Serious Fun with Flexagons

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Volume 164

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Les Pook

Serious Fun with Flexagons

A Compendium and Guide

Springer

Les Pook
Department of Mechanical Engineering
University College London
Torrington Place
London WC1E 7JE
UK
les.pook@tesco.net

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About the Author

Leslie Philip (Les) Pook was born in Middlesex, England in 1935. He obtained a B.Sc. in metallurgy from the University of London in 1956. He started his career at Hawker Siddeley Aviation Ltd., Coventry in 1956. In 1963 he moved to the National Engineering Laboratory, East Kilbride, Glasgow. In 1969, while at the National Engineering Laboratory, he obtained a Ph.D. in mechanical engineering from the University of Strathclyde. Dr. Pook moved to University College London in 1990. He retired formally in 1998 but remained professionally active in the fields of metal fatigue and fracture mechanics, and is affiliated to University College London as a visiting professor. He now has more tine to pursue long standing interests in recreational mathematics, including flexagons, and in horology, especially synchronous electric clocks. He is a Fellow of the Institution of Mechanical Engineers and a Fellow of the Institute of Materials, Minerals and Mining. Les married his wife Ann in 1960. They have a daughter, Stephanie, and a son, Adrian.

Preface

Flexagons are rings of hinged polygons that have the intriguing property of displaying different pairs of faces when they are flexed. Workable paper models of flexagons are easy to make and entertaining to manipulate. Flexagons have a surprisingly complex mathematical structure, and just how a flexagon works is not obvious on casual examination of a paper model. The aesthetic appeal of flexagons is in their dynamic behaviour rather than the static appeal of, say, polyhedra. One of the attractions of flexagons is that it is possible to explore their dynamic properties experimentally as well as theoretically. Flexagons may be appreciated at three different levels. Firstly as toys or puzzles, secondly as a recreational mathematics topic, and finally as a subject of serious mathematical study.

My book *Flexagons Inside Out* was published in 2003 by Cambridge University Press. Since then there has been an upsurge in interest in flexagons. Enthusiasts can keep in touch through the *Flexagon Lovers Group*, hosted by Yahoo, and moderated with a light touch by Ann Schwartz. Details of some interesting flexagons have been posted by Group members, and I have enjoyed some stimulating exchanges with other members of the Group. The amount of new information available means that *Flexagons Inside Out* is now outdated. Further geometric analysis has also lead to a much better understanding of the behaviour of flexagons, and has in turn led to the discovery of previously unknown flexagons, some of them with entertaining dynamic properties.

Most of the material in the book is new. It is arranged in a logical order appropriate for a textbook on the geometry of flexagons. Extensive cross references are included so that individual chapters do not have to be read in order. Definitions are included in the index so that they can be easily located. It is assumed that the reader already has an interest in flexagons, and has some knowledge of elementary geometry. The book is written so that it can be enjoyed at both the recreational mathematics level, and at the serious mathematics level. In general, detailed proofs are long and tedious, so they are not included. Where there is uncertainty over the accuracy of a conclusion this is made clear in the text. Basic material from *Flexagons Inside Out* is referenced only where needed for clarity but, where appropriate, new material is fully referenced. There are a few errors in *Flexagons Inside Out*, and these are corrected in the present book. In some ways the book is an updated version of the 1962 book length report *Flexagons* by Conrad and Hartline, which is available on the Internet.

A feature of the book is a compendium of over 100 nets for the construction of paper models of some of the more interesting flexagons. These are reproduced at approximately half full size. Many of the nets have not previously been published. The flexagons have been chosen to complement the text, with particular emphasis on demonstrating relationships between different types of flexagon. Three spectacular examples are included. These are the octopus flexagon, the hexa-dodeca-flexagon and the thrice threefold flexagon. Detailed instructions for assembling and manipulating individual flexagons are included for the benefit of those who wish to enjoy flexagons without going into the mathematics. Photographs of some flexagons are included to assist assembly and manipulation.

Most flexigators who move on from making up flexagons from published nets try folding up promising looking nets to see what happens. This bottom up approach has led to the discovery of some interesting flexagons. The top down approach used in this book makes it possible to analyse and understand the dynamic properties of any flexagon. It is also makes it possible to design flexagons having desired properties. Manipulating paper models of the resulting flexagons often reveals unexpected properties that were not predicted theoretically.

January 2009 Les Pook

I do not know how far it is possible to convey to any one who has not experienced it, the peculiar interest, the peculiar satisfaction that lies in a sustained research when one is not hampered by want of money.

H G Wells, *Tono-Bungay*

Contents

Chapter 1
Introduction

1.1 General Features

Flexagons are a twentieth century discovery (Gardner 1965, 2008; Pook 2003). Arthur H Stone, a postgraduate student at Princeton University in America, discovered them in 1939 while folding strips of paper. Figure 1.1a is a photograph of a trihexaflexagon, which was the first type of flexagon to be discovered. The black and white photographs of flexagons in this book are nearly all of models made either from coloured card, coloured origami paper, or from origami duo paper, which is differently coloured on its two surfaces. The appearance of some flexagons is shown as a line diagram such as the ring of four squares shown in Fig. 1.1b.

Workable paper (or card) models of flexagons are easy to make and entertaining to manipulate. They have the intriguing property of displaying different pairs of faces, sometimes in cyclic order, when they are flexed. Flexagons have a surprisingly complex mathematical structure, and just how a flexagon works is not obvious on casual examination of a paper model. The aesthetic appeal of flexagons is in their dynamic behaviour rather than the static appeal of, say, polyhedra. One of the attractions of flexagons is that it is possible to explore their dynamic properties experimentally as well as theoretically. Manipulation of paper models often reveals configurations that have not been predicted theoretically.

A flexagon is a motion structure that has an infinity of states (positions). An umbrella is an everyday example of a motion structure. An edge flexagon consists of a band of identical polygons hinged at common edges by edge hinges. The individual polygons in a flexagon, called leaves, are usually identical (congruent) and are usually regular convex polygons. However, some flexagons consist of other types of convex polygons, and leaves are not always identical. If one hinge of a band is disconnected the band can be laid flat and used as a *net* to construct a flexagon. Nets are sometimes called templates or friezes. A band of 8 edge hinged squares that has been cut and laid flat as the net for a square even edge flexagon is shown in Fig. 1.2. Assembly and flexing instructions for this flexagon are given in Section 1.4.2. This particular flexagon is a twisted band. This can be seen by disconnecting a hinge

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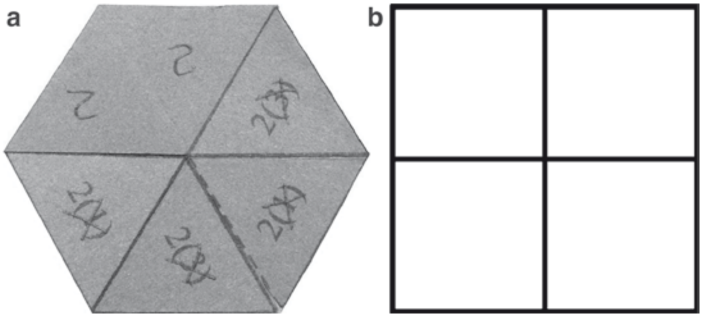


Fig. 1.1 (**a**) A trihexaflexagon as a flat regular even edge ring of six equilateral triangles. (**b**) A flat regular even edge ring of four squares (Les Pook, Flexagons inside out, 2003, © Cambridge University Press 2003, reprinted with permission)

of a paper model, and gently pulling the ends of the band apart. Some edge flexagons are untwisted bands.

A motion structure usually has certain characteristic positions that can conveniently be defined as main positions. For example, a fully opened umbrella is in a main position. A main position of an even edge flexagon is a position that is, in appearance, an untwisted even edge ring of an even number ($2n$) of polygons that are hinged together by edge hinges. Similarly, a main position of an odd edge flexagon is, in appearance, an untwisted odd edge ring of an odd number (n) of polygons that are hinged together. Thus, a main position has two faces. Some flexagons have more than one type of main position. In this book ring refers to the appearance of a flexagon and band to its topological structure. The topological structure of a flexagon is an invariant. This means that the topological structure is always the same no matter what position the flexagon is in.

The trihexaflexagon is an example of an even edge flexagon. A main position is, in appearance, an even edge ring of six equilateral triangles (Fig. 1.1a). The ring is flat and it is a regular edge ring in the sense that all the triangles are the same distance from the centre of the ring. The two sector first order fundamental square even edge flexagon (Fig. 1.2) is, as its name implies, another even edge flexagon. A main position is, in appearance, a flat regular even edge ring of four squares (Fig. 1.1b). The even edge rings shown in Fig. 1.1 are the two possible ways in which an even number of regular convex polygons can be arranged about a point in a plane.

The polygons visible in main positions of a flexagon are called pats. A pat can be either a folded pile of leaves or a single leaf. The pats in a main position of the trihexaflexagon are alternately single leaves and folded piles of two leaves. The pats in a main position of the two sector first order fundamental square even edge flexagon are alternately single leaves and fan folded piles of three leaves. The hinge angle is the angle between the two edge hinges of a polygon, leaf or pat (Fig. 1.3). For example, the hinge angle of the squares shown in Fig. 1.2 is 90°, and the two

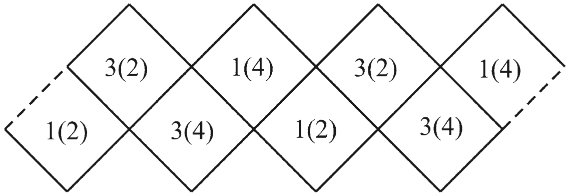


Fig. 1.2 The two sector first order fundamental square even edge flexagon as a band of hinged squares that has been cut and laid flat for use as a net. To assemble the flexagon crease the lines between the squares to form hinges, transfer the number in brackets on the upper face of each square to the reverse face, fold together pairs of squares numbered 3 and 4, and join the ends of the net (dashed lines) using transparent adhesive tape

hinges intersect at a vertex of a square. The two hinges do not always intersect, as shown for a regular hexagon in the figure.

In a flat regular edge ring of regular convex polygons the sum of the hinge angles at the centre of the ring is 360°. Regular edge rings of regular convex polygons, and main positions of flexagons that have the same appearance, are not always flat, and the sum of the hinge angles can be greater or less than 360°. The angle deficit is 360° minus the sum of the hinge angles, and is called the curvature of the ring (Demaine and O'Rourke 2007). In a slant ring the curvature is positive, and in a skew ring it is negative. For example, Fig. 1.4 shows a slant regular odd edge ring of five equilateral triangles. Its curvature is $360° - 5 \times 60° = 60°$. The curvature of the skew regular even edge ring of four regular hexagons shown in Fig. 1.5 is $360° - 4 \times 120° = -120°$.

A vertex flexagon is a band of identical polygons hinged at common vertices by point hinges. Bands can be twisted or untwisted. Point hinges are impossible in a paper model, but short paper strips provide a workable approximation. There are two families of vertex flexagons. These are skeletal flexagons and point flexagons. Skeletal flexagons are not satisfactory as paper models, but are included because of their theoretical interest. Point flexagons are special cases of skeletal flexagons, and are satisfactory as paper models.

A main position of an even skeletal flexagon, is, in appearance, an untwisted even vertex ring of an even number ($2n$) of polygons that are hinged together by point hinges. The point hinges mean that the rings can always be laid flat, and the curvature is indeterminate. A flexagon as a flat regular even vertex ring of four equilateral triangles is shown in Fig. 1.6. Theoretically, vertices of adjacent equilateral triangles coincide, but in the paper model they are separated and connected by narrow strips. This particular ring has an open centre. The two sector first order fundamental even skeletal flexagon is shown collapsed into a twisted band in Fig. 1.7.

A main position of a point flexagon is, in appearance, a polygon vertex pair rather than a vertex ring. The two polygons are connected either by a pair of point hinges or by a single point hinge. This is shown for equilateral triangles in Fig. 1.8.

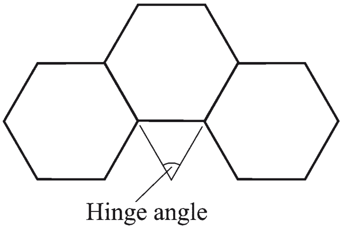


Fig. 1.3 Definition of hinge angle

Fig. 1.4 A flexagon as a slant regular odd edge ring of five equilateral triangles

Fig. 1.5 A flexagon as a skew regular even edge ring of four regular hexagons

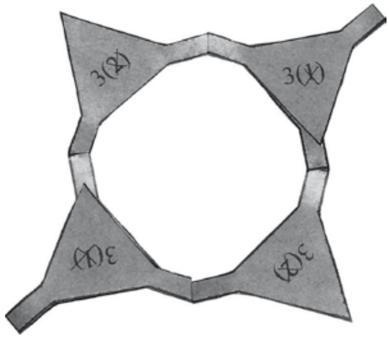


Fig. 1.6 A flexagon as a flat regular even vertex ring of four equilateral triangles. Point hinges approximated by paper strips

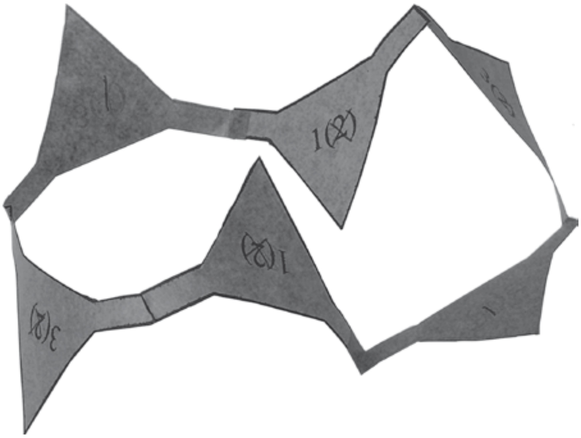


Fig. 1.7 The two sector first order fundamental even skeletal flexagon as an open band

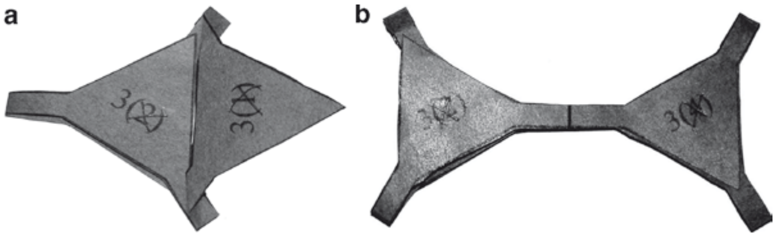


Fig. 1.8 A flexagon as an equilateral triangle vertex pair. Point hinges approximated by paper strips. (**a**) Connected by a pair of point hinges. (**b**) Connected by a single point hinge

Flexagons, in general, exist in infinite series. Usually, only a few members of a series of flexagons are satisfactory as paper models. In this book it is therefore taken as understood that only some early members of a series are being described, for example the first order fundamental even edge flexagons listed in Table 4.1.

1.2 Terminology

Terminology is always a problem in any developing field and appears to be a particular problem with flexagons (Pook 2007). Inevitably, people develop terminology to simplify descriptions of features they are investigating. Equally inevitably, terminologies developed by different people differ, and sometimes conflict. In this book, definitions of descriptive terms and notations are given and indexed when needed, not always when they are first used. Some combinations of descriptive terms are not separately defined or indexed. Terminology has been chosen so as to maintain a balance between clarity and mathematical rigour. As far as is possible usage follows previous practice, but some terms and notations differ from those used in Pook (2003).

The following conventions are used are used when describing flexagons and their characteristics. A family of flexagons is a group of flexagons with some characteristics in common. A characteristic flex for a family of flexagons is a flex that can be used to flex all the members of the family. Most of the descriptions of the dynamic properties of flexagons are based on the use of a characteristic flex. A variety of flexagon is a group of flexagons within a family, all of which have a main position appearance in common. A type of flexagon is a particular flexagon within a variety. All flexagons exist as an enantiomorphic (mirror image) pair. The two members of an enantiomorphic pair are different types of flexagon but are not usually considered to be distinct types of flexagon (Pook 2003). Hence, terminology used does not, in general, distinguish between the two enantiomorphs of a flexagon.

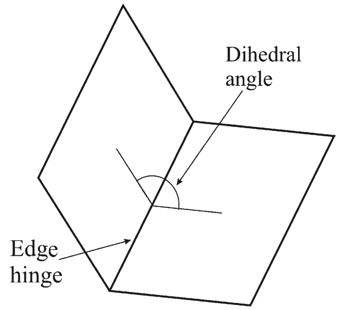
In general, leaves in nets shown are numbered to identify the faces which can be displayed on a flexagon: all the leaves visible on a face of a main position have the same number. Face numbering sequences are arbitrary so different sequences can be applied to the same flexagon. Two face numbering sequences are only regarded as distinct if one cannot be transformed into the other by substitution on a one to one basis. For example, adding 7 to each of the face numbers shown in Fig. 1.2 does not result in a second distinct face numbering sequence. Numbers on nets shown are assigned so as to make descriptions as simple as possible, and also to make it possible to write general assembly instructions (Section 1.4.1). In some special situations faces are also identified using letters, or Roman numerals, or both. Other markings are sometimes added to simplify assembly and flexing.

In geometry, a distinction is often made between an ideal mathematical object and an imperfect physical model of the object. For example, in geometry a line is defined as having zero width, whereas any real line drawn on a piece of paper must have a finite width. Fortunately, paper models of many types of flexagon do approximate closely to a mathematical ideal, and it is not usually necessary to make a distinction between an ideal flexagon and the corresponding paper model.

Mathematically, an idea leaf is a flat polygon that is rigid, of zero thickness, and consists of its one dimensional edges plus its two dimensional interior (Cromwell 1997). Because an ideal leaf is of zero thickness an ideal pat is also of zero thickness. An ideal flexagon consists of a band of ideal leaves that are hinged together by ideal hinges. In an ideal edge hinge the dihedral angle between the two planes containing two hinged leaves may vary between 0°and 360° without constraint. The dihedral angle (Coxeter 1963) is the angle on a section which cuts both planes at 90° (Fig. 1.9). Each leaf is hinged to two other leaves. The angle between the two hinges is the hinge angle (Fig. 1.3). An ideal edge flexagon is an ideal flexagon with ideal leaves which are hinged together by ideal edge hinges. Similarly, an ideal point flexagon is an ideal flexagon with ideal leaves which are hinged together by ideal point hinges.

The concept of an ideal edge flexagon is similar to that of rigid origami in which only a finite number of creases is permitted, between which the paper must stay

Fig. 1.9 Definition of dihedral angle for an edge hinge



rigid and flat (Demaine and O'Rourke 2007). Leaves are always flat in main positions of edge flexagons, but in some edge flexagons leaves have to be bent in order to flex from one main position to another. In origami terms the leaves are rolled using an infinity of creases. Whether flexes that require leaf bending are legitimate is a matter of taste. A pragmatic approach, used in this book, is that a flex is legitimate provided that it can be carried out in a paper model without too much difficulty. In this approach an ideal flexible leaf is inextensible and of zero thickness, but with some flexibility. Theoretically, the flexibility of paper could be quantified, but in practice this is not helpful.

From a mechanical engineering viewpoint an ideal edge flexagon is a three dimensional linkage (Pook 2003). The formal definition of a linkage (Macmillan 1950) is that it is an assembly of coupled rigid bodies (links) whose freedom of movement is restricted, after the fixture of one link in space, by the constraint imposed by their couplings. Demaine and O'Rourke (2007) give an equivalent definition. The number of degrees of freedom possessed by a linkage is the number of independent parameters needed to completely determine its configuration. For example, two polygons connected by an edge hinge have one degree of freedom in which the dihedral angle changes. In any practical linkage the number of links is finite, so the number of degrees of freedom is also finite. In other words, the links can only follow a finite number of paths relative to each other. In an ideal edge flexagon the links are the rigid leaves, each of which is coupled to two neighbouring leaves by ideal hinges along common edges. An ideal edge flexagon is therefore what is known as a hinged linkage.

Some ideal even edge flexagons, including the hexahexaflexagon (Section 11.2.2) have large numbers of degrees of freedom (Pook 2003), and can therefore by flexed into numerous main positions, in most of which face numbers become mixed up. Possible positions have been investigated in detail for some even edge flexagons, for example by McLean (1979), and by Mitchell (2002). Large numbers of degrees of freedom mean that unwanted positions may occur if a paper model is not flexed correctly. A flexagon that has been accidentally flexed into an unwanted position is said to be muddled. It is easy to get some types badly muddled. It is usually difficult to see how to return an accidentally muddled flexagon to a wanted position.

Avoidance of muddling is the reason why some authors give very detailed instructions on the manipulation of models of some types of even edge flexagon. Such instructions usually include the implicit requirement that rotational symmetry be maintained during flexing. This artificially limits the number of degrees of freedom, and hence unwanted positions are avoided.

Paper models of edge flexagons only approximate to ideal edge flexagons. Paper has finite thickness, and is not rigid. This has two main consequences. Firstly, the finite thickness sometimes makes manipulation of some edge flexagons difficult. Secondly, leaves can be bent during flexing. If this is regarded as permissible then, in some edge flexagons, this extends the range of possible flexes. However, in main positions leaves are always flat. Leaves do not have to be bent while flexing the two sector first order fundamental square even edge flexagon (Fig. 1.2).

Two polygons connected at a common vertex by an ideal point hinge have two degrees of freedom. An ideal point hinge between two polygons is a special case of a Hooke's joint, as used in motor vehicle drivelines (Dunkerley 1910). In an initial position both polygons lie in the same plane, as shown in Fig. 1.10 for two triangles lying in the *x*-*y* plane. In one degree of freedom the triangles can be rotated relative to each other about the *y*-axis. This is equivalent to the degree of freedom of an edge hinge. In the other degree of freedom the triangles can be rotated relative to each other about the *y*-axis. Possible combinations of the two rotations are restricted by interference between the two triangles. By definition, the triangles cannot be twisted relative to each other. Ideal point flexagons are linkages.

In a compound edge ring alternate polygons are the same distance from the centre of the ring, and alternate hinge angles are the same. By definition, a compound edge ring must be even. A flat compound edge ring of 8 squares is shown in Fig. 1.11. The heavy lines indicate that there is no square in the centre. The curvature of a ring with a hollow centre is calculated in the same way as the curvature of a ring in which the polygons have a common vertex. A more complicated example is the flat compound edge ring of 16 regular octagons shown in Fig. 1.12. The vertices of eight of the hinge angles are on the outside of the ring so these are taken as negative when calculating the curvature.

An irregular edge ring of regular convex polygons is a ring that is neither regular nor compound. For example, Fig. 1.13 shows a flat irregular edge ring of 12 equilateral triangles.

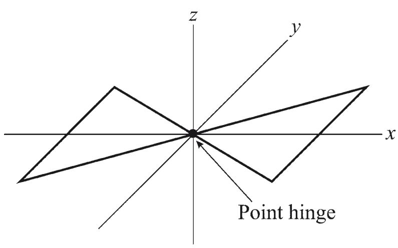


Fig. 1.10 A point hinge connecting two triangles in the *x*-*y* plane

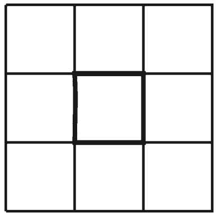


Fig. 1.11 A flat compound edge ring of eight squares (Les Pook, Flexagons inside out, 2003, © Cambridge University Press 2003, reprinted with permission)

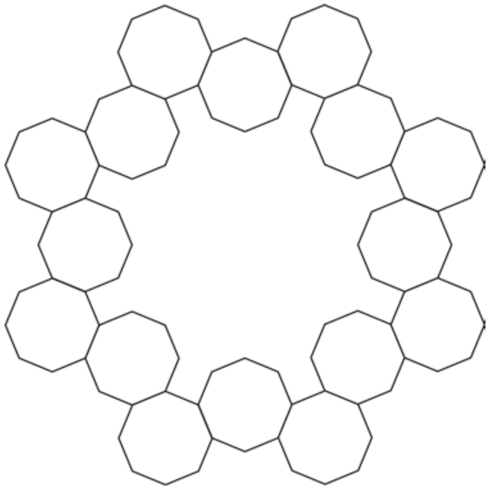


Fig. 1.12 A flat compound edge ring of 16 regular octagons

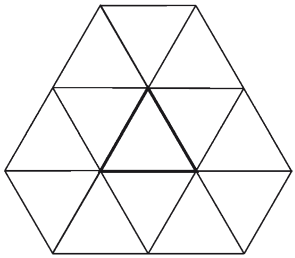


Fig. 1.13 A flat irregular even edge ring of 12 equilateral triangles

1.3 Outline of Book

There is an infinity of different types of flexagon, so no book on flexagons can be comprehensive. Flexagons whose nets are given in the text were chosen primarily because they are interesting to manipulate. Most are reasonably easy to handle. They have also be chosen to illustrate points made in the text, with particular emphasis on demonstrating relationships between different types of flexagon.

Hinged rings of polygons that have the same appearance as main positions of flexagons are discussed in Chapter 2, including some geometric constraints that restrict permissible rings. Nets used in the construction of flexagons vary widely in appearance, and some are very irregular. However, there are certain fundamental nets, such as the net shown in Fig. 1.2, that have a high degree of symmetry. These fundamental nets are used in the construction of fundamental flexagons, and are described in Chapter 3.

Fundamental edge flexagons, such as the two sector first order fundamental square even edge flexagon (Section 1.4.2) are constructed from fundamental edge nets, and main positions are, in appearance, regular edge rings. Fundamental edge flexagons are described in Chapter 4. Broadly, they are the equivalent of regular polyhedra in that they are constructed from identical regular convex polygons, and have a high degree of symmetry both in structure and in dynamic properties. Fundamental skeletal flexagons and fundamental point flexagons are constructed from fundamental vertex nets, and are described in Chapter 5. Fundamental skeletal flexagons are related to fundamental edge flexagons. Thus, broadly, fundamental skeletal flexagons are also the equivalent of regular polyhedra. Fundamental point flexagons are a special case of fundamental skeletal flexagons. Fundamental compound flexagons are constructed from fundamental edge nets, and are described in Chapter 6. Main positions are, in appearance, compound edge rings, for example the flat compound edge ring of eight squares shown in Fig. 1.11.

In a fundamental flexagon all the main positions that appear as a cycle of main positions is traversed have the same appearance and the same pat structure. However, in an irregular cycle flexagon the pat structure, but not the appearance of main positions, varies as a cycle is traversed. Irregular cycle flexagons are described in Chapter 7.

A precursor flexagon is a flexagon that is modified in some way to form a different type of flexagon. For example, deletion of one or more faces from a precursor flexagon leads to a degenerate flexagon. Most of the flexagons described in Chapters 4–7 can be used as precursors. Degenerate flexagons are described in Chapter 8. A feature of some degenerate flexagons is that they are easier to handle than the precursor flexagons.

Irregular ring even edge flexagons are even edge flexagons with main positions that are, in appearance, irregular even edge rings, for example the flat irregular even edge ring of 12 equilateral triangles shown in Fig. 1.13. Irregular ring fundamental even edge flexagons are made from fundamental edge nets. They are described in Chapter 9, together with degenerate versions.

All the flexagons described in Chapters 4–9 are made from regular convex polygons. However, flexagons can be made from any convex polygon, they are not restricted to regular convex polygons, although only a limited range of irregular shapes results in irregular polygon flexagons whose paper models are reasonably easy to handle. Some of these are described in Chapter 10. Silver flexagons, made from 45°–45°–90° triangles, and bronze flexagons, made from 30°–60°–90° triangles, are of particular interest.

The flexagons described in Chapters 4–10 are all so-called solitary flexagons, for example the two sector first order fundamental square even edge flexagon (Fig. 1.2). Complex flexagons, described in Chapter 11, are made from two or more solitary flexagons. For example, two sector first order fundamental square even edge flexagons can be linked to form a complex flexagon. Its dynamic properties

Fig. 1.14 The stella octangula, a compound of two regular tetrahedra

include features of the dynamic properties of the precursor flexagons. Complex flexagons can also incorporate parts of solitary flexagons. Most of the flexagons for which nets have been published are complex flexagons, and include some spectacular examples. For this reason it would have been better to have introduced the concept of a complex flexagon earlier in the book. However, material on solitary flexagons in previous chapters is needed as a preliminary to discussion of complex flexagons. Complex flexagons are broadly equivalent to compound polyhedra, such as the well known stella octangula, which is a compound of two regular tetrahedra (Cromwell 1997), and is shown in Fig. 1.14. Complex flexagons include some of the most interesting types of flexagon.

The miscellaneous flexagons and related structures described in Chapter 12 do not fit conveniently into the classification schemes used in earlier chapters, and include some interesting examples.

1.4 Making Flexagons

Geometric and aesthetic aspects of flexagons can be fully appreciated only by manipulating models. For videos and animations of flexagons being flexed see, for example, Highland Games (2008), Moseley (2008), Sherman (2008a, b), and YouTube (2008).

The nets used to construct flexagons are usually strips. If the polygons are regular, with edge hinges, then nets can be defined as a sequence of hinge angles (Moseley 2008). For example, the hinge angles in the net for the two sector first order fundamental square even edge flexagon (Fig. 1.2) are alternately +90° and −90°. Definition in terms of hinge angles has to be done if nets are generated by computer, but published nets are usually presented as line diagrams, without hinge angle data.

The appearance of flexagon models can be improved by colouring and decorating the faces, or by making them from coloured paper or card. A recent suggestion is to use transparent coloured material for the leaves (Shuttleworth 2006). Numerous decorative schemes have been used on various types of flexagon. Some of the decorative schemes exploit symmetries of flexagons both to create an attractive appearance and to create puzzles, for example Moseley (2008). There are books, for example Mitchell (1999) and Pedersen and Pedersen (1973), that include attractively decorated cut out nets for several types of flexagon. These have the disadvantage that making up the flexagons destroys the book.

1.4.1 General Assembly Instructions

Nets included in Chapters 4–12 are shown at approximately half full size and are satisfactory if 80 g/m^2 printer paper is used. This is a good compromise between rigidity and thickness. When flexing involves bending leaves, the use of origami paper, which is more flexible and creases well, can make flexing easier. A problem with origami paper is that printer ink tends to run and show through. However, origami paper takes pencil well. Some models are neater if made from 160 g/m^2 card with nets enlarged to about three times the size shown in the text. Models of most flexagons can be conveniently kept in transparent A5 (210 × 149 mm) size pockets kept in A5 ring binders. A recommended flexagon for a first attempt is described in the next section.

To assemble nets shown in the book, use the general scheme below. Where needed, different or additional instructions are included in captions for nets. Photographs showing assembly are included for some flexagons. All flexagons exist as enantiomorphic pairs (Section 1.2). The net for one enantiomorph can be converted into the net for the other by interchanging the markings on the faces of each leaf. Sometimes models of flexagons do not flex smoothly. If this is a problem ensure that all the hinges are well creased. If this does not work try using thinner paper or larger leaves. With edge flexagons try trimming a small amount, say 1 mm, from the edges of the net.

1. Make the specified number of copies and cut them out. Cut along any heavy lines.
2. Crease the lines between leaves to form hinges. For point and skeletal flexagons crease the lines across the strips joining leaves. Ensure that adjacent leaves superimpose correctly when folded together.
3. Transfer the numbers, and any other markings, that are in brackets on the upper face of a leaf to the reverse face, and delete from the upper face.
4. If more than one copy is specified join them end to end. For edge hinges join at the dashed lines, using transparent adhesive tape. If the hinges are short tape both sides. For point hinges glue tabs together. When copies have an odd number of leaves turn alternate copies over before joining.
5. Fold leaves with the same number together until only leaves numbered 1 and 2 are visible. An appropriate order is usually obvious, but if in doubt start with the highest number and work downwards. Some point flexagons have to be interleaved during assembly. For instructions see Sections 5.4.2 and 5.6.2.
6. Join the ends of the net to complete assembly.

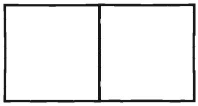


Fig. 1.15 A square edge pair

1.4.2 The Two Sector First Order Fundamental Square Even Edge Flexagon

The two sector first order fundamental square even edge flexagon is recommended as a first attempt at making a flexagon. Its net is shown in Fig. 1.2. The flexagon and its dynamic properties are described in more detail in Sections 4.2.1, 4.2.2, 4.2.4 and 4.2.5. The net is shown half full size. It could either be copied or drawn on squared paper.

As assembled, the flexagon is in a main position with leaves numbered 1 visible on one face and leaves numbered 2 on the other. In appearance, it is a flat regular even edge ring of four squares (Fig. 1.1b). It can be traversed around a cycle of four main positions by using the twofold pinch flex, which is its characteristic flex. Start by folding the flexagon in two, so that only leaves numbered 2 are visible, to reach an intermediate position. This is, in appearance, a square edge pair (Fig. 1.15). There are two ways of folding the flexagon in two so that leaves numbered 2 are visible, only one of which works. Then open the flexagon about the opposite long edge to reach another main position in which leaves numbered 2 and 3 are visible, thus completing a twofold pinch flex. Next, repeat the twofold pinch flex by folding in two so that leaves numbered 3 are visible in an intermediate position, and unfold to reach a main position in which leaves numbered 3 and 4 are visible. Repeat again, folding so that leaves numbered 4 are visible, and unfold so that leaves numbered 1 and 4 are visible. Finally, complete the traverse by folding so that leaves numbered 1 are visible, and unfold to return to the initial main position in which leaves numbered 1 and 2 are visible.

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Chapter 2
Polygon Rings

2.1 Introduction

In general, a main position of a flexagon is a position that is, in appearance, a ring of convex polygons (Section 1.1). Consequently, an understanding of the properties of polygon rings is needed for an understanding of some of the properties of flexagons. Polygon rings are clearly defined geometric objects that exist in infinite series. In this chapter it is taken as understood that only the first few members of an infinite series are being described. Polygon rings are described as flat, slant and skew (Section 1.1). These descriptions can also be applied to main positions of flexagons, and they are described as flat main positions, slant main positions and skew main positions.

All the polygon rings described in this chapter, and in other chapters, are hinged. What is meant by an even edge ring, an odd edge ring, and an even vertex ring is defined in Section 1.1. A compound edge ring and an irregular edge ring are defined in Section 1.2. Various aspects of flat edge rings of regular polygons have been discussed by several authors (Conrad and Hartline 1962; Hirst 1995; Dunlap 1997/1998; Griffiths 2001; Pook 2003). Polygon rings illustrated in this chapter have been chosen to complement points made in the text. Polygon rings in subsequent chapters complement descriptions of flexagons.

Most polygon rings are linkages (Section 1.2). A linkage has an infinity of possible positions (states). As a convention, it is assumed that a polygon ring which is a linkage is arranged to be flat, whenever possible, and also that it is arranged to be as symmetrical as possible. This has been done for the edge rings shown in Figs. 1.1, 1.4, 1.5 and 1.11–1.13. The additional degree of freedom in point hinges means that all vertex rings can be laid flat (Fig. 1.6).

Positions that are, in appearance, multiple polygons and combinations appear during the flexing of some flexagons. These are related to polygon rings, and are described in the next two sections.

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2.1.1 Multiple Polygons

During flexing some flexagons pass through intermediate positions that are, in appearance, a multiple polygon which consists of polygons hinged together, either at a common edge, a common vertex, or at pairs of common vertices. Depending on the number of polygons these multiple polygons are described as polygon pairs, polygon triples, etc. As a convention, it is a assumed that a multiple polygon is arranged to be as symmetrical as possible. Figure 2.1 shows a flexagon as an equilateral triangle edge triple, Fig. 2.2 another flexagon as an equilateral triangle vertex triple connected at a common vertex, Fig. 1.4a flexagon as an equilateral triangle vertex pair connected at a pair of common vertices and at a single common vertex, and Fig. 1.15a square edge pair.

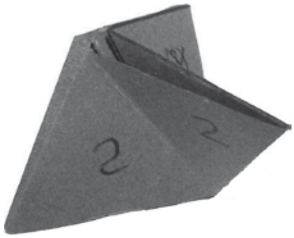

Fig. 2.1 A flexagon as an equilateral triangle edge triple

Fig. 2.2 A flexagon as an equilateral triangle vertex triple connected at a common vertex. Point hinges approximated by paper strips