Deduction, Computation, Experiment

Exploring the Effectiveness of Proof

Rossella Lupacchini, Giovanna Corsi (Eds.)

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Rossella Lupacchini
Giovanna Corsi
Dipartimento di Filosofia
Università degli Studi di Bologna

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Preface

This volume is located in a cross-disciplinary field bringing together mathematics, logic, natural science and philosophy. Reflection on the effectiveness of proof brings out a number of questions that have always been latent in the informal understanding of the subject. What makes a symbolic construction significant? What makes an assumption reasonable? What makes a proof reliable? Gödel, Church and Turing, in different ways, achieve a deep understanding of the notion of effective calculability involved in the nature of proof. Turing's work in particular provides a "precise and unquestionably adequate" definition of the general notion of a formal system in terms of a machine with a finite number of parts. On the other hand, Eugene Wigner refers to the unreasonable effectiveness of mathematics in the natural sciences as a miracle.

Where should the boundary be traced between mathematical procedures and physical processes? What is the characteristic use of a proof as a computation, as opposed to its use as an experiment? What does natural science tell us about the effectiveness of proof? What is the role of mathematical proofs in the discovery and validation of empirical theories? The papers collected in this book are intended to search for some answers, to discuss conceptual and logical issues underlying such questions and, perhaps, to call attention to other relevant questions.

Can every 'real' proof be translated into a 'formal' proof? Although Hilbert and Gentzen's positive answer is widely shared, there are also reasons for disagreement. To deals with this matter Carlo Cellucci addresses two fundamental questions - Why proof? What is a proof? - which he settles by contrasting the notion of axiomatic proof with the notion of analytic proof.

The contribution by Andrea Cantini concentrates on the nature and role of formal proofs. It is argued that formal proofs do not target certainty or formalistic foundations. Recent results in proof theory are considered in order to illustrate the role of formal proofs in exploring ideas and clarifying foundational questions in mathematics. The question is raised to what extent are proofs for mathematics what experimental procedures are for empirical sciences?

Closely related to this topic is the question as to what role mathematics should play in certain physical theories lacking both of rigorous mathematical structures and of experimental verifications. The notion of "theoretical mathematics", as a synthesis of theoretical physics and mathematics, is elucidated by Annalisa Marzuoli. By creating "toy models", *i.e.* simplified models of complex physical systems, not only can mathematics set up a basis for testing physical theories, but mathematical proofs may become more compelling than experiments.

On the other hand, the significance and the diverse degrees of involvement of 'experimental' methods in mathematics are investigated by Gabriele Lolli. Examples are taken throughout the history of mathematics to throw doubt on the empiricist view of mathematics as a "quasi-empirical" science and to maintain the distinctive symbolic character of mathematics.

These same problems have made it necessary for mathematics to be concerned about proofs produced by machines. Dag Prawitz's paper explores conceptual questions as to the use of deductive machinery to verify the correctness of computer programs and to the running of programs on computers to produce proofs.

Giovanna Corsi presents a Gentzen-style calculus as a case study for the discussion of typical metatheoretical properties: when is that a proof-tree is closed? when is a proof-tree cut-free? when is it analytic?

The intercalation method for proof search is extended from pure first-order logic to parts of mathematics by Wilfried Sieg and Clinton Field. By interweaving general logical strategies with specific mathematical heuristic, proofs of significant theorems are found in a fully automated way. They present a solution for Gödel's incompleteness theorems.[1]

New perspectives in computational complexity theory are examined by Ugo Dal Lago and Simone Martini in the frame of the "Curry-Howard correspondence". By adopting a kind of proof-theoretical approach, the so-called "Implicit Computational Complexity" takes into account single machine-free models of computation and analyses complexity classes with respect to language constraints.

The contribution by Mario Rasetti suggests how one might benefit from quantum information tools in order to elicit wider fields than mere computation. In particular, the problem of 'combing' finite groups brings to light relevant connections between language-theoretical and algorithmic-structural issues. To open up more comprehensive fields a complex blend of notions from different theoretical regions - such as formal languages, finite groups and quantum computation - is needed.

From an 'outside' point of view, *i.e.* the detached view of the philosopher of science, Dag Westerståhl discusses the conflict between classical and intuitionistic mathematics. By focusing on proofs rather than on explanations

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[1] This paper is reprinted from *Annals of Pure and Applied Logic* **133** (2005) pp. 319-338

of meaning, mutual understanding between classical and intuitionistic mathematicians may be significantly improved. 'Distinguishability' may help to clarify how intuitionists and classical mathematicians understand proofs by grading classical and constructive proofs.

Rossella Lupacchini's paper examines how a search for 'more effective' distinguishability leads quantum theory in complex Hilbert spaces and opens up new computational paths. Specifically the bilateral symmetry involved in the very notion of distinguishability, emerging out of quantum probability amplitudes, can provide an argument for the irreducibility of quantum structures to classical ones.

While there is a classical standard model of computability on natural numbers, *i.e.* in the style of the Turing machine, there are several nonequivalent theories of computability for real numbers. In order to illustrate different approaches, Guido Gherardi investigates the computability of the wave equation as a key example of a partial differential equation system in theoretical physics. Here the difficulty of reconciling the discrete nature of computability with the continuous character of motion equations stands out.

A reflection on the nature of incompleteness and its crucial role in the evaluation of the effectiveness of both mathematics and physics is developed by Francis Bailly and Giuseppe Longo. The phenomenon of incompleteness is analysed in the context of Gödel's logical results as well as of quantum theory. A general constructivist approach to knowledge underlies this attempt to achieve a 'unified' understanding of apparently unrelated theoretical issues.

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Giovanna Corsi
Rossella Lupacchini

Contents

List of Contributors

Francis Bailly
Physique, CNRS, Meudon
9, Rue Jules Hetzel
92190 Meudon (France)
francis.bailly@noos.fr

Andrea Cantini
Università di Firenze
Dipartimento di Filosofia
Via Bolognese 52
50139 Firenze (Italy)
cantini@philos.unifi.it

Carlo Cellucci
Università di Roma 'La Sapienza'
Dipartimento di Studi Filosofici ed Epistemologici
Via Carlo Fea 2
00161 Roma (Italy)
carlo.cellucci@uniroma1.it

Giovanna Corsi
Università di Bologna
Dipartimento di Filosofia
Via Zamboni 38
40126 Bologna (Italy)
giovanna.corsi@unibo.it

Ugo Dal Lago
Università di Bologna
Dipartimento di Scienze dell'Informazione
Mura Anteo Zamboni 7
40127 Bologna (Italy)
dallago@cs.unibo.it

Clinton Field
Carnegie Mellon University
Department of Philosophy
Baker Hall 135
Pittsburgh PA 15213-3890 (US)

Guido Gherardi
Università di Bologna
Dipartimento di Filosofia
Via Zamboni 38
40126 Bologna (Italy)
guido.gherardi@unibo.it

Gabriele Lolli
Università di Torino
Dipartimento di Matematica
Via Carlo Alberto 10
10123 Torino (Italy)
gabriele.lolli@unito.it

Giuseppe Longo
Ecole Normale Supérieure
CNRS et Département d'Informatique
45, rue d'Ulm
75230 Paris Cedex 05 (France)
Giuseppe.Longo@ens.fr

Rossella Lupacchini
Università di Bologna
Dipartimento di Filosofia
Via Zamboni 38
40126 Bologna (Italy)
rossella.lupacchini@unibo.it

Simone Martini
Università di Bologna
Dipartimento di Scienze dell'Informazione
Mura Anteo Zamboni 7
40127 Bologna (Italy)
martini@cs.unibo.it

Annalisa Marzuoli
Università di Pavia
Dipartimento di Fisica Nucleare e Teorica
Via Bassi 6
27100 Pavia (Italy)
Annalisa.Marzuoli@pv.infn.it

Dag Prawitz
Stockholms Universitet
Filosofiska institutionen
106 91 Stockholm (Sweden)
dag.prawitz@philosophy.su.se

Mario Rasetti
Politecnico di Torino
Dipartimento di Fisica
C.so Duca degli Abruzzi 24
10129 Torino (Italy)
mario.rasetti@polito.it
Fondazione ISI
Via Settimio Severo 65
10133 Torino (Italy)
rasetti@isi.it

Wilfried Sieg
Carnegie Mellon University
Department of Philosophy
Baker Hall 135
Pittsburgh PA 15213-3890 (US)
sieg@cmu.edu

Dag Westerståhl
Göteborgs Universitet
Filosofiska institutionen
Box 200, SE 405 30 Göteborg (Sweden)
dag.westerstahl@phil.gu.se

Why Proof? What is a Proof?

Carlo Cellucci

Dipartimento di Studi Filosofici ed Epistemologici
Università di Roma 'La Sapienza' (Italy)
carlo.cellucci@uniroma1.it

1 The Hilbert-Gentzen Thesis

This paper is concerned with real proofs as opposed to formal proofs, and specifically with the ultimate reason of real proofs ('Why Proof?') and with the notion of real proof ('What is a Proof?').

Several people believed and still believe that real proofs can be represented by formal proofs. A recent example is provided by Macintyre who claims that "one could go on to translate" all "classical informal proofs into formal proofs of some accepted formal system", where such translations "do map informal proofs to formal proofs" [44, p. 2420].

This view is to a certain extent implicit in Frege – to a certain extent only, because for Frege in a sense "every inference is non-formal in that the premises as well as the conclusions have their thought-contents which occur in this particular manner of connection only in that inference" [18, p. 318].

Anyway, the view that real proofs can be represented by formal proofs is explicitly stated by Hilbert and Gentzen.

For Hilbert claims that formal proofs are "carried out according to certain definite rules, in which the technique of our thinking is expressed" [34, p. 475]. These are "the rules according to which our thinking actually proceeds". They "form a closed system that can be discovered and definitively stated".

Similarly, Gentzen claims that formal proofs in his natural deduction systems have "a close affinity to actual reasoning" [22, p. 80]. They reflect "as accurately as possible the actual logical reasoning involved in mathematical proofs" [22, p. 74].

Thus one may state the following:

> *Hilbert-Gentzen Thesis*. Every real proof can be represented by a formal proof.

Although the Hilbert-Gentzen thesis is widely held, there are various reasons for thinking that it is inadequate. To discuss this matter we must answer the questions: Why proof? What is a proof?

2 What is a proof?

As already mentioned, ‘Why proof?’ is a question about the ultimate reason of real proofs. Such question is strictly connected to the question ‘What is a proof?’, for the ultimate reason of real proofs depends on what real proofs are, so one can expect that an answer to the question ‘What is a proof?’ will yield an answer to the question ‘Why proof?’.

Of course, this holds only if ‘Why proof?’ is meant as a question about the ultimate reason of real proofs, not as a question about their possible uses, which are multifarious. (Thirty-nine such uses are listed in Lolli [42]). Here ‘Why proof?’ will be meant in that sense.

There are two distinct answers to the question ‘What is a proof?’, which yield two essentially different and indeed alternative notions of real proof. All known apparently different notions of proof can be reduced to such two notions.

A) *The notion of axiomatic proof.* Proofs are deductive derivations of propositions from primitive premisses that are true in some sense of ‘true’. They start from given primitive premisses and go down to the proposition to be proved. Their aim is to give a foundation and justification of the proposition.

B) *The notion of analytic proof.* Proofs are non-deductive derivations of plausible hypotheses from problems, in some sense of ‘plausible’. They start from a given problem and go up to plausible hypotheses. Their aim is to discover plausible hypotheses capable of giving a solution to the problem.

The notion of axiomatic proof was first stated by Aristotle in *Posterior Analytics* and then modified by Pascal, Pieri, Hilbert, Padoa in that order (see Cellucci [4, Chs. 4-5]). It is a very familiar one and so does not seem to require further explanation.

The notion of analytic proof was first stated by Plato in *Meno* and *Phaedo* (see [4, pp. 270-308]). It is less familiar and so requires some explanation. The main points seem to be the following.

1) A problem is any open question.

2) A hypothesis is any means that can be used to solve a problem.

3) A hypothesis is said to be plausible if and only if it is compatible with the existing data – that is, all mathematical notions and results available at that moment – in the sense that, comparing the arguments for and the arguments against the hypothesis on the basis of the existing data, the arguments for prevail over those against. (A typical example is provided by the discussions

concerning the plausibility of the axiom, or rather hypothesis, of choice at the beginning of the twentieth century).

4) The process by which problems are solved is an application of the analytic method, which can be described as follows. One looks for some hypothesis that is a sufficient condition for solving the problem. The hypothesis is obtained from the problem, and possibly other data, by some non-deductive inference: inductive, analogical, diagrammatic, metaphorical, metonymical, by generalization, by specialization, by variation of the data, and so on. (On different kinds of non-deductive inferences for finding hypotheses, see Cellucci [6, pp. 235-295]. The hypothesis must not only be a sufficient condition for solving the problem, but must also be plausible, that is, compatible with the existing data. However the hypothesis, in turn, is a problem that must be solved, and will be solved in the same way. That is, one will look for another hypothesis that is a sufficient condition for solving the problem posed by the former hypothesis, it is obtained from it, and possibly other data, by some non-deductive inference, and must be plausible. And so on, *ad infinitum.* Thus the solution of a problem is a potentially infinite process.

In the course of this process the statement of the problem may be modified to a certain extent to make it more precise, or may even be radically changed as new data emerge. Thus the development of the statement of the problem and the development of the solution of the problem may proceed in parallel.

The analytic method is both a method of discovery and a method of justification. It involves two distinct processes, the formulation of candidates for hypothesis by means of non-deductive inferences, and the choice among such candidates on the grounds of their plausibility. Such choice is necessary because non-deductive inferences can yield different conclusions from the very same premisses. To choose among such conclusions one must carefully assess the arguments for and the arguments against each of them on the basis of the existing data. Such assessment is a process of justification, so justification is part of discovery. (For more on the analytic method, see [4], [6], [7], [11]).

The axiomatic method is what results from the analytic method when the hypotheses stated at a certain stage are considered as an absolute starting point, for which no justification is given. Thus the axiomatic method is an unjustified truncation of the analytic method.

One of the oldest examples of analytic proof concerns the problem of the duplication of the cube a^3. Hippocrates of Chios solved it by showing that the hypothesis ‘One can find two mean proportionals x and y in continued proportion between a and $2a$’ is a sufficient condition for its solution.Then Menaechmus solved the problem posed by such hypothesis by showing that a certain other hypothesis is a sufficient condition for its solution. And so on.

A recent example of analytic proof concerns Fermat’s Problem. Ribet solved it by showing that the Taniyama-Shimura conjecture – or hypothesis – is a sufficient condition for its solution. For “let E be an elliptic curve over $\mathbf{Q}$. The Taniyama-Shimura Conjecture (also known as the Weil-Taniyama Conjecture) states that E is modular” [55, p. 123]. Ribet showed: “Conjecture of

Taniyama-Shimura $\Longrightarrow$ Fermat's Last Theorem" [55, p. 127]. Thus, as stated above, Ribet solved Fermat's problem by showing that the Taniyama-Shimura hypothesis is a sufficient condition for its solution. Then Wiles and Taylor solved the problem posed by the Taniyama-Shimura hypothesis by showing that certain other hypotheses are a sufficient condition for its solution. And so on.

But already one can hear the objection: Surely Ribet did not solve Fermat's Problem, for his alleged solution depended on a hypothesis that at the time had not been proved yet! (One could hear a similar objection about Hippocrates of Chios's solution of the problem of the duplication of the cube).

Now, if Ribet did not solve Fermat's Problem because his solution depended on a hypothesis, the Taniyama-Shimura hypothesis, that at the time had not been proved yet, then Wiles and Taylor have not solved Fermat's problem because their solution depends on a hypothesis, the axioms of set theory, that to this very day has not been proved yet. (Similarly as regards Hippocrates of Chios and Menaechmus. Ribet stands to Hippocrates of Chios as Wiles and Taylor stand to Menaechmus).

One can also hear the objection: A so-called 'analytic proof' is not a proof. Admittedly, working backwards to find the needed ingredients to prove a potential theorem is a standard method in the work of mathematicians. But this cannot justifiably be referred to as proof, for what is customarily understood by 'proof' is a sequence of arguments to justify an assertion.

Now, at each stage in the development of an analytic proof, only a finite piece of the proof is given and, reading it top down rather than bottom up, one has a sequence of arguments that justifies an assertion. The sequence justifies it because hypotheses must be plausible. Reading the sequence top down rather than bottom up is inessential, it is just a matter of convention.

3 Analytic and axiomatic proof

The point of analytic proof can be seen in terms of Gödel's first incompleteness theorem.

Suppose that you want to solve a problem, say, of elementary number-theory and want to find a hypothesis to solve it. By Gödel's result there is no guarantee that the hypothesis can be derived from the axioms of Peano Arithmetic, so you must be prepared to look for hypotheses of any kind, concerning objects of any mathematical field. Think, for instance, of Fermat's problem, a problem concerning natural numbers that, as we have already mentioned, Ribet solved using a hypothesis, the Taniyama-Shimura, concerning elliptic curves over $\mathbf{Q}$ – the set of rational numbers.

Moreover, again by Gödel's result, there is no guarantee that the hypothesis can be derived from any known axioms. Think, for instance, of Gödel's suggestion that we might need new infinity axioms to solve number-theoretic problems (see [23, p. 269]). Thus solving a problem generally consists in looking for hypotheses in an open, that is, not predetermined space.

The point of analytic proof can be also seen, perhaps more vividly, in terms of Hamming's statement: "If the Pythagorean theorem were found to not follow from postulates, we would again search for a way to alter the postulates until it was true. Euclid's postulates came from the Pythagorean theorem, not the other way" [26, p. 87]. In mathematics "you start with some of the things you want and you try to find postulates to support them" [27, p. 645]. The idea that you simply lay down some arbitrary postulates and then make deductions from them "does not correspond to simple observation" [26, p. 87].

In addition to giving alternative answers to the question 'What is a Proof?', the notions of axiomatic proof and analytic proof give alternative answers to the question 'Why Proof?'. For the ultimate reason of axiomatic proof is to give a foundation and justification of a proposition, and the ultimate reason of analytic proof is to discover plausible hypotheses capable of giving a solution to a problem.

Since their ultimate reasons are different, the notions of axiomatic and analytic proof play different roles in the development of mathematics.

Axiomatic proof, being meant to give a foundation and justification of an already acquired proposition, is not intrinsically fruitful for the creation of new mathematics.

On the contrary, analytic proof has a great heuristic value, not only because it is meant to discover plausible hypotheses capable of giving a solution to a problem, but also because such hypotheses may belong to areas of mathematics different from the one to which the problem belongs. Thus they may establish connections between the problem and concepts and results of other areas of mathematics. This may reveal unexpected relations between different areas, which may suggest new perspectives and new problems and so may be very fruitful for the development of mathematics. As Grosholz says, these new perspectives and problems may allow one "to explore the analogies among disparate things, a practice which in the formal sciences tends to generate new intelligible things" [25, p. 49].

The notions of axiomatic and analytic proof yield alternative notions of mathematical theory.

In terms of axiomatic proof, a mathematical theory is a closed set of primitive premisses and propositions obtained from them by deductive inferences – a closed set, because primitive premisses are predetermined and given once for all, and the propositions belonging to the set in question are entirely determined by the primitive premisses. Briefly, a mathematical theory is a closed system.

In terms of analytic proof, a mathematical theory is an open set of problems and hypotheses for their solution obtained from the problems by non-deductive inferences – an open set, because hypotheses are not predetermined or given once for all and the solution of a problem may generate new problems.

Briefly, a mathematical theory is an open system. (For more on these notions of closed and open system, see [4, pp. 309-347], [5], [6, Chs. 7 and 26]).

4 Analytic and analytic-synthetic method

The analytic method must not be confused with the analytic-synthetic method. While the analytic method is a method for finding hypotheses to solve given problems, the analytic-synthetic method is a method for finding deductions of given propositions from given primitive premisses (axioms, rules, definitions), thus it is only a heuristic pattern within axiomatized mathematics.

In the analytic-synthetic method, to find a deduction of a given proposition from given primitive premisses, one looks for premisses from which that proposition will follow, then one looks for premisses from which those premisses will follow, and so on until one arrives at some primitive premisses among the given ones. If this process is successful, then inverting the path direction – that is, repeating the steps in inverse order – one gets a deduction of the given proposition from the given primitive premisses, as desired.

While in the analytic method finding a solution of a given problem is a potentially infinite process, in the analytic-synthetic method finding a deduction of a given proposition from given primitive premisses is a finite process.

Actually, there are two versions of the analytic-synthetic method, originally described by Aristotle and Pappus, respectively, which differ as to the direction of the analysis. In Aristotle's version the direction is upward, in Pappus's version it is downward (see [4, pp. 289-299], [11, Ch. 15]). Here we need only consider Aristotle's version, that is the one described above. (For more on the analytic-synthetic method, see Hintikka-Remes [37], Knorr [40], Mäenpää [45], Timmermans [59]).

5 Frege's Thesis

Supporters of the notion of axiomatic proof assume that all proofs come under that notion. This is due to the influence of Frege, who sharply separates the context of discovery from the context of justification, limiting logic to the latter and confining the context of discovery to individual psychology (see [19, p. 5], [17, p. 3]). Contemporary supporters of the notion of axiomatic proof follow Frege. Thus one may state the following:

Frege's Thesis. Every real proof is an axiomatic proof.

The Hilbert-Gentzen Thesis can be viewed as an extreme form of Frege's Thesis, for it implies that every real proof not only is an axiomatic proof but is also, 'up to representation', a formal proof. Azzouni states such implication by saying that "ordinary mathematical proofs indicate (one or another) mechanically checkable derivation of theorems from the assumptions those ordinary mathematical proofs presuppose" [2, p. 105]. So "it's derivations, derivations in one or another algorithmic system, which underlie what's characteristic of mathematical practice" [2, p. 83]. (For a critical appraisal of Azzouni's views, see Rav [54]).

6 Proofs as means of discovery or justification

While Hilbert and Gentzen build on Frege's Thesis, Aristotle who, as we have already mentioned, first stated the notion of axiomatic proof, would have rejected it. For, although Aristotle sharply distinguishes between the procedure by which new propositions are obtained and the procedure by which propositions already obtained are organized and presented, he considers both such procedures as belonging to logic. Aristotle views the former as the procedure of the working mathematician, the latter as the procedure for teaching and learning propositions already obtained, and attributes only the latter to the notion of axiomatic proof.

For Aristotle states that the procedure by which new propositions are obtained consists in a method that will tell us "how we may always find a deduction to solve any given problem, and by what way we may reach the primitive premisses adequate to each problem" (Aristotle, *Analytica Priora*, A 27, 43a 20-22). The method "is useful with respect to the first elements in each science" (Aristotle, *Topica*, A 2, 101a 36-37). For, "being used in the investigation, it directs to the primitive premisses of all sciences" (*ibid.*, A 2, 101b 3-4). On the other hand, the procedure by which propositions already obtained are organized and presented consists in the axiomatic method, for "we know things through demonstrations", where demonstration is "scientific deduction" (Aristotle, *Analytica Posteriora*, A 2, 71b 17-19). That is, it is a deduction which proceeds "from premisses that are true and primitive" (*ibid.*, A 2, 71b 20-21). The importance of demonstration depends on the fact that "all teaching and intellectual learning" is obtained by means of it, in particular "the mathematical sciences are acquired in this way" (*ibid.*, A 1, 71a 1-4).

Aristotle's description of the procedure by which propositions already obtained are organized and presented corresponds to the notion of axiomatic proof. On the other hand, his description of the procedure by which new propositions are obtained does not correspond to that notion. It does not correspond to the notion of analytic proof either because, for Aristotle, the process by which new propositions are obtained is finite. Rather, it corresponds to the procedure by which deductions of given propositions from given primitive premisses are obtained in – Aristotle's version of – the analytic-synthetic method.

That, for Aristotle, the procedure by which new propositions are obtained is finite depends on his argument that infinite regress is inadmissible, otherwise one could prove everything, including falsehood. To stop infinite regress Aristotle assumes that there must be some primitive premisses that are true, and must also be known to be true, otherwise one would be unable to tell whether something is a demonstration.

For Aristotle claims that, if the series of premisses "did not terminate and there was always something above whatever premiss has been taken, then there would be demonstrations of all things" (Aristotle, *Analytica Posteriora*, A 22, 84a 1-2.). Thus there must be premisses that must be "primitive and indemonstrable, because otherwise there would be no scientific knowledge", and moreover "must be true, because it is impossible to know what is not the case" (*ibid.*, A 2, 71b 25-27). In addition to being true, primitive premisses must also be known to be true, for "if it is impossible to know the primitive premisses, then it is impossible to have scientific knowledge of what proceeds from them absolutely and properly" (*ibid.*, A 2, 72b 13-14). And to know the primitive premisses amounts to knowing that they are true, for "grasping and stating" them "is truth" (Aristotle, *Metaphysica*, Θ 10, 1051b 24).

Moreover, Aristotle claims that we know that primitive premisses are true by intuition. For since "there cannot be scientific knowledge of the primitive premisses, and since nothing except intuition can be truer than scientific knowledge, it will be intuition that apprehends the primitive premisses" (Aristotle, *Analytica Posteriora*, B 19, 100b 10-12). So "it is intuition that grasps the unchangeable and first terms in the order of proofs" (Aristotle, *Ethica Nicomachea*, Z 11, 1143b 1-2).

However reasonable such Aristotle's claims may appear, nevertheless they are untenable.

Aristotle's claim that, since nothing except intuition can be truer than scientific knowledge, it will be intuition that apprehends the primitive premisses, is untenable because intuition is an unreliable source of knowledge. Kripke states: "I think" that intuition "is very heavy evidence in favor of anything, myself. I really don't know, in a way, what more conclusive evidence one can have about anything, ultimately speaking" [41, p. 42]. Actually just the opposite is true. Being completely subjective and arbitrary, intuition cannot be used as evidence for anything. One really doesn't know what less conclusive evidence one could have about anything, ultimately speaking. For instance, Frege considered his paradoxical Basic Law V completely intuitive since, in his opinion, it "is what people have in mind, for example, where they speak of the extensions of concepts" [18, p. 4]. But Russell's paradox showed that Frege's intuition was wrong. On the other hand, completely counterintuitive propositions, the so-called 'monsters', have been proved in various parts of mathematics. (On intuition and 'monsters', see [6, Ch. 12] and [11, Ch. 8]).

Moreover, Aristotle's claim that, if the series of premisses did not terminate, then there would be demonstrations of all things, is untenable because in the analytic method, which involves a potentially infinite regress, premisses – that is, hypotheses – must be plausible, that is, compatible with the existing data, so there can only be demonstrations of things using plausible premisses.

Of course, a price has to be paid for that. Since plausible premisses are not certain, the things proved by demonstrations are not certain, so mathematics is not certain. But, in view of the unreliability of intuition, there is no alternative to that. As Xenophanes said, “as for certain truth, no man has known it, nor will he know it” for “all is but a woven web of guesses” [16, 21 B 34]. And yet knowledge, uncertain knowledge, is possible, for “with due time, through seeking, men may learn and know things better” [16, 21 B 18].

7 The status of the Hilbert-Gentzen Thesis

In the light of what has been stated above, the status of the Hilbert-Gentzen Thesis can be assessed as follows.

1) If by ‘proof’ one means ‘analytic proof’, then the Hilbert-Gentzen Thesis is obviously inadequate because formal proofs don’t represent analytic proofs.

2) If by ‘proof’ one means ‘axiomatic proof’, then the Hilbert-Gentzen Thesis is inadequate because, for instance, even the very first proof in Hilbert’s *Grundlagen der Geometrie* cannot be represented by a formal proof since it makes an essential use of properties of a figure (see [9] and [11, Ch. 9]). This belies Hilbert’s claim that “a theorem is only proved when the proof is completely independent of the figure” [36, p. 75]. Admittedly, one can give a purely formal proof of the same result, but this involves replacing the use of the figure by the use of additional primitive premisses (see Meikle-Fleuriot [46]). Then the resulting formal proof is essentially different from, and hence cannot be considered a representation of, Hilbert’s proof. Generally the use of figures is crucial in mathematics. As Grosholz says, “number and figure are the Adam and Eve of mathematics” [25, p. 47].

3) If by ‘proof’ one means ‘axiomatic proof’, then the Hilbert-Gentzen Thesis is inadequate also for the the more basic reason that the notion of axiomatic proof itself is inadequate. It is widely believed that the axiomatic method “guarantees the truth of a mathematical assertion” [56, p. 135]. This belief depends on the assumption that proofs are deductive derivations of propositions from primitive premisses that are true, in some sense of ‘true’. Now, as we will presently see, generally there is no rational way of knowing whether primitive premisses are true. Thus either primitive premisses are false, so the proof is invalid, or primitive premisses are true but there is no rational way of knowing that they are true, then one will be unable to see whether something is a proof, and hence will be unable to distinguish proofs from non-proofs. In both cases, the claim that the axiomatic method guarantees the truth of a mathematical assertion is untenable.

8 The truth of primitive premisses

We have claimed that generally there is no rational way of knowing whether primitive premisses are true. This can be seen as follows.

That primitive premisses are true can be meant in several distinct senses. The main ones are the following: 1) truth as possession of a model; 2) truth as consistency; 3) truth as convention.

8.1 Truth as possession of a model

Primitive premisses are true in the sense that they have a model, that is, there is a domain of objects in which they are true.

For instance, Tarski says that we "arrive at a definition of truth and falsehood simply by saying that a sentence is true" in a given domain "if it is satisfied by all objects" in that domain, "and false otherwise" [58, p. 353]. Then a sentence is true if and only if there is a domain of objects in which it is true.

But, if primitive premisses are true in the sense that they have a model, then to know that they are true one must be able to prove that they have a model. However, by Gödel's second incompleteness theorem, the sentence 'Primitive premisses have a model' will not be provable from such primitive premisses but only from a proper extension of them, whose primitive premisses have a model. However, by Gödel's second incompleteness theorem, the sentence 'The primitive premisses of the proper extension have a model' will not be provable from such primitive premisses but only from a proper extension of them, whose primitive premisses have a model. And so on, *ad infinitum*.

Thus there is no rational way of knowing whether primitive premisses are true in the sense that they have a model.

8.2 Truth as consistency

Primitive premisses are true in the sense that they are consistent, that is, no contradiction is provable from them.

For instance, Hilbert says that, "if arbitrarily given axioms do not contradict one another with all their consequences, then they are true" [35, p. 39]. Thus "'non-contradictory' is the same as 'true' " [33, p. 122].

But, if primitive premisses are true in the sense that they are consistent, then to know that they are true one must be able to prove that they are consistent. However, by Gödel's second incompleteness theorem, the sentence 'The primitive premisses are consistent' will not be provable from such primitive premisses but only from a proper extension of them, whose primitive premisses are consistent. However, by Gödel's second incompleteness theorem, the sentence 'The primitive premisses of the proper extension are consistent' will not be provable from such primitive premisses but only from a proper extension of them, whose primitive premisses are consistent. An so on, *ad infinitum*.

Thus there is no rational way of knowing whether primitive premisses are true in the sense that they are consistent.

8.3 Truth as convention

Primitive premisses are true in the sense that they are conventions, that is, they may be chosen arbitrarily, subject to no condition whatsoever.

For instance, Carnap says that “it is not our business to set up prohibitions, but to arrive at conventions” [3, p. 51]. Primitive premisses “may be chosen quite arbitrarily” and this “choice, whatever it may be, will determine what meaning is to be assigned to the fundamental logical symbols” [3, p. xv]. Thus “no question of justification arises at all, but only the question of the syntactical consequences to which one or the other of the choices leads”. A sentence is said to be ‘determinate’ if its truth or falsity is settled by the syntactical consequence relation alone, which thus provides “a complete criterion of validity for mathematics” [3, p. 100]. Sentences “may be divided into logical and descriptive, i.e. those which have a purely logical, or mathematical, meaning” and those which express “something extralogical – such as empirical” facts, “properties, and so forth” [3, p. 177]. Then “every logical sentence is determinate; every indeterminate sentence is descriptive” [3, p. 179].

But, if primitive premisses are true in the sense that they are conventions, then to know that they are true one must know that they are true with respect to the meaning their choice assigns to the fundamental logical symbols. However, by Gödel’s first incompleteness theorem, there are sentences of Peano Arithmetic, say, that are indeterminate and hence descriptive, so for Carnap they express something extralogical. This means that the primitive premisses of Peano Arithmetic don’t fully determine the meaning of the fundamental logical symbols, which will then be partly extralogical. Thus, to know that the primitive premisses of Peano Arithmetic are true involves considering something extralogical, say, some empirical facts.

To overcome this problem Carnap considers the possibility of expanding the primitive premisses of Peano Arithmetic by adding an inference rule with infinitely many premisses, the ω-rule, which allows one to infer $\forall x A(x)$ from $A(0), A(1), A(2), ...$ and makes all sentences of Peano Arithmetic determinate. Carnap claims that “there is nothing to prevent the practical application of such a rule” [3, p. 173]. But the syntactical consequence relation resulting from this addition is not recursively enumerable, and hence a fortiori, in Carnap’s parlance, it is indefinite. For, according to Carnap, “every definite” relation “can be calculated”, whereas in this case there exists no “definite method by means of which this calculation” can “be achieved” [3, p. 46]. So the ω-rule yields “a method of deduction which depends upon indefinite individual steps” [3, p. 100].

Thus any choice of primitive premisses for Peano Arithmetic either will not fully determine the meaning of the fundamental logical symbols – which will then be partly extralogical – or will yield an indefinite syntactical consequence relation.

Moreover, by Gödel’s second incompleteness theorem, one cannot know whether primitive premisses are consistent. This is problematic for, if primitive premisses were inconsistent, then it would be worthless to know that a sentence is a syntactical consequence of them. Since one cannot know whether primitive premisses are consistent, the only ground one would have to believe that their syntactical consequences include no contradiction would be inductive, that is, it would consist in the fact that until then no contradiction has been drawn from them. But then induction, not convention, would be the basis of the choice of primitive premisses.

Thus we may conclude that there is no rational way of knowing whether primitive premisses are true in the sense that they are conventions.

This is the substance of Plato's criticism of the axiomatic method. Plato asks: "When a man does not know the principle, and when the conclusion and intermediate steps are also constructed out of what he does not know, how can he imagine that such a fabric of convention can ever become science?" (Plato, *Republic*, VII 533 c 4-5). Carnap has no answer to this question. (For more on Plato's criticism of the axiomatic method, see [4, pp. 286-291]).

9 Decline and fall of axiomatic proof

As we have already said, to stop infinite regress Aristotle assumes that there must be primitive premisses that are true and are also known to be true. By what we have just seen, however, such primitive premisses cannot exist, for there is generally no rational way of knowing whether primitive premisses are true, in any sense of 'true'.

Thus the very foundation on which Aristotle and his modern followers wanted to build an alternative to analytic proof breaks down. Axiomatic proof is no viable alternative to analytic proof since it is inadequate. One is not justified in using it for generally there is no rational way of knowing whether the starting points of axiomatic proofs are true, in any sense of 'true'.

Axiomatic proof is inadequate also because there is no non-circular way of proving that deduction from primitive premisses is truth-preserving, that is, such that, if primitive premisses are true, then the propositions deduced from them are also true (see [8] and [11, Ch. 26]).

In addition to implying that axiomatic proof is inadequate, the fact that generally there is no rational way of knowing whether primitive premisses are true has another important consequence. It entails that primitive premisses of axiomatic proofs are simply 'accepted opinions', *endoxa* in Aristotle's parlance, or rather plausible propositions in the sense explained above. Thus they have the same status as hypotheses in analytic proofs. Then the notion of axiomatic proof collapses into that of analytic proof.

Even some supporters of the axiomatic method acknowledge that. For instance, Pólya states that analogy and other non-deductive inferences "not only help to shape the demonstrative argument and to render it more understandable, but also add to our confidence to it. And so we are led to suspect that a good part of our reliance on demonstrative reasoning may come from plausible reasoning" [52, p. 168].

Thus Frege's Thesis depends on a misunderstanding. An instance of such misunderstanding is the claim that Wiles and Taylor solved Fermat's Problem. What they actually solved is the problem posed by the Taniyama-Shimura hypothesis.

Admittedly, in the last century most mathematicians have thought themselves to be pursuing axiomatic proof. But, as the case of Fermat's problem shows, they weren't. Their belief to be pursuing axiomatic proof has been a matter of trend and fashion, so essentially a sociological fact: a result of the predominance of the ideology of the Göttingen School and the Bourbaki School over the mathematicians of the last century (see [4, Ch. 5]).

Mathematicians who think themselves to be pursuing axiomatic proof don't seem to be generally aware that Frege's Thesis together with Hilbert-Gentzen Thesis would make mathematics trivial. For then there would be an algorithm that in principle could generate all possible proofs from given axioms in systematic manner, checking each time if the final proposition is the proposition to be proved. Thus theorem proving would become an activity requiring no intelligence.

Some supporters of axiomatic proof seem however to be aware of that, at least to a certain extent. For instance Rota, while maintaining that the axiomatic method guarantees the truth of a mathematical assertion, says that the "identification of mathematics with the axiomatic method has led to a widespread prejudice among scientists that mathematics is nothing but a pedantic grammar, suitable only for belaboring the obvious" [56, p. 142].

10 Proving and re-proving

Even if a rational way of knowing whether primitive premisses are true generally existed, the notion of axiomatic proof would have other basic defects.

For instance, in terms of that notion one cannot explain why, once a proof of a proposition has been found, mathematicians look for alternative proofs.

Several research papers in mathematics are concerned not with proving but with re-proving. For instance, well over four hundred distinct proofs of the Pythagorean Theorem have been given, a Fields Medal has been awarded to Selberg for producing a new proof of a theorem, the prime-number theorem, for which a proof was already known, and so on.

Now, if proofs were meant to provide a foundation and justification of a proposition, once a proof has been found and hence a foundation and justification has been given, what would be the point of looking for other proofs, even hundreds of them? No adequate answer to this question can be given in terms of the notion of axiomatic proof.

A suitable answer can be given only in terms of the notion of analytic proof, by which, to solve a problem, one may use several distinct hypotheses.

For a problem may have several sides, so one may look at it from several distinct perspectives, each of which may suggest a distinct hypothesis, thus a different proof and hence a different explanation. (On the notion of mathematical explanation involved here, see [10]). As we have already pointed out, this may have a great heuristic value, so it may be very fruitful for the development of mathematics. (For other approaches to the question of re-proving, see [56, Ch. XI], Avigad [1], Dawson [13]).

11 Mathematics and intuition

Since generally there is no rational way of knowing whether primitive premisses are true, supporters of axiomatic proof may only resort to assuming that there is an irrational faculty, intuition, by which one can grasp mathematical concepts and see that primitive premisses are true of them – an irrational faculty, because intuition is a faculty of which no account can be given.

This is the solution that, as we have seen, Aristotle suggested and most supporters of axiomatic proof have since adopted.

For instance, Gödel claims that ultimately for the "axioms there exists no other" foundation except that they "can directly be perceived to be true" by means of "an intuition of the objects falling under them" [24, pp. 346-347].

However, appealing to intuition not only bases mathematical knowledge on an irrational – and completely unreliable – faculty, but reduces proofs to rhetorical flourishes.

This is made quite clear by Hardy, who states that a mathematician is "in the first instance an observer, a man who gazes at a distant range of mountains and notes down his observations" [28, p. 18]. If "he sees a peak" and "wishes someone else to see it, he points to it, either directly or through the chain of summits which led him to recognize it himself". When "his pupil also sees it, the research, the argument, the proof is finished". Seeing a peak corresponds to having an intuition of certain mathematical objects. That mathematical activity consists in seeing peaks and pointing to them entails – Hardy argues – that "there is, strictly, no such thing as mathematical proof; that we can, in the last analysis, do nothing but point"; that proofs are merely "gas, rhetorical flourishes designed to affect psychology".

That appealing to intuition bases mathematical knowledge on an irrational and completely unreliable faculty and reduces proofs to rhetorical flourishes, conflicts with the intended aim of axiomatic proof to give a foundation and justification of a proposition.

Moreover, appealing to intuition is inconclusive. For suppose that you have an intuition of the concept of set S which tells you that your axioms of set theory T are true of S. By Gödel's first incompleteness theorem there is a sentence A of T which is true of S but is unprovable in T. Then the theory $T \cup \{\neg A\}$ is consistent, so it has a model, say S'. Thus $\neg A$ is true of S', and hence A is false of S'. Then S and S' are both models of T, so they are both concepts of set, but A is true of S and false of S'. Therefore S and S' cannot be isomorphic, so S and S' are essentially different.

Now suppose that, by reflecting on the way S' has been obtained, you get an intuition of the concept of set S' which tells you that the axioms of T are true of S'. Then you have two distinct intuitions, one ensuring that S is the genuine concept of set, the other one ensuring that S' is the genuine concept of set. Since S and S' are essentially different, this raises the question: Which of S and S' is the genuine concept of set? Intuition gives no answer.

This confirms that axiomatic proof is inadequate. As Hersh says, "the view that mathematics is in essence derivations from axioms is backward. In fact, it's wrong" [32, p. 6]. Only analytic proof is adequate, so axiomatic proof is not on a par with it.

12 Mathematics and evolution

Axiomatic proof is not on a par with analytic proof also in another respect. While axiomatic proof is simply a way of organizing and presenting results already obtained and so, as Aristotle says, is essentially aimed at teaching and learning, analytic proof goes deeply into the nature of organisms for it reflects the way in which they mainly solve their problems, starting from the most basic one: survival.

All organisms survive by making hypotheses on the environment by a process that is essentially an application of the analytic method. Thus analytic proof is based on the procedure by which organisms provide for their most basic needs. As our hunting ancestors solved their survival problem by making hypotheses about the location of preys on the basis of hints – crushed or bent grass and vegetation, bent or broken branches or twigs, mud displaced from streams, and so on – provided by them, mathematicians solve mathematical problems by making hypotheses for the solution of problems on the basis of hints provided by them.

Some of the hypotheses on the environment are chosen by natural selection and are embodied in the biological structure of organisms, and some of them concern mathematical properties of the environment. As a result, all organisms have at least some of the following innate capabilities: space sense, number sense, size sense, shape sense, order sense. Such capabilities are mathematical in kind. They have a biological function and are a result of biological evolution that has selected and embodied them in organisms.

Mathematical capabilities embodied in organisms, also non-human ones, can even be rather sophisticated.

For instance, if standing on a beach with a dog at the water's edge you throw a tennis ball into the waves diagonally, the dog will not plunge into the water immediately swimming all the way to the ball. It will run part of the way along the water's edge, and only then will plunge into the water and swim out to the ball. For, since the dog's running speed is greater than the dog's swimming speed, the dog will choose to plunge into the water at a point that will minimize the time of travel to the target. Such point can be determined by calculus, and the point actually chosen by the dog broadly agrees with the one given by calculus (see Pennings [51]). Does that mean that dogs know calculus? Of course not. They are capable of choosing an optimal point thanks to natural selection, which gives a definite survival advantage to organisms that exhibit better judgment. Thus the calculation required to determine an optimal point is not made by the dog but has been made by nature through natural selection. It is thanks to natural selection that dogs are able to solve this calculus problem. (For further examples of mathematical capabilities embodied in non-human organisms, see Devlin [15]).

Natural selection has hardwired organisms to perform certain mathematical operations building mathematics in several features of their biological structure, such as locomotion and vision, which require some sophisticated embodied mathematics. Such mathematical operations are essential to escape from danger, to search for food, to seek out a mate.

One may then distinguish a 'natural mathematics', that is, the mathematics embodied in organisms as a result of natural selection, from 'artificial mathematics', that is, mathematics as discipline. (Devlin calls artificial mathematics 'abstract mathematics', but 'artificial mathematics' seems more suitable here since it expresses that it is a mathematics that is not a natural product, being not a direct result of biological evolution but rather a human creation [15, p. 249]).

Natural mathematics, however, is necessarily limited since biological evolution is slow. On the contrary, artificial mathematics has developed relatively fast in the past five thousand years or so since it is a result of cultural evolution, which is relatively fast. This raises serious doubts about Cooper's claim that artificial "mathematics must itself be evolutionarily reducible" [12, p. 135]. It seems more reasonable to conclude that artificial mathematics cannot be reduced to natural mathematics.

The fact that natural mathematics is a result of biological evolution, whereas artificial mathematics is a result of cultural evolution, leads to a program of interpreting mathematics in terms of biological and cultural evolution. Although, of course, such program cannot be carried out here, some of its preconditions can be briefly discussed.

13 Mathematics and logic

Natural mathematics is based on natural logic, which is that natural capability to solve problems that all organisms have and is a result of biological evolution. On the other hand, artificial mathematics is based on artificial logic, which is a set of techniques invented by organisms to solve problems and is a result of cultural evolution.

Unlike the distinction between natural mathematics and artificial mathematics, the distinction between natural logic and artificial logic is not a new one. A similar distinction was made in the sixteenth century, for instance, by Ramus, and was still alive two centuries later when Kant used it in his logic lectures (see [39, pp. 252, 434, 532]). At that time, however, artificial logic was restricted to deductive inferences. But the notion of analytic proof requires that artificial logic include non-deductive inferences. Natural logic too requires non-deductive inferences, since the process by which all organisms provide for their most basic needs is essentially an application of the analytic method. However, natural logic requires not only non-deductive inferences but also non-propositional unconscious inferences, for the latter are essential, for instance, in vision. (On the role of non-propositional unconscious inference in vision, and generally on the characters of natural and artificial logic as intended here, see [11, Chs. 16-17]).

Since natural and artificial logic are based on two different forms of evolution, biological and cultural evolution, they are distinct. That, however, does not mean that they are opposed. For artificial logic ultimately depends on capabilities of organisms that are a result of biological evolution. Moreover, both natural and artificial logic depend on the very same basic procedure: the analytic method. The latter then provides a link between natural and artificial logic, and hence between natural mathematics and artificial mathematics.

14 Logic and reason

The main aim of natural logic is to find hypotheses on the environment to the end of survival. This implies that there is a strict connection between logic and the search of means for survival and that, since generally all organisms seek survival, natural logic does not belong to humans only but to all organisms.

On the contrary, logic has been traditionally viewed as the organ of reason meant as a higher faculty belonging to humans only, which allows them to overcome the limitations of their biological constitution, limitations within which animals and plants are instead constrained. In particular, such higher faculty has been supposed to be capable of intuitively and directly apprehending certain primitive truths, and specifically certain primitive premisses which are the necessary basis of any demonstrative reasoning.

But reason is not such a higher faculty, it is rather the capability of choosing means adequate to a given end. As Russell says, ‘reason’ “signifies the choice of the right means to an end that you wish to achieve” [57, p. 8]. Thus, in conformity with the original meaning of ‘ratio’, reason is a relation between means and ends. Then nothing is rational in itself but only relative to a given end. Now, since the primary end of all organisms is survival, the choice of means adequate to that end can be viewed as an expression of the faculty of reason, which then does not belong to humans only.

One might think that the concept of reason could be made less relative by stating that 'rational' – that is, 'compliant with reason' – is what is compliant with human nature. That, however, would not solve the problem of explaining what reason is but would simply refer it back to the problem of explaining what human nature is.

Now human nature is the result of two factors, biological evolution and cultural evolution. In explaining what human nature is biological evolution plays an important role, for our biological structure has a basic importance in determining what we are.

This view is fiercely opposed by those who, like Heidegger, deny that "the essence of man consists in being an animal organism", claiming that "the aberration of biologism" consists in considering the body of man as that of "an animal organism", and that the fact "that the physiology and biochemistry of man as an organism can be investigated in a natural scientific way is no proof that the essence of man lies in this organicity, that is, in the scientifically explained body" [26, p. 324].

But these claims are unjustified, because our biological structure really plays an essential role in determining what we are. For instance, monozygotic twins, when separated at birth and grown up in distinct environments with no possibility of mutual communication, have similar personalities, their behaviours resemble under several respects, they even take similar positions on the most disparate questions.

Those who deny that our biological structure has a basic importance in determining what we are, claim that the behaviour of humans is not largely governed by biological functions shared by all humans. There is no biological basis of our most important behaviours, the latter are a result of cultural evolution.

But the claim that our most important behaviours are a result of cultural evolution is not in conflict with the claim that our biological structure has a basic importance in determining what we are, for cultural evolution develops on the basis of biological evolution. Culture is not an ethereal substance independent of our biological structure. It depends on the neural networks with which biological evolution has provided us, for it is a product of our biological structure and so is bound to it. To separate cultural from biological evolution is to neglect what the subject of cultural evolution is: a biological organism which is an outcome of biological evolution.

Since biological and cultural evolution are what determines human nature, they are the relative terms with which we must commensurate rationality. Of course, only relative terms, for there is nothing necessary in biological evolution or in cultural evolution. In particular, biological evolution does not work by design: it has gone that way but could have gone otherwise. Thus, if 'rational' is what is compliant with human nature, there is nothing absolute in rationality. 'Rational' is a term relative to the contingent character of human nature, which is a contingent result of biological evolution and cultural evolution.

To view logic as the organ of reason, meant as a higher faculty belonging to humans only, is to misjudge the nature of reason. Logic can be said to be the organ of reason, though of a reason intended not as a higher faculty but as the capability of choosing means adequate to a given end, starting from survival, and hence as belonging to all organisms. Natural logic is the organ of reason for it provides all organisms with means adequate to their ends.

Here ‘organisms’ are supposed to include not only animals but also plants. Some of them, when attacked by herbivores, implement sophisticated defense strategies. They produce complex polymers that reduce plant digestibility, or toxins that repel or even kill the herbivores. They use other insects against the herbivores, emitting volatile organic compounds that attract other carnivorous insects which kill the attacking herbivores. These volatile organic compounds may be also perceived by neighboring yet-undamaged plants to adjust their defensive phenotype according to the present risk of attack, thus they function as external signal for within-plant communication (see Heil and Silva Bueno [31]).

15 Logic and evolution

That natural logic belongs to all organisms does not mean that non-human organisms choose means adequate to their ends on the basis of learned logical cognitions. But several humans too do not choose means adequate to their ends on the basis of learned logical cognitions. They use logical means such as induction, the cause-effect relation, the identity principle, and generally make inferences, without having attended to any logic course. They are capable of using logical means because biological evolution has designed them to do so.

Not only biological evolution has designed humans to use logical means, but natural logic, in addition to being a means for survival, is itself a result of natural selection. The natural logic system we have inherited is such that, on average, it increases the possibility of surviving and reproducing in the environment in which our most ancient ancestors evolved. Thus the first and deepest origin of reason and logic is natural selection, which has provided humans with those capabilities that have allowed them to survive.

The importance of reason and logic stems from the fact that the world changes continually and irregularly, so organisms are confronted all the time with the need to adapt to new situations. To deal with them they need logic, which helps them to cope with new situations, thus increasing their overall adaptive value.

The logic useful to this end is not only natural logic but also artificial logic, though an artificial logic including not only deductive propositional inferences, but also non-deductive and non-propositional inferences.

Biological evolution has embodied a series of informations in organisms concerning their evolutionary past, and also suitable kinds of behaviour by which they are able to cope with situations similar to those that already