaussian processes and the relationship between corresponding multiple integrals and Hermite polynomials. In Chapter 9 we describe Hermitian random measures and spectral representations and define the Hermite processes. In Chapter 10 we introduce Charlier polynomials and relate them to multiple integrals with respect to a Poisson random measure. Chapter 11 and Chapter 12 deal with CLTs on Wiener and Poisson chaos respectively. There are two appendices[2]. Appendix A describes the *Mathematica* commands. These are also listed in the Contents. Finally, Appendix B contains tables of moments and cumulants.

[2] The index does not include the appendices.

1.2 Some related topics

In this survey, we choose to follow a very precise path, namely starting with the basic properties of partition lattices and diagrams, and develop from there as many as possible of the formulae associated with products, moments and cumulants in the theory of stochastic integration with respect to completely random measures. In order to keep the length of the present work within bounds, several crucial topics are not included (or are just mentioned) in the discussion to follow. One remarkable omission is of course a complete discussion of the connections between multiple stochastic integrals and orthogonal polynomials. This topic is partially treated in Chapters 8 and 10 below, in the particular case of Gaussian and Poisson fields. For recent references on more general stochastic processes (such as Lévy processes), see e.g. the monograph by Schoutens [137] and the two papers by Solé and Utzet [143, 144]. Other related (and missing) topics are detailed in the next list, whose entries are followed by a brief discussion.

- *Wick products*. Wick products are intimately related to chaotic expansions. A complete treatment of this topic can be found e.g. in Janson's book [46].
- *Malliavin calculus.* See the two monographs by Nualart [93, 94] for Malliavin calculus in a Gaussian setting. The monograph by Di Nunno, Øksendal and Proske [21] provides an introduction to Malliavin calculus with applications to finance. A good introduction to Malliavin calculus for Poisson measures is contained in the classic papers by Nualart and Vives [100], Privault [120] and Privault and Wu [125], as well as in the recent monograph by Privault [122]. A fundamental connection between Malliavin operators and limit theorems has been first pointed out in [96]. See [82, 86, 106] for further developments.
- *Hu-Meyer formulae.* Hu-Meyer formulae connect Stratonovich multiple integrals and multiple Wiener-Itô integrals. See [94] for a standard discussion of this topic in a Gaussian setting. Hu-Meyer formulae for general Lévy processes can be naturally obtained by means of the theory described in the present book: see the excellent paper by Farré *et al.* [28] for a complete treatment of this point.
- *Stein's method.* Stein's method for normal and non-normal approximation can be a very powerful tool in order to obtain central and non-central limit theorems for non-linear functionals of random fields. In Chapter 3, we will only scratch the surface of this topic, by proving two basic results related to Stein's method, namely the Stein's Lemma for the normal distribution, and the Chen-Stein Lemma for the Poisson distribution. Some further discussion is contained in Chapter 11. See [147] for a classic reference on the subject and [15] for an exhaustive recent monograph. See [86, 87, 90] for several limit theorems involving functionals of Gaussian fields, obtained by means of Stein's method and Malliavin calculus. See [106, 114] for applications of Stein's method to functionals of Poisson measures.
- *Free probability.* The properties of the lattice of (non-crossing) partitions and the corresponding Möbius function are crucial in free probability. See the monograph by Nica and Speicher [80] for a valuable introduction to the combinatorial aspects

of free probability. See Anshelevich [3, 4] for some instances of a “free” theory of multiple stochastic integration. The paper [53], by Kemp *et al.*, establishes some explicit connections between limit theorems in free probability and the topics discussed in Chapter 11 below.

2

The lattice of partitions of a finite set

In this chapter we recall some combinatorial results concerning the lattice of partitions of a finite set. These objects play an important role in the obtention of the *diagram formulae* presented in Chapter 5. The reader is referred to Stanley [146, Ch. 3] and Aigner [2] for a detailed presentation of (finite) partially ordered sets and Möbius inversion formulae.

2.1 Partitions of a positive integer

Given an integer $n \geq 1$, we define the set $\Lambda(n)$ of *partitions* of n as the collection of all vectors of the type $\lambda = (\lambda_1, ..., \lambda_k)$ $(k \geq 1)$, where:

$$\begin{aligned} &\text{(i) } \lambda_j \text{ is an integer for every } j = 1, ..., k; \\ &\text{(ii) } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1; \\ &\text{(iii) } \lambda_1 + \cdots + \lambda_k = n. \end{aligned} \tag{2.1.1}$$

We call k the *length* of λ. It is sometimes convenient to write a partition $\lambda = (\lambda_1, ..., \lambda_k) \in \Lambda(n)$ in the form

$$\boxed{\lambda = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \,.}$$

This representation (which encodes all information about λ) simply indicates that, for every $i = 1, ..., n$, the vector λ contains exactly r_i (≥ 0) components equal to i. Clearly, if $\lambda = (\lambda_1, ..., \lambda_k) = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \in \Lambda(n)$, then

$$1r_1 + \cdots + nr_n = n \tag{2.1.2}$$

and $r_1 + \cdots + r_n = k$. We will sometimes use the (more conventional) notation

$$\boxed{\lambda \vdash n \quad \text{instead of} \quad \lambda \in \Lambda(n).}$$

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Example 2.1.1 (i) If $n = 5$, one can, for example, have $5 = 4 + 1$ or $5 = 1 + 1 + 1 + 1 + 1$. In the first case the length is $k = 2$, with $\lambda_1 = 4$ and $\lambda_2 = 1$, and the partition is $\lambda = \left(1^1 2^0 3^0 4^1 5^0\right)$. In the second case, the length is $k = 5$ with $\lambda_1 = ... = \lambda_5 = 1$, and the partition is $\lambda = \left(1^5 2^0 3^0 4^0 5^0\right)$.

(ii) One can go easily from one representation to the other. Thus $\lambda = \left(1^2 2^3 3^0 4^2\right)$ corresponds to

$$n = (1 \times 2) + (2 \times 3) + (3 \times 0) + (4 \times 2) = 16,$$

that is, to the decomposition $16 = 4 + 4 + 2 + 2 + 2 + 1 + 1$, and thus to

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = (4, 4, 2, 2, 2, 1, 1).$$

Remark. Fix $n \geq 2$ and $k \in \{1, ..., n\}$. Consider a partition of n of length k, say $\lambda = (\lambda_1, ..., \lambda_k)$, and write it in the form $\lambda = (1^{r_1} 2^{r_2} \cdots n^{r_n})$. Then, one has necessarily that $r_j = 0$ for every $j > n - k + 1$ (or, equivalently, $n \leq j + k - 2$). Indeed, if $r_j > 0$ for such a j, then there would exist $\lambda_{a^*} \in \lambda$ such that $n \leq \lambda_{a^*} + k - 2$. This contradicts the inequality $n = \lambda_1 + \cdots \lambda_k \geq \lambda_{a^*} + k - 1$, which results from the fact that λ is composed of k strictly positive integers whose sum equals n. This fact implies that, for fixed $n \geq 2$ and $k \in \{1, ..., n\}$, every partition $\lambda \in \Lambda(n)$ of length k has the form

$$\lambda = \left(1^{r_1} 2^{r_2} \cdots (n - k + 1)^{r_{n-k+1}} (n - k + 2)^0 \cdots n^0\right), \tag{2.1.3}$$

thus yielding immediately the following statement.

Proposition 2.1.2 *There exists a bijection, say β, between the subset of $\Lambda(n)$ composed of partitions with length k and the collection of all vectors $(r_1, ..., r_{n-k+1})$ of nonnegative integers such that*

$$r_1 + \cdots + r_{n-k+1} = k \quad \textit{and} \quad 1r_1 + 2r_2 + \cdots + (n - k + 1)\, r_{n-k+1} = n. \tag{2.1.4}$$

The bijection is obtained as follows: if $(r_1, ..., r_{n-k+1})$ verifies (2.1.4), then β^{-1} $(r_1, ..., r_{n-k+1})$ is the element of $\Lambda(n)$ of length k given by (2.1.3).

A vector of nonnegative integers $(r_1, ..., r_m)$ verifying $r_1 + \cdots + r_m = k$ is customarily called a *weak m-composition* of k (see for example Stanley [146, p. 15]).

Example 2.1.3 Consider the case $n = 4$ and $k = 2$. Then, there exist only two vectors (r_1, r_2, r_3) of nonnegative integers satisfying (2.1.4) (that is, such that $r_1 + r_2 + r_3 = 2$ and $r_1 + 2r_2 + 3r_3 = 4$), namely $(1, 0, 1)$ and $(0, 2, 0)$. These vectors correspond respectively to the partitions $\lambda_1 = (3, 1) = (1^1 2^0 3^1 4^0)$ and $\lambda_2 = (2, 2) = (1^0 2^2 3^0 4^0)$. Proposition 2.1.2 implies that λ_1 and λ_2 are the only elements of $\Lambda(4)$ having length 2.

2.2 Partitions of a set

Let b denote a finite nonempty set and let

$$\boxed{\mathcal{P}(b) \text{ be the set of } \textit{partitions} \text{ of } b.}$$

By definition, an element π of $\mathcal{P}(b)$ is a collection of nonempty and disjoint subsets of b (called *blocks*), such that their union equals b. The symbol $|\pi|$ indicates the number of blocks (or the *size*) of the partition π.

Notation. For each pair $i, j \in b$ and for each $\pi \in \mathcal{P}(b)$, we write

$$\boxed{i \sim_\pi j}$$

whenever i and j belong to the same block of π.

We now define a partial ordering on $\mathcal{P}(b)$. For every $\sigma, \pi \in \mathcal{P}(b)$, we write

$$\boxed{\sigma \leq \pi}$$

if and only if

$$\boxed{\text{each block of } \sigma \text{ is contained in a block of } \pi.}$$

Thus,

$$\boxed{\text{If} \quad \sigma \leq \pi, \quad \text{then} \quad |\sigma| \geq |\pi|.}$$

Borrowing from the terminology used in topology one also says that π is *coarser* than σ. It is clear that $\leq$ is a *partial ordering relation*, that is, $\leq$ is a *binary* relation on $\mathcal{P}(b)$, which is also *reflexive*, *transitive* and *antisymmetric*, that is:

(i) $\sigma \leq \sigma$, for every $\sigma \in \mathcal{P}(b)$ (reflexivity);
(ii) if $\sigma \leq \pi$ and $\pi \leq \rho$, then $\sigma \leq \rho$ (transitivity);
(iii) if $\sigma \leq \pi$ and $\pi \leq \sigma$, then $\sigma = \pi$ (antisymmetry);

(see also Stanley [146, pp. 97-98]).

Example 2.2.1 (i) If $b = \{1,2,3,4,5\}$, $\pi = \{\{1,2,3\},\{4,5\}\}$ and $\sigma = \{\{1,2\},\{3\},\{4,5\}\}$, then, $\sigma \leq \pi$ because each block of σ is contained in a block of π. We have $3 = |\sigma| > |\pi| = 2$.

(ii) If $\pi = \{\{1,2,3\},\{4,5\}\}$ and $\sigma = \{\{1,2\},\{3,4,5\}\}$, then π and σ are not ordered.

Moreover, the relation $\leq$ induces on $\mathcal{P}(b)$ a *lattice* structure. Recall that a lattice is a partially ordered set such that each pair of elements has a least upper bound and a greatest lower bound (see the forthcoming remark for a definition of these two notions, as well as [146, p. 102]). In particular, the partition

$$\sigma \wedge \pi, \quad \sigma, \pi \in \mathcal{P}(b),$$

called *meet* of σ and π, is the partition of b such that each block of $\sigma \wedge \pi$ is a nonempty intersection between one block of σ and one block of π. On the other hand, the partition

$$\sigma \vee \pi, \quad \sigma, \pi \in \mathcal{P}(b),$$

called *join* of σ and π, is the element of $\mathcal{P}(b)$ whose blocks are constructed by taking the non-disjoint unions of the blocks of σ and π, that is, by taking the union of those blocks that have at least one element in common.

Remarks. (a) Whenever $\pi_1 \leq \pi_2$, one has $|\pi_1| \geq |\pi_2|$. In particular, $|\sigma \wedge \pi| \geq |\sigma \vee \pi|$.

(b) The partition $\sigma \wedge \pi$ is the greatest lower bound associated with the pair (σ, π). As such, $\sigma \wedge \pi$ is completely characterized by the property of being the unique element of $\mathcal{P}(b)$ such that: (i) $\sigma \wedge \pi \leq \sigma$, (ii) $\sigma \wedge \pi \leq \pi$, and (iii) $\rho \leq \sigma \wedge \pi$ for every $\rho \in \mathcal{P}(b)$ such that $\rho \leq \sigma, \pi$.

(c) Analogously, the partition $\sigma \vee \pi$ is the least upper bound associated with the pair (σ, π). It follows that $\sigma \vee \pi$ is completely characterized by the property of being the unique element of $\mathcal{P}(b)$ such that: (i) $\sigma \leq \sigma \vee \pi$, (ii) $\pi \leq \sigma \vee \pi$, and (iii) $\sigma \vee \pi \leq \rho$ for every $\rho \in \mathcal{P}(b)$ such that $\sigma, \pi \leq \rho$.

Example 2.2.2 (i) Take $b = \{1,2,3,4,5\}$. If $\pi = \{\{1,2,3\},\{4,5\}\}$ and $\sigma = \{\{1,2\},\{3\},\{4,5\}\}$, then, as noted above, $\sigma \leq \pi$ and we have

$$\sigma \wedge \pi = \sigma \quad \text{and} \quad \sigma \vee \pi = \pi.$$

A graphical representation of π, σ, $\sigma \wedge \pi$ and $\sigma \vee \pi$ is:

$\pi =$	1 2 3	4 5	
$\sigma =$	1 2	3	4 5
$\sigma \wedge \pi =$	1 2	3	4 5
$\sigma \vee \pi =$	1 2 3	4 5	

(ii) If $\pi = \{\{1,2,3\},\{4,5\}\}$ and $\sigma = \{\{1,2\},\{3,4,5\}\}$, then π and σ are not ordered and

$$\sigma \wedge \pi = \{\{1,2\},\{3\},\{4,5\}\} \quad \text{and} \quad \sigma \vee \pi = \{b\} = \{\{1,2,3,4,5\}\}.$$

A graphical representation of π, σ, $\sigma \wedge \pi$ and $\sigma \vee \pi$ is:

$$\pi = \begin{array}{|c|c|}\hline 1\ 2\ 3 & 4\ 5 \\ \hline\end{array}$$

$$\sigma = \begin{array}{|c|c|}\hline 1\ 2 & 3\ 4\ 5 \\ \hline\end{array}$$

$$\sigma \wedge \pi = \begin{array}{|c|c|c|}\hline 1\ 2 & 3 & 4\ 5 \\ \hline\end{array}$$

$$\sigma \vee \pi = \begin{array}{|c|}\hline 1\ 2\ 3\ 4\ 5 \\ \hline\end{array}$$

(iii) A convenient way to build $\sigma \vee \pi$ is to do it in successive steps. Take the union of two blocks with a common element and look at it as a new block of π. See if it shares an element with another block of σ. If yes, repeat. For instance, suppose that $\pi = \{\{1,2\},\{3\},\{4\}\}$ and $\sigma = \{\{1,3\},\{2,4\}\}$. Then, π and σ are not ordered and

$$\sigma \wedge \pi = \{\{1\},\{2\},\{3\},\{4\}\} \quad \text{and} \quad \sigma \vee \pi = \{\{1,2,3,4\}\}.$$

One now obtains $\sigma \vee \pi$ by noting that the element 2 is common to $\{1,2\} \in \pi$ and $\{2,4\} \in \sigma$, and the "merged" block $\{1,2,4\}$ shares the element 1 with the block $\{1,3\} \in \sigma$, thus implying the conclusion. A graphical representation of π, σ, $\sigma \wedge \pi$ and $\sigma \vee \pi$ is:

$$\pi = \begin{array}{|c|c|c|}\hline 1\ 2 & 3 & 4 \\ \hline\end{array}$$

$$\sigma = \begin{array}{|c|c|}\hline 1\ 3 & 2\ 4 \\ \hline\end{array}$$

$$\sigma \wedge \pi = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$$

$$\sigma \vee \pi = \begin{array}{|c|}\hline 1\ 2\ 3\ 4 \\ \hline\end{array}$$

Notation. When displaying a partition π of $\{1, ..., n\}$ ($n \geq 1$), the blocks $b_1, ..., b_k \in \pi$ will always be listed in the following way: b_1 will always contain the element 1, and

$$\min\{i : i \in b_j\} < \min\{i : i \in b_{j+1}\}, \ j = 1, ..., k-1.$$

Also, the elements within each block will be always listed in increasing order. For instance, if $n = 6$ and the partition π involves the blocks $\{2\}, \{4\}, \{1,6\}$ and $\{3,5\}$, we will write $\pi = \{\{1,6\},\{2\},\{3,5\},\{4\}\}$.

Definition 2.2.3 *The **maximal element** of $\mathcal{P}(b)$ is the trivial partition*

$$\boxed{\hat{1} = \{b\}.}$$

*The **minimal element** of $\mathcal{P}(b)$ is*

the partition $\hat{0}$, such that each block of $\hat{0}$ contains exactly one element of b.

Observe that $\left|\hat{1}\right| = 1$ and $\left|\hat{0}\right| = |b|$, and also $\hat{0} \leq \hat{1}$. Thus if $b = \{1,2,3\}$, then

$$\hat{0} = \{\{1\},\{2\},\{3\}\} \quad \text{and} \quad \hat{1} = \{1,2,3\}.$$

Definition 2.2.4 *The **partition segment** (or **interval**) $[\sigma,\pi]$ in $\mathcal{P}(b)$, with $\sigma \leq \pi$, is the following subset of partitions of b:*

$$\boxed{[\sigma,\pi] = \{\rho \in \mathcal{P}(b) : \sigma \leq \rho \leq \pi\}.}$$

Plainly,

$$\mathcal{P}(b) = \left[\hat{0},\hat{1}\right].$$

2.3 Partitions of a set and partitions of an integer

We now focus on the notion of *class*, which associates to a segment of partitions a partition of an integer.

Definition 2.3.1 *The **class** of a segment $[\sigma,\pi]$ ($\sigma \leq \pi$), denoted $\lambda(\sigma,\pi)$, is defined as the partition of the integer $|\sigma|$ given by*

$$\lambda(\sigma,\pi) = (1^{r_1}2^{r_2}\cdots|\sigma|^{r_{|\sigma|}}), \tag{2.3.5}$$

where r_i, $i = 1,...,|\sigma|$, indicates the number of blocks of π that contain exactly i blocks of σ. We stress that necessarily $|\sigma| \geq |\pi|$, and also

$$|\sigma| = 1r_1 + 2r_2 + \cdots + |\sigma|\,r_{|\sigma|} \quad \textit{and} \quad |\pi| = r_1 + \cdots + r_{|\sigma|}.$$

The lenght of $\lambda(\sigma,\pi)$ equals $|\pi|$.

Example 2.3.2 (i) If $\pi = \{\{1,2,3\},\{4,5\}\}$ and $\sigma = \{\{1,2\},\{3\},\{4,5\}\}$, then since $\{1,2\}$ and $\{3\}$ are contained in $\{1,2,3\}$ and $\{4,5\}$ in $\{4,5\}$, we have 1 block of π (namely $\{4,5\}$) containing 1 block of σ and 1 block of π (namely $\{1,2,3\}$) containing 2 blocks of σ. Thus $r_1 = 1$, $r_2 = 1$, $r_3 = 0$, that is, $\lambda(\sigma,\pi) = (1^1 2^1 3^0)$, corresponding to the partition of the integer $3 = 2+1$.

(ii) In view of (2.1.1), one may suppress the terms $r_i = 0$ in (2.3.5), and write for instance $\lambda(\sigma,\pi) = (1^1 2^0 3^2) = (1^1 3^2)$ for the class of the segment $[\sigma,\pi]$, associated with the two partitions $\sigma = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$ and $\pi = \{\{1\},\{2,3,4\},\{5,6,7\}\}$.

From now on, we let

$$\boxed{[n] = \{1, \cdots, n\}, \quad n \geq 1.} \tag{2.3.6}$$

With this notation, the maximal and minimal element of the set $\mathcal{P}([n])$ are given, respectively, by

$$\boxed{\hat{1} = \{[n]\} = \{\{1, ..., n\}\} \text{ and } \hat{0} = \{\{1\}, ..., \{n\}\}.} \tag{2.3.7}$$

Now fix a set b, say $b = [n] = \{1, \cdots, n\}$ with $n \geq 1$ and consider the partition $\hat{0} = \{\{1\}, ..., \{n\}\}$. Then, for a fixed $\lambda = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \vdash n$, the number of partitions $\pi \in \mathcal{P}(b)$ such that $\lambda\left(\hat{0}, \pi\right) = \lambda$ is given by

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} \triangleq \begin{bmatrix} n \\ r_1, ..., r_n \end{bmatrix} = \frac{n!}{(1!)^{r_1} r_1! (2!)^{r_2} r_2! \cdots (n!)^{r_n} r_n!}. \tag{2.3.8}$$

This is the number of partitions π containing exactly r_1 blocks of size 1, r_2 blocks of size 2, ..., r_n blocks of size n. Equation (2.3.8) follows from the following fact. Fix $i = 1, ..., n$. If π contains r_i blocks of size i, then the elements in each block can be permuted within the block, yielding $(i!)^{r_i}$ possibilities and, in addition, the posiiton of the r_i blocks can be permuted as well, yielding $r_i!$ possiblities (see, for example, [146] for more details). The requirement that $\lambda\left(\hat{0}, \pi\right) = \lambda = (1^{r_1} 2^{r_2} \cdots n^{r_n})$ simply means that, for each $i = 1, ..., n$, the partition π must have exactly r_i blocks containing i elements of b. Recall that the integers $r_1, ..., r_n$ must satisfy (2.1.2), namely $1r_1 + \cdots + nr_n = n$.

Example 2.3.3 (i) For any finite set b, one has always that

$$\lambda\left(\hat{0}, \hat{1}\right) = \left(1^0 2^0 \cdots |b|^1\right),$$

because $\hat{1}$ has only one block, namely b, and that block contains $|b|$ blocks of $\hat{0}$.

(ii) Fix $k \geq 1$ and let b be such that $|b| = n \geq k+1$. Consider $\lambda = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \vdash n$ be such that $r_k = r_{n-k} = 1$ and $r_j = 0$ for every $j \neq k, n-k$. For instance, if $n = 5$ and $k = 2$, then $\lambda = \left(1^0 2^1 3^1 4^0 5^0\right)$. Then, each partition $\pi \in \mathcal{P}(b)$ such that $\lambda\left(\hat{0}, \pi\right) = \lambda$ has only one block of k elements and one block of $n-k$ elements. To construct such a partition, it is sufficient to specify the block of k elements. This implies that there exists a bijection between the set of partitions $\pi \in \mathcal{P}(b)$ such that $\lambda\left(\hat{0}, \pi\right) = \lambda$ and the collection of the subsets of b having exactly k elements. In particular, (2.3.8) gives

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(iii) Let $b = [7] = \{1, ..., 7\}$ and $\lambda = \left(1^1 2^3 3^0 4^0 5^0 6^0 7^0\right)$. Then, (2.3.8) implies that there are exactly $\frac{7!}{3!(2!)^3} = 105$ partitions $\pi \in \mathcal{P}(b)$, such that $\lambda\left(\hat{0}, \pi\right) = \lambda$. One of these partitions is $\{\{1\}, \{2,3\}, \{4,5\}, \{6,7\}\}$. Another is $\{\{1,7\}, \{2\}, \{3,4\}, \{5,6\}\}$.

(iv) Let $b = [5] = \{1, ..., 5\}$, $\sigma = \{\{1\}, \{2\}, \{3\}, \{4,5\}\}$ and $\pi = \{\{1,2,3\}, \{4,5\}\}$. Then, $\sigma \leq \pi$ and the set of partitions defined by the interval $[\sigma, \pi]$ is $\{\sigma, \pi, \rho_1, \rho_2, \rho_3\}$, where

$$\begin{aligned}\rho_1 &= \{\{1,2\}, \{3\}, \{4,5\}\} \\ \rho_2 &= \{\{1,3\}, \{2\}, \{4,5\}\} \\ \rho_3 &= \{\{1\}, \{2,3\}, \{4,5\}\}.\end{aligned}$$

The partitions ρ_1, ρ_2 and ρ_3 are not ordered (i.e., for every $1 \leq i \neq j \leq 3$, one cannot write $\rho_i \leq \rho_j$), and are built by taking unions of blocks of σ in such a way that they are contained in blocks of π. Moreover, $\lambda(\sigma, \pi) = \left(1^1 2^0 3^1 4^0 5^0\right)$, since there is exactly one block of π containing one block of σ, and one block of π containing three blocks of σ.

(v) This example is related to the techniques developed in Chapter 6. Fix $n \geq 2$, as well as a partition $\gamma = (\gamma_1, ..., \gamma_k) \in \Lambda(n)$ such that $\gamma_k \geq 1$. Recall that, by definition, one has that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k$ and $\gamma_1 + \cdots + \gamma_k = n$. Now consider the segment $\left[\hat{0}, \pi\right]$, where

$$\begin{aligned}\hat{0} &= \{\{1\}, \{2\}, ..., \{n\}\}, \text{ and} \\ \pi &= \{\{1, ..., \gamma_1\}, \{\gamma_1 + 1, ..., \gamma_1 + \gamma_2\}, ..., \{\gamma_1 + \cdots + \gamma_{k-1} + 1, ..., n\}\}.\end{aligned}$$

Then, the jth block of π contains exactly γ_j blocks of $\hat{0}$, for every $j = 1, ..., k$, implying that the class of the segment $[\hat{0}, \pi]$ coincides with $(\gamma_1, ..., \gamma_k)$. For instance, when $\gamma_1 > \gamma_2 > ... > \gamma_k$ (that is, all the γ_i's are different), one has that the class $\lambda\left(\hat{0}, \pi\right)$ is such that $\lambda\left(\hat{0}, \pi\right) = \left(\gamma_k^1 \gamma_{k-1}^1 \cdots \gamma_1^1\right) = \gamma$, after suppressing the indicators of the type r^0.

The following statement is a consequence of (2.3.8) and of Proposition 2.1.2.

Proposition 2.3.4 *Let $n \geq 1$ and $k \in \{1, ..., n\}$. Then, the number of partitions of $[n]$ having exactly k blocks is given by*

$$S(n,k) \triangleq \sum_{r_1, \ldots, r_{n-k+1}} \frac{n!}{(1!)^{r_1} r_1! \, (2!)^{r_2} r_2! \cdots (n-k+1)!^{r_{n-k+1}} r_{n-k+1}!}, \qquad (2.3.9)$$

where the sum runs over all vectors on nonnegative integers $(r_1, ..., r_{n-k+1})$ satisfying $r_1 + \cdots + r_{n-k+1} = k$ and $1r_1 + 2r_2 \cdots + (n-k+1) r_{n-k+1} = n$.

Proof. By Definition 2.3.1, a partition $\pi \in \mathcal{P}([n])$ has k blocks if and only if $\lambda\left(\hat{0}, \pi\right)$ has length k. Since $\lambda\left(\hat{0}, \pi\right)$ is a partition of the integer n, then using Proposition 2.1.2 one deduces that $\pi \in \mathcal{P}([n])$ has k blocks if and only if $\lambda\left(\hat{0}, \pi\right)$ has the form of the right-hand side of (2.1.3), for some vector on nonnegative integers $(r_1, ..., r_{n-k+1})$ satisfying $r_1 + \cdots + r_{n-k+1} = k$ and $1r_1 + 2r_2 \cdots + (n-k+1)\, r_{n-k+1} = n$. The proof is concluded by using (2.3.8). ■

Remark. One defines customarily $S(0,0) = 1$ and, for $n \geq 1$, $S(n,0) = S(n,k) = 0$, for every $k > n$. The integers

$$S(n,k), \quad n,k \geq 0, \tag{2.3.10}$$

defined by these conventions and by (2.3.9), are called the *Stirling numbers of the second kind*. See, for example, [158, Ch. 8] and [146, p. 33] for some exhaustive presentations of the properties of Stirling numbers. See Section 2.4 for a connection with Bell and Touchard polynomials.

Example 2.3.5 (i) For every $n \geq 1$, one has that $S(n,1) = 1$, that is, there exists only one partition of $[n]$ containing exactly one block (i.e., the trivial partition $\{[n]\}$). To see that this is consistent with (2.3.9) in the case $k = 1$, observe that the only integer solution to the system

$$r_1 + \cdots + r_n = 1, \text{ and } 1r_1 + 2r_2 \cdots + n\, r_n = n,$$

is given by $r_1 = \cdots = r_{n-1} = 0$ and $r_n = 1$, and in this case

$$\frac{n!}{(1!)^{r_1}\, r_1!\, (2!)^{r_2}\, r_2! \cdots (n!)^{r_n} r_n!} = 1.$$

By a similar route, one also checks that $S(n,n) = 1$.

(ii) Fix $n \geq 3$. We want to compute $S(n,2)$, that is, the number of partitions of $[n]$ containing exactly two blocks. This case corresponds to $k = 2$, and one has therefore to consider the system

$$r_1 + \cdots + r_{n-1} = 2, \text{ and } 1r_1 + 2r_2 \cdots + (n-1)\, r_{n-1} = n.$$

When n is even, this system has exactly $n/2$ solutions, obtained by choosing either $r_{n/2} = 2$ and $r_l = 0$ elsewhere, or $r_j = r_{n-j} = 1$ and $r_l = 0$ elsewhere, for some $j = 1, ..., n/2 - 1$. On the other hand, when n is odd the system has exactly $(n-1)/2$ solutions, obtained by choosing $r_j = r_{n-j} = 1$ and $r_l = 0$ elsewhere, for some $j = 1, ..., (n-1)/2$. Using (2.3.9), we therefore deduce that

$$S(n,2) = \sum_{j=1}^{(n/2)-1} \binom{n}{j} + \frac{1}{2}\binom{n}{n/2}, \quad \text{if } n \text{ is even,}$$

$$S(n,2) = \sum_{j=1}^{(n-1)/2} \binom{n}{j}, \quad \text{if } n \text{ is odd.}$$

For instance $S(3,2) = \binom{3}{1} = 3$, $S(4,2) = \binom{4}{1} + \frac{1}{2}\binom{4}{2} = 4 + 3 = 7$, and

$$S(5,2) = \binom{5}{1} + \binom{5}{2} = 5 + 10 = 15.$$

The following statement contains a useful identity.

Proposition 2.3.6 *Fix $n \geq 1$. Let f be a function on $\mathcal{P}([n])$ such that there exists a function h on $\Lambda(n)$ (the set of the partitions of n) verifying $f(\pi) = h(\lambda(\hat{0}, \pi))$ (that is, f only depends on the class $\lambda(\hat{0}, \pi)$), one has*

$$\sum_{\pi=\{b_1,\dots,b_k\}\in\mathcal{P}([n])} f(\pi) = \sum_{\lambda=(1^{r_1}2^{r_2}\cdots n^{r_n})\vdash n} \begin{bmatrix} n \\ \lambda \end{bmatrix} h(\lambda).$$

The proof of Proposition 2.3.6 is elementary and left to the reader.

2.4 Bell polynomials, Stirling numbers and Touchard polynomials

We will now connect some of the objects presented in the previous sections (in particular, the Stirling numbers of the second kind introduced in (2.3.9)) to the remarkable classes of Bell and Touchard polynomials. These polynomials will appear later in the book, as they often provide a neat way to express combinatorial relations between moment and cumulants of random variables. See, for example, [13, Ch. 11] for an exhaustive discussion of these objects, as well as [119, Ch. 1] and [146, p. 33] and the references therein.

Definition 2.4.1 *Fix $n \geq 1$. For every $k \in \{1, ..., n\}$, the* **partial Bell polynomial** *of index (n,k) is the polynomial in $n-k+1$ variables given by*

$$B_{n,k}(x_1, ..., x_{n-k+1}) = \sum_{r_1,\dots,r_{n-k+1}} \frac{n!}{r_1!r_2!\cdots r_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{r_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{r_{n-k+1}} \tag{2.4.11}$$

$$= \sum_{\substack{\lambda=(1^{r_1}2^{r_2}\cdots n^{r_n})\vdash n \\ \lambda \text{ has length } k}} \begin{bmatrix} n \\ \lambda \end{bmatrix} x_1^{r_1} \times \cdots \times x_{n-k+1}^{r_{n-k+1}}, \tag{2.4.12}$$

where the first sum runs over all vectors on nonnegative integers $(r_1, ..., r_{n-k+1})$ satisfying $r_1 + \cdots + r_{n-k+1} = k$ and $1r_1 + 2r_2 \cdots + (n-k+1)\, r_{n-k+1} = n$, and the symbol $\begin{bmatrix} n \\ \lambda \end{bmatrix}$ is defined in (2.3.8). The nth **complete Bell polynomial** *is the polynomial*