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David V. Cruz-Uribe
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Weights, Extrapolation and the Theory of Rubio de Francia

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David V. Cruz-Uribe
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Authors
David V. Cruz-Uribe
Department of Mathematics
Trinity College
300 Summit Street
Hartford, CT 06106-3100, USA
david.cruzuribe@trincoll.edu

José María Martell
Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
c/Serrano 121
28006 Madrid, Spain
chema.martell@icmat.es

Carlos Pérez
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
41080 Sevilla, Spain
carlosperez@us.es

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Preface

This monograph has been a labor of many years, and it reflects many of the themes that have dominated our joint research: weighted norm inequalities, Rubio de Francia extrapolation and its generalizations, the Calderón-Zygmund decomposition, etc. Though there does not exist a single starting point for our work, there was one seminal event. In 1998, the first author, with the generous support of the Ford Foundation and the Dean of Trinity College, was able to spend a month in Madrid. The last day of his visit found him and the third author double-parked at Barajas airport while he explained an idea he had had the night before. Twelve hours later, when he arrived home and checked his email he found three long messages waiting. These led to a new way of thinking about weak type inequalities and contained the germ of the extrapolation results proved in Chapter 8.

Over the years we have benefited from conversations with and ideas from many other mathematicians. We particularly want to thank: Jose García-Cuerva, advisor of the second author; Javier Duoandikoetxea, for his many insights on extrapolation and especially for sharing with us his new proof of extrapolation with sharp constants; Janine Wittwer and Michael Wilson, whose work on the dyadic square function informed our own; C.J. Neugebauer, for his groundbreaking work on two-weight extrapolation; and Javier Parcet, for his work in collaboration with the second author on the generalized Calderón-Zygmund decomposition contained in the Appendix. And finally, we must acknowledge our debt to the late José Luis Rubio de Francia, who created the theory of extrapolation.

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Personal Dedications

I want to thank my children, Nicolás, Antonio and Francisco, for accepting my extended absences while this book was written. I also want to thank Donald Sarason and Christoph Neugebauer for the faith they showed in me as a mathematician.
DCU

I want to thank my wife Isabel, and my sons Chema and Javier for filling my life with happiness.
JMM

I would like to thank my family, Cristina, Sergio and Adriana for their support.
CP

Preliminaries

We assume that the reader knows real analysis as presented in standard works such as Rudin [197] or Royden [192]. The reader should also be familiar with the basics of harmonic analysis as contained in the first few chapters of Duoandikoetxea [68] or Grafakos [91]. Here we review some notation and basic results that will be used throughout this monograph.

We will work primarily in $\mathbb{R}^n$. The norm in $\mathbb{R}^n$ will be denoted by $|\cdot|$ and Lebesgue measure by dx. Given a measurable set E, $|E|$ will also denote the Lebesgue measure of E, and χ_E will denote the characteristic function of the set E. Given a cube Q, $\ell(Q)$ will denote the side-length of Q, so that $\ell(Q)^n = |Q|$. Given Q and $\lambda > 0$, λQ will denote the cube with the same center as Q and such that $\ell(\lambda Q) = \lambda\ell(Q)$.

By a weight we will mean a non-negative function u that is positive on a set of positive measure. If a condition is given on a weight involving an integral, we will implicitly assume that the integral is finite. Given a weight u and a measurable set E, let

$$u(E) = \int_E u(x)\,dx.$$

A weight u is called a doubling measure if there exists a constant $C > 0$ such that for all cubes Q, $u(2Q) \leq Cu(Q)$. If $|E| > 0$, define

$$\fint_E u(x)\,dx = \frac{1}{|E|}\int_E u(x)\,dx;$$

if $|E| = 0$, set it equal to 0.

The collection of smooth functions of compact support will be denoted by C_c^∞.

For $1 \leq p < \infty$, L^p will denote the Banach function space with norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{1/p}.$$

Given a measurable set E we define the localized L^p norm on E by

$$\|f\|_{p,E} = \left(\fint_E |f(x)|^p\,dx\right)^{1/p}.$$

For $1 < p < \infty$, p' denotes the conjugate exponent of p:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The space $L^{p'}$ is the dual space of L^p. Given a locally integrable weight u, $L^p(u)$ will denote the Banach function space with norm

$$\|f\|_{L^p(u)} = \left(\int_{\mathbb{R}^n} |f(x)|^p u(x)\, dx\right)^{1/p}.$$

The dual space of $L^p(u)$ is $L^{p'}(u)$. Given a vector-valued function $f = \{f_i\}$ and q, $1 \leq q < \infty$, let

$$\|f\|_{\ell^q} = \left(\sum_{i=1}^{\infty} |f_i|^q\right)^{1/q}.$$

We say that $f = \{f_i\} \in L^p(u)$ if $\|f(\cdot)\|_{\ell^q} \in L^p(u)$.

Given p, q, $1 \leq p$, $q < \infty$, and a pair of weights (u, v), an operator T is of strong type (p, q) if there exists a constant C such that for all $f \in L^p(v)$,

$$\|Tf\|_{L^q(u)} \leq C\|f\|_{L^p(v)}.$$

We also denote this by $T : L^p(u) \to L^q(u)$. If $u = v$ and $p = q$, then we say that T is bounded on $L^p(u)$, and we denote the infimum of the constant C by $\|T\|_{L^p(u)}$.

Given a locally integrable weight u, $L^{p,\infty}(u)$, $1 \leq p < \infty$, will denote the Lorentz space with quasi-norm

$$\|f\|_{L^{p,\infty}(u)} = \sup_{\lambda > 0} \lambda u(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/p}.$$

An operator T is of weak type (p, q) if there exists a constant C such that, for all $f \in L^p(v)$,

$$\|Tf\|_{L^{q,\infty}(u)} \leq C\|f\|_{L^p(v)},$$

or equivalently, for all $\lambda > 0$,

$$u(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq C\left(\frac{1}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx\right)^{q/p}.$$

We denote this by $T : L^p(v) \to L^{q,\infty}(u)$.

Given a locally integrable function f, the Hardy-Littlewood maximal function, Mf, is defined by

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. An operator that is pointwise equivalent is gotten if the supremum is taken over all cubes in $\mathbb{R}^n$, cubes centered at x, or balls. The maximal operator M is a bounded operator on L^p, $1 < p < \infty$, and satisfies the weak $(1,1)$ inequality $M : L^1 \to L^{1,\infty}$.

For each $j \in \mathbb{Z}$, define the set

$$\mathcal{D}_j = \{[0, 2^{-j})^n + k : k \in \mathbb{Z}^n\};$$

the set of dyadic cubes $\mathcal{D}$ is the union $\cup_j \mathcal{D}_j$.

Finally, throughout this monograph, C, c, etc. will denote positive constants whose values may change even in a chain of inequalities. Specific values that the constants depend on will be noted as necessary.

Part I

One-Weight Extrapolation

Chapter 1

Introduction to Norm Inequalities and Extrapolation

The extrapolation theorem of Rubio de Francia is one of the deepest results in the study of weighted norm inequalities in harmonic analysis: it is simple to state but has profound and diverse applications. The goal of this book is to give a systematic development of the theory of extrapolation, one which unifies known results and expands them in new directions. In addition, we want to show how extrapolation theory, broadly defined, can be applied to the theory of weighted norm inequalities. We describe new and simpler proofs of known results, and then prove new results and show how these lead to additional open questions.

The primary audience for our work is researchers and graduate students who are working on weighted norm inequalities and related topics. However, we believe that many of our results will be useful to mathematicians working in other areas of harmonic analysis and partial differential equations. While the more technical results and proofs will require specialized knowledge to be fully understood, we have striven to make the broad outline of the theory and the statement of our main results accessible to a broader audience. The minimum we have assumed and the basic notation we use is given in the Preliminaries at the beginning of the book.

In this chapter, to put our results in context, we first review the history of the theory of weighted norm inequalities. Unfortunately, no recent survey of the field exists, but beyond the specific articles we cite below, we refer the reader to the books by García-Cuerva and Rubio de Francia [88], Duoandikoetxea [68] and Grafakos [92], and the early survey articles by Muckenhoupt [150] and Dynkin and Osilenker [72]. We then describe the theory of extrapolation and summarize the subsequent chapters. A more detailed overview of the contents of Part I is given in the second half of Chapter 2.

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1.1 Weighted norm inequalities

By a weight we mean a non-negative, locally integrable function that is positive on a set of positive measure. The integrability condition can be relaxed, but for simplicity we consider here this important special case. The basic problems in the study of weighted norm inequalities are to prove estimates of the form

$$\int_{\mathbb{R}^n} |Tf(x)|^p u(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx$$

or

$$u(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \le \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx,$$

where $1 \le p < \infty$, and T is an operator, usually one of the classical operators of harmonic analysis: i.e., a maximal function, singular integral, square function, etc. These problems divides naturally into two classes: when we have a single weight function w (i.e., $u = v = w$) and when we have a pair of weights (u, v). These are referred to as one-weight and two-weight inequalities.

The theory of one-weight inequalities began with the study of power weights of the form $u(x) = |x|^a$. See for example, Stein [213, 214] (see also Soria and Weiss [212]). Shortly thereafter came the celebrated Helson-Szegö theorem [101], which characterized one-weight inequalities for the conjugate function (i.e., the periodic Hilbert transform on the unit circle) using complex analysis.

A period of sustained research in this area began in the 1970s with the work of Muckenhoupt and others. In [148] Muckenhoupt introduced the A_p weights (now often referred to as Muckenhoupt weights): for $1 < p < \infty$, $w \in A_p$ if there exists a constant K such that for every cube $Q \subset \mathbb{R}^n$,

$$⨍_Q w(x)\,dx \left(⨍_Q w(x)^{1-p'}\,dx \right)^{p-1} \le K < \infty.$$

(A variant of this condition was introduced earlier by Rosenblum [191].) The weight w is in A_1 if there exists a constant K such that for almost every $x \in \mathbb{R}^n$, $Mw(x) \le Kw(x)$, where M is the Hardy-Littlewood maximal operator. In each case the infimum of all such K is denoted by $[w]_{A_p}$. The prototypical A_p weights are the power weights: for all $a \in \mathbb{R}$, $|x|^a \in A_1$ if and only if $-n < a \le 0$, and for $p > 1$, $|x|^a \in A_p$ if and only if $-n < a < (p-1)n$.

The centrality of the A_p condition is shown by the following result.

Theorem 1.1. *Given p, $1 \le p < \infty$, and $w \in A_p$, then*

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \le \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx,$$

where T is the Hardy-Littlewood maximal operator, the Hilbert transform, or a Riesz transform. If $p > 1$, then the corresponding strong type inequality holds:

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.$$

Furthermore, the A_p condition is necessary: if the strong or weak (p,p) inequality holds for a weight w and one of these operators, then $w \in A_p$.

The sufficiency of the A_p weights for the Hardy-Littlewood maximal operator to be bounded on $L^p(w)$ was proved by Muckenhoupt [148]; for the Hilbert transform by Hunt, Muckenhoupt and Wheeden [104]; and for Riesz transforms (and indeed for any singular integral with a sufficiently smooth kernel) by Coifman and Fefferman [25]. Their proof involved proving an intermediate inequality: for $0 < p < \infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \le C \int_{\mathbb{R}^n} Mf(x)^p w(x)\, dx, \tag{1.1}$$

where w satisfies the so-called A_∞ condition. (This is defined in Theorem 1.3 below.) The necessity of the A_p condition for the maximal operator is also due to Muckenhoupt, and a similar argument works for the Hilbert transform [104]. This argument can be extended to show that if all the Riesz transforms are bounded on $L^p(w)$, then $w \in A_p$. (See [88].) The fact that if a single Riesz transform is bounded, then $w \in A_p$ is due to Stein [216].

Similar results were soon proved for a variety of other operators, and there now exists an extensive literature on one-weight norm inequalities. For a partial list, we refer the reader to [68, 72, 88, 92] and the references they contain. A common approach has been to prove inequalities that are similar to (1.1), and we will collectively refer to these as Coifman-Fefferman inequalities.

One important variation that emerged was the class of "dyadic" A_p weights, A_p^d. This condition is defined as the general A_p conditions but with the cubes restricted to dyadic cubes. This class is the appropriate one to consider for a variety of dyadic operators. For more on this subject, we refer the reader to the lecture notes by Pereyra [169] and the references they contain.

Central to the original proofs of Theorem 1.1 is the rich structure of the A_p weights. Several properties are immediate consequences of the definition.

Proposition 1.2. *The Muckenhoupt weights have the following properties:*

(a) *for $1 < p < \infty$, $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$;*

(b) *if $1 \le p < q < \infty$, then $A_p \subset A_q$;*

(c) *given $w_1, w_2 \in A_1$, for $1 < p < \infty$, $w_1 w_2^{1-p} \in A_p$.*

Property (a) follows at once from the definition; property (b) from Hölder's inequality; and property (c) from the fact that if $w \in A_1$, then for almost every $x \in Q$,

$$\fint_Q w(y)\, dy \le Mw(x) \le [w]_{A_1} w(x).$$

Beyond the elementary results of Proposition 1.2, Muckenhoupt weights have many deeper properties. We begin with a definition: define the class of weights A_∞

by

$$A_\infty = \bigcup_{p\geq 1} A_p.$$

Theorem 1.3. *The Muckenhoupt weights can be characterized by the following properties:*

(a) *$w \in A_\infty$ if and only if there exist constants C, $\delta > 0$ such that given any cube Q and any measurable set $E \subset Q$,*

$$\frac{w(E)}{w(Q)} \leq C \left(\frac{|E|}{|Q|}\right)^\delta;$$

(b) *$w \in A_\infty$ if and only if for some $s > 1$, $w \in RH_s$: there exists a constant K such that for every cube Q,*

$$\left(\fint_Q w(x)^s \, dx\right)^{1/s} \leq K \fint_Q w(x)\, dx;$$

(c) *If $w \in A_p$, $p > 1$, there exists ϵ, $0 < \epsilon < p-1$, such that $w \in A_{p-\epsilon}$;*

(d) *If $w \in A_p$, $p > 1$, there exist w_1, $w_2 \in A_1$ such that $w = w_1 w_2^{1-p}$.*

The A_∞ condition was discovered independently by Coifman and Fefferman [25] and Muckenhoupt [149]. The RH_s condition is referred to as the reverse Hölder inequality and was also proved by Coifman and Fefferman [25]. (The RH_s classes were considered independently by Gehring [89].) The A_p condition itself is a kind of reverse Hölder inequality, since the opposite inequality,

$$1 \leq \fint_Q w(x)\, dx \left(\fint_Q w(x)^{1-p'}\, dx\right)^{p-1},$$

is a consequence of Hölder's inequality. Property (c) follows immediately from the reverse Hölder inequality for the weight $w^{1-p'}$; it was first proved directly by Muckenhoupt [148]. Property (d) is referred to as the Jones factorization theorem; it was first conjectured by Muckenhoupt at the Williamstown conference in 1979 (see [150]) and proved by P. Jones at the same conference [111]. A much simpler proof was later given by Coifman, Jones and Rubio de Francia [26]. (We will say more about this proof below.) The expression "the factorization theorem" usually refers to both property (d) and its much simpler converse, property (c) in Proposition 1.2 above, but we will reserve this name for property (d). No standard terminology exists for property (c) in Proposition 1.2, but we will refer to it as "reverse factorization."

In the 1970s the rapid progress in the study of one-weight norm inequalities initially fed hopes that the corresponding problems for two-weight inequalities would soon be solved as well. The immediate candidate for a condition on a pair of weights (u,v) was the two-weight A_p condition: for $p>1$, $(u,v)\in A_p$ if

$$\fint_Q u(x)\,dx\left(\fint_Q v(x)^{1-p'}\,dx\right)^{p-1}\leq K<\infty,$$

and $(u,v)\in A_1$ if $Mu(x)\leq Kv(x)$. (In particular, given any weight u, $(u,Mu)\in A_1$.) Muckenhoupt [148] noted that the same proof as in the one-weight case immediately shows that for all p, $1\leq p<\infty$, $(u,v)\in A_p$ if and only if the maximal operator satisfies the weak (p,p) inequality. However, it was soon discovered that while the two-weight A_p condition is necessary for the strong (p,p) inequality for the maximal operator and the strong and weak type inequalities for the Hilbert transform, it is not sufficient. (See Muckenhoupt and Wheeden [155].)

This led Muckenhoupt and Wheeden [147] to focus not on the structural or geometric properties of A_p weights but on their relationship to the maximal operator, in particular, the fact that $w\in A_p$ was necessary and sufficient for the maximal operator to be bounded on $L^p(w)$ and $L^{p'}(w^{1-p'})$. This led them to make the following conjecture which is still open: given a pair of weights (u,v), a sufficient condition for the Hilbert transform to satisfy the strong (p,p) inequality $H:L^p(v)\to L^p(u)$, $1<p<\infty$, is that the maximal operator satisfy the pair of inequalities

$$M:L^p(v)\to L^p(u), \tag{1.2}$$

$$M:L^{p'}(u^{1-p'})\to L^{p'}(v^{1-p'}). \tag{1.3}$$

(Even though M is not a linear operator, inequality (1.3) is referred to as the dual of (1.2).) Additionally, Muckenhoupt and Wheeden conjectured that if the dual inequality (1.3) holds for a pair (u,v), then the weak (p,p) inequality $H:L^p(v)\to L^{p,\infty}(u)$ also holds. For the weak $(1,1)$ inequality, they conjectured that

$$u(\{x\in\mathbb{R}^n:|Hf(x)|>\lambda\})\leq\frac{C}{\lambda}\int_{\mathbb{R}^n}|f(x)|Mu(x)\,dx.$$

Each of these conjectures can be generalized naturally to other singular integrals. All of them are in the spirit of Calderón and Zygmund, whose philosophy was that to control a singular integral one should control the maximal operator.

The study of two-weight norm inequalities has proved to be considerably more difficult than it is in the one-weight case. As B. Muckenhoupt recently noted [152], fundamental problems in the two-weight case, including the conjectures just discussed, remain open. Progress has been made, but slowly, and seemingly small improvements in results have required the development of sophisticated techniques. In many cases interesting results have been proved but they have remained isolated and could not be developed further. Cotlar and Sadosky [29, 30], working in the spirit of the Helson-Szegö theorem, gave a necessary and sufficient condition on a pair of weights for the conjugate function to satisfy a two-weight strong (p,p)

inequality. Leckband [122] and Fujii [82] found two-weight conditions that generalized the A_p condition by incorporating measure-theoretic properties similar to the A_∞ condition. Rakotondratsimba [186, 187, 188] gave A_p type conditions for weights that are radial and monotone.

Currently there are two major approaches to two-weight norm inequalities, which we will refer to as "testing conditions" and "A_p bump conditions." The latter are central to our understanding of norm inequalities and extrapolation theory, and so determine the point of view we have adopted in this book. However, though testing conditions do not play a direct role in our work, we want to describe them before discussing our own. We do so for two reasons: first, they are very important in the study of weighted norm inequalities and an area of active research today. Second, despite its importance, we believe that this approach has some shortcomings and we want to highlight these to suggest to the reader the advantages of our approach. We do not claim that the A_p bump conditions are "better" in any normative sense: we just want to illustrate the reasons why we prefer one over the other.

Testing conditions were originally introduced by Sawyer [201]. He proved that a necessary and sufficient condition on a pair of weights (u, v) for the strong (p, p) inequality, $1 < p < \infty$,

$$\int_{\mathbb{R}^n} Mf(x)^p u(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx,$$

is that for every cube Q,

$$\int_Q M(v^{1-p'}\chi_Q)(x)^p u(x)\,dx \le C \int_Q v(x)^{1-p'}\,dx. \tag{1.4}$$

The necessity of this condition is immediate: simply apply the norm inequality to the family of test functions $v^{1-p'}\chi_Q$. We denote the fact that a pair of weights satisfies (1.4) by writing $(u, v) \in S_p$.

Sawyer [207, 208] later extended this approach to linear operators with positive kernels, for instance, the fractional integral operator I_α, $0 < \alpha < n$. Because of linearity, strong (p, p) inequalities with respect to the weights (u, v) are equivalent to strong (p', p') inequalities for the weights $(v^{1-p'}, u^{1-p'})$. This led naturally to two testing conditions: one from the L^p inequality and one from the dual $L^{p'}$ inequality. More precisely, I_α satisfies the (p, p) inequality $I_\alpha : L^p(v) \to L^p(u)$ if and only if the weights (u, v) satisfy

$$\int_Q I_\alpha(v^{1-p'}\chi_Q)(x)^p u(x)\,dx \le C \int_Q v(x)^{1-p'}\,dx, \tag{1.5}$$

$$\int_Q I_\alpha(u\chi_Q)(x)^{p'} v(x)^{1-p'} \le C \int_Q u(x)\,dx. \tag{1.6}$$

Sawyer [204] also proved that the dual testing condition (1.6) was necessary and sufficient for the fractional integral operator to satisfy the weak (p, p) inequality $I_\alpha : L^p(v) \to L^{p,\infty}(u)$.

These results for fractional integrals led to the following conjectures: if T is a singular integral operator (e.g., the Hilbert transform), then $T : L^p(v) \to L^p(u)$ if and only if

$$\int_Q |T(v^{1-p'}\chi_Q)(x)|^p u(x)\,dx \le C \int_Q v(x)^{1-p'}\,dx, \tag{1.7}$$

$$\int_Q |T(u\chi_Q)(x)|^{p'} v(x)^{1-p'} \le C \int_Q u(x)\,dx. \tag{1.8}$$

Testing conditions such as these are referred to generically as Sawyer-type conditions.

After the original work of Sawyer, no progress was made on these conjectures until the groundbreaking work of Nazarov, Treil and Volberg. They realized that there is a close connection between Sawyer type conditions and the testing conditions that are part of the $T1$ theorem of David and Journé [58] (also see [92]). As part of their work on the Vitushkin conjecture, they developed a theory of singular integrals on non-homogeneous spaces (e.g., $\mathbb{R}^n$ with a non-doubling measure) including a Tb theorem. (See [157, 159, 160].) Building on these ideas they have been able to prove L^2 Sawyer-type conditions for several operators. In [158] they proved that Sawyer-type conditions were necessary and sufficient for families of Haar multipliers H_a to satisfy $H_a : L^2(v) \to L^2(u)$ with uniform bounds. (Haar multipliers are dyadic singular integral operators that are "localized" and so easier to deal with. They provide a good model for singular integrals, and more general dyadic operators can be used to approximate Hilbert and Riesz transforms—see [105, 180, 181, 182, 183, 184].) They also proved that the single testing condition (1.7) was necessary and sufficient for the dyadic square function to satisfy $S_d : L^2(v) \to L^2(u)$. In [227] they proved that (1.7), (1.8) and a stronger version of the two-weight A_2 condition—the so-called invariant A_2 condition—are necessary and sufficient for the Hilbert transform to satisfy $H : L^2(v) \to L^2(u)$, provided that u and v satisfy doubling conditions. In [161] they proved L^2 Sawyer-type conditions for individual Haar multipliers and other dyadic operators (without doubling conditions).

Despite the elegance of these results, we believe that they have some drawbacks. First, it is not clear if they are the correct conditions when $p \ne 2$. In [158], Nazarov, Treil and Volberg noted (without proof) that their results for families of Haar multipliers are not true when $p \ne 2$. In addition, for operators other than singular integrals we have reason to believe that Sawyer-type conditions may not be the correct ones to consider. As we discuss in detail Chapter 10 below, in the two-weight case the dyadic square function behaves very differently for $p \le 2$ and for $p > 2$; this in turn suggests that the Sawyer-type condition (which is in some sense the same for all p) may not be sufficient when $p > 2$.

Second, we believe that Sawyer-type conditions are of limited utility in practice. Given a pair of weights (u, v), it seems almost as difficult to check whether the Sawyer-type conditions hold for a specific operator as it does to prove a strong type norm inequality. Conversely, it seems equally difficult to construct examples of weights that satisfy them. Moreover, unlike the A_p weights, the Sawyer-type conditions and the weights that satisfy them are bound to individual operators: if the operator is changed, the work of finding or checking pairs of weights must be started over.

Our approach to two-weight norm inequalities is quite different: our goal has been to find two-weight, A_p-type conditions that are sufficient for large classes of operators. Our work has close connections with the deep conjecture of Muckenhoupt and Wheeden discussed above, which we will explain below. Its proximate origins are in our work to generalize an often overlooked paper by Neugebauer [163]. To best understand his result, we first restate the A_p condition, $1 < p < \infty$, in terms of localized L^p norms: $(u, v) \in A_p$ if for every cube Q,

$$\|u^{1/p}\|_{p,Q}\|v^{-1/p}\|_{p',Q} \le K < \infty.$$

Neugebauer showed that given a pair of weights (u, v), there exist $w \in A_p$ and positive constants c_1, c_2 such that $c_1u(x) \le w(x) \le c_2v(x)$ if and only if there exists $r > 1$ such that for every cube Q,

$$\|u^{1/p}\|_{rp,Q}\|v^{-1/p}\|_{rp',Q} \le K < \infty. \tag{1.9}$$

From this condition we immediately get a large number of two-weight norm inequalities as corollaries to the analogous one-weight results. In particular, we have that the two inequalities (1.2) and (1.3) hold for the maximal operator. We refer to (1.9) as an A_p bump condition.

An immediate question was whether this condition could be weakened and still get that the maximal operator satisfies $M : L^p(v) \to L^p(u)$. This was answered in [174], where it was shown that a sufficient condition was that the pair of weights satisfies

$$\|u^{1/p}\|_{p,Q}\|v^{-1/p}\|_{B,Q} \le K < \infty,$$

where the norm on the right-hand term is a normalized Orlicz space norm and B is a Young function that satisfies an easily checked growth condition. We defer the statement of this condition to Chapter 5, Theorem 5.14, as it requires some additional definitions; intuitively, it says that $B(t)$ is "infinitesimally" larger than $t^{p'}$. For example, we could take $B(t) = t^{rp'}$, $r > 1$; in this, case we see that we can eliminate one "bump" from Neugebauer's condition. But we can also take $B(t) = t^{p'} \log(e + t)^{p'-1+\delta}$, where $\delta > 0$. The centrality of this growth condition is shown by the fact that it is necessary for the maximal operator to be bounded—see Remark 5.15 below.

As an immediate consequence of this result, we have that the maximal operator satisfies inequalities (1.2) and (1.3) provided that the pair of weights (u, v) satisfy the A_p bump condition

$$\|u^{1/p}\|_{A,Q}\|v^{-1/p}\|_{B,Q} \le K < \infty, \tag{1.10}$$

where the Young functions A, B satisfy the appropriate growth conditions. This led naturally to the following version of the conjecture of Muckenhoupt and Wheeden: a sufficient condition on the pair of weights (u, v) for any singular integral to satisfy $T : L^p(v) \to L^p(u)$ is that (1.10) holds. Moreover, our version of their conjecture for weak (p, p) inequalities is that $T : L^p(v) \to L^{p,\infty}(u)$ if the pair satisfies

$$\|u^{1/p}\|_{A,Q}\|v^{-1/p}\|_{p',Q} \le K < \infty.$$

In a series of papers over the past fifteen years [33, 47, 53, 55, 56, 171] we have made considerable progress on these conjectures, and in the final two chapters of this book we expand upon our earlier work.

In contrast to the testing conditions discussed above, it is usually straightforward to determine if a pair of weights satisfies (1.10), though we must admit that it can be computationally tedious depending on the Young functions A and B. Moreover, it is very easy to construct examples of pairs (u, v) that satisfy (1.10)—see (1.12) below.

1.2 The theory of extrapolation

With the theory of weighted norm inequalities as a foundation, we can now discuss the extrapolation theorem of Rubio de Francia. Here we state the essential version of the theorem, though we defer the proof to Chapter 2 below.

Theorem 1.4. *Given an operator T, suppose that for some p_0, $1 \le p_0 < \infty$, and every $w \in A_{p_0}$, there exists a constant C depending on $[w]_{A_{p_0}}$ such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)\, dx.$$

Then for every p, $1 < p < \infty$, and every $w \in A_p$ there exists a constant depending on $[w]_{A_p}$ such that

$$\int_{\mathbb{R}^n} |Tf(x)|^{p} w(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^{p} w(x)\, dx.$$

The extrapolation theorem was an unexpected and very surprising result. It was discovered by Rubio de Francia (see [193, 194, 195]) whose background in functional analysis gave him a very different perspective on the theory of weighted norm inequalities. (For this background, see the survey articles by Torrea *et al.* [223].) The philosophy underlying this result was pithily summarized by Rubio de Francia's colleague Antonio Cordoba [84]:

There are no L^p spaces, only weighted L^2.

Beyond the original work of Rubio de Francia, there are a number of proofs of Theorem 1.4 and we will discuss these in more detail in Chapter 2. A key feature of many of these proofs is the iteration algorithm of Rubio de Francia: given a positive, sublinear operator T that is bounded on $L^p(w)$, define a new operator $\mathcal{R}$ by

$$\mathcal{R}h = \sum_{k=0}^{\infty} \frac{T^k h}{2^k \|T\|_{L^p(w)}^k}.$$

(This was first referred to as the Rubio de Francia algorithm in [13].) The crucial, but deceptively simple property of the iteration algorithm is that it is almost "invariant" under the operator T: more precisely,

$$T(\mathcal{R}h) \leq 2\|T\|_{L^p(w)}\mathcal{R}h.$$

The iteration algorithm is central to our own work and we discuss it in more detail in subsequent chapters. Here, we want to note that it is also used in a central way in the simplest proofs of the Jones factorization theorem (Theorem 1.3 above; see [88, 92]), and this shows that there is a very deep connection between extrapolation and factorization.

Given the ongoing work of creating a theory of two-weight norm inequalities parallel to the one-weight theory, it is not surprising that a number of authors considered two-weight extrapolation and factorization theorems: see Neugebauer [163, 164], Bloom [13], Hernández [102], Ruiz and Torrea [198], and Segovia and Torrea [209, 210]. In every case these authors worked with pairs of weights (u, v) such that the maximal operator satisfied inequality (1.2) and the dual inequality (1.3); in other words, there is a close connection between their results and the Muckenhoupt-Wheeden conjecture for singular integrals discussed in the previous section. This points to one drawback of these results: given the current state of knowledge they cannot be applied, since it is not possible to prove the "base case" (e.g., weighted L^2 inequalities) needed to use extrapolation. A different approach to two-weight extrapolation using pairs $(u, v) \in A_1$ was developed in [54] and was implicit in [53].

The importance of the Rubio de Francia extrapolation theorem lies not only in its intrinsic beauty, but also in its powerful applications. There have been many; here we describe four in detail. Additional applications that arise from our generalizations of one-weight extrapolation are given in Chapters 3 and 4. The first important application was to rough singular integrals. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, let $\Omega \in L^\infty(S^{n-1})$ be such that $\int_{S^{n-1}} \Omega(x)\,dx = 0$ and let $h \in L^\infty(\mathbb{R}_+)$. We consider the singular integral T with kernel

$$K(x) = h(|x|)\frac{\Omega(x/|x|)}{|x|^n}.$$

T is bounded on L^p, $1 < p < \infty$; when $h \equiv 1$ this follows from the method of rotations (see [68]); for general h this was proved by R. Fefferman [77]. Duoandikoetxea and Rubio de Francia [71] proved that T is bounded on $L^p(w)$, $1 < p < \infty$, for

$w \in A_p$. Key to the proof was the extrapolation theorem, since this reduced the problem to proving that T is bounded on $L^2(w)$, $w \in A_2$. For this case they used the Fourier transform, square function estimates and interpolation with change of measure to deduce the weighted inequality from the unweighted L^2 estimate.

Another important application of extrapolation was given by R. Fefferman and Pipher [78]. They were considering singular integral operators $T_{\mathcal{Z}}$ on $\mathbb{R}^3$ that commute with the family of multiparameter dilations $\phi_{s,t}(x, y, z) = (sx, ty, stz)$, s, $t > 0$. The closely related maximal operators $M_{\mathcal{Z}}$ are defined as the supremum of averages over rectangles in the Zygmund basis $\mathcal{Z}$ whose side-lengths are of the form (s, t, st). Unweighted estimates for such operators were proved by Ricci and Stein [190]. Fefferman and Pipher proved one-weight estimates on $L^p(w)$, where $w \in A_{p,\mathcal{Z}}$, the A_p class defined with respect to rectangles in $\mathcal{Z}$. Again central to the proof was Rubio de Francia extrapolation (which they noted could be extended to weights in $A_{p,\mathcal{Z}}$): they showed that in $L^2(w)$ the proof reduced to a certain square function estimate, but this approach does not work when $p \neq 2$. They also proved sharp embedding theorems for these operators in Orlicz spaces close to L^1. They showed that these followed from sharp L^p estimates for the Hilbert transform; implicit in their proof of these sharp estimates is a duality argument that is reminiscent of our approach to both one and two-weight extrapolation.

The third application is to elliptic differential equations. The Beltrami equation in the plane is $f_z - \mu f_{\bar{z}} = 0$, where μ is a bounded function such that $\|\mu\|_\infty = k < 1$. Astala, Iwaniec and Saksman [6] showed that solutions of this equation are continuous if $f \in W^{1,q}_{loc}$ for $q > k + 1$ and that there were discontinuous solutions if $q < k + 1$. Further, they showed that solutions in $W^{1,1+k}_{\text{loc}}$ were continuous if the Beurling-Ahlfors operator (a complex-valued analog of the Hilbert transform) satisfied a certain sharp weighted A_p estimate. This estimate was proved by Petermichl and Volberg [184]; via extrapolation they reduced the problem to weighted L^2 estimates, which they proved using Bellman function techniques.

As a final application we consider our work on a conjecture by Sawyer. In [205] he proved that if u, $v \in A_1$, then

$$uv(\{x \in \mathbb{R} : M(fv)(x) > \lambda v(x)\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}} |f(x)|u(x)v(x)\, dx; \tag{1.11}$$

this inequality arose naturally when trying to prove weighted norm inequalities by combining interpolation with change of measure (see Stein and Weiss [217]) and the Jones factorization theorem. Sawyer conjectured that this inequality is true if the maximal operator is replaced by the Hilbert transform. This conjecture was proved in [45] and was shown to hold in higher dimensions for Calderón-Zygmund singular integral operators. The proof consisted of several steps. First, inequality (1.11) was extended to higher dimensions for the dyadic maximal operator. Then, using a version of Rubio de Francia extrapolation adapted to this kind of inequality, it was shown that it holds in $\mathbb{R}^n$ for both the Hardy-Littlewood maximal operator and singular integrals.

1.3 The organization of this book

The starting point for this book is a new and much simpler proof of the Rubio de Francia extrapolation theorem. It does not require cases depending on the size of p, and it only uses very elementary structural properties of weights and the fact that the maximal operator is bounded on $L^p(w)$. In some sense, we are able to make clear "what is really going on" in the proof, and this provides a springboard for a number of generalizations of the extrapolation theorem. We had already begun to do so in earlier work (see, for instance, [40, 44, 46, 57]); here we develop these extensions systematically.

Our material divides naturally into two parts. In Part I we consider the one-weight theory. In Chapter 2 we briefly describe earlier proofs of Theorem 1.4 and then give our own proof. We analyze the proof to highlight the key features—the boundedness of the maximal operator, duality, and reverse factorization. We then describe the numerous generalizations which our approach makes possible. These generalizations are developed and proved in Chapters 3 and 4. In Chapter 3 we focus on weighted L^p results, and in Chapter 4 we show that Rubio de Francia extrapolation can be generalized to prove norm inequalities for operators in large families of Banach function spaces. These generalizations may be succinctly captured by expanding upon Cordoba's remark given above:

There are no Banach function spaces, only weighted L^2.

At the end of both of these chapters we sketch a number of applications of our extrapolation theory. These include a new approach to Coifman-Fefferman inequalities that avoids the so-called good-λ inequalities, vector-valued inequalities, modular inequalities for singular integrals, and norm inequalities for operators on the variable Lebesgue spaces.

In Part II we treat two-weight extrapolation and factorization theory. To a certain extent this half of the book is independent of Part I, though the reader should consult Chapter 2 to get a better sense of our overall philosophy. Our approach to two-weight extrapolation grew naturally out of our approach to weighted norm inequalities; in fact, some special cases of our extrapolation results were implicit in earlier work [33, 53, 54]. We show that one can extrapolate in the scale of weights that satisfy (1.10): given an operator T, suppose that for some p_0 and Young functions A, B in a certain class, $\|Tf\|_{L^{p_0}(u_0)} \le C\|f\|_{L^{p_0}(v_0)}$ whenever (u_0, v_0) satisfy (1.10) with p replaced by p_0. Then given any p we give sufficient conditions on Young functions A and B so that $\|Tf\|_{L^p(u)} \le C\|f\|_{L^p(v)}$ whenever (u, v) satisfy (1.10).

The presence of the Orlicz space norms in (1.10) causes our proofs to be more technical than the proofs in the one-weight case in Part I. Further, to get the sharpest possible results we diverge considerably from the specific proofs given in Part I. Nevertheless, the proofs in the two-weight case rely on the same essential ingredients: boundedness of the maximal operator, duality, and reverse factorization.

The material in Part II is organized as follows: in Chapter 5 we gather preliminary information about Young functions, Orlicz spaces, and Orlicz maximal operators that is needed in subsequent chapters. In particular, we characterize the Young functions such that (1.10) implies that the maximal operator satisfies (1.2) and the dual inequality (1.3).

In Chapter 6 we discuss factorization in the two-weight setting. We first define the appropriate A_1-type weights and prove a reverse factorization theorem for weights that satisfy (1.10). Since there is a close connection between reverse factorization, factorization and extrapolation, we also develop a two-weight factorization theory for weights that satisfy (1.10). We introduce an important new class of weights—the so-called factored weights,

$$(\tilde{u}, \tilde{v}) = \big(w_1 (M_\Psi w_2)^{1-p}, (M_\Phi w_1) w_2^{1-p}\big), \tag{1.12}$$

where M_Φ and M_Ψ are Orlicz maximal operators—that are gotten from the reverse factorization theorem and which satisfy (1.10). These weights are of particular interest in applications since we can prove a number of results for this special class that generalize known results in surprising ways, and these lead to new conjectures for two-weight inequalities in general.

Chapters 7 and 8 are the theoretical heart of Part II. In Chapter 7 we prove the main two-weight extrapolation theorems. This chapter is unavoidably technical, both because of the nature of the conditions on the weights and because we wanted to develop our results in a fairly general setting. To clarify the situation we give a number of examples and special cases. In Chapter 8 we further develop the theory of two-weight extrapolation, focusing particularly on endpoint results and rescaling such as we used to develop the so-called A_∞ extrapolation in Chapter 3.

Throughout Chapters 5–8 we will primarily consider A_p-type conditions and maximal operators defined with respect to arbitrary cubes. However, unless we specifically say otherwise, all of our results hold when we restrict ourselves to operators and conditions defined in terms of dyadic cubes. At certain points we will point out when other, stronger, results hold, but in Chapter 10 we will often apply results from these chapters to the dyadic case without comment.

In the last two chapters we give applications of extrapolation to the study of two-weight norm inequalities. In Chapter 9 we consider three kinds of operators: the sharp maximal operator, Calderón-Zygmund singular integrals, and fractional integral operators. For the sharp maximal operator, we give a two-weight inequality which is a generalization of the Fefferman-Stein inequality [76] and compare our result to another two-weight version due to Fujii [81]. We then use these results to develop a two-weight theory of Coifman-Fefferman inequalities. For singular and fractional integrals we give conjectures for sharp conditions for two-weight weak and strong type inequalities that are based on the original conjectures of Muckenhoupt and Wheeden. We then review the known results, showing in some cases that they are easy consequences of extrapolation, and then prove new results, including new results for pairs of factored weights.