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The Author

Prof. Ali Hasan Nayfeh
Virginia Polytechnic Institute and
State University
Department of Engineering
Science and Mechanics
Blacksburg, VA 24061
USA
anayfeh@vt.edu

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To my youngest son Nader

Contents

Preface *XI*

Introduction *1*

1 SDOF Autonomous Systems *7*
1.1 Introduction *7*
1.2 Duffing Equation *9*
1.3 Rayleigh Equation *13*
1.4 Duffing–Rayleigh–van der Pol Equation *15*
1.5 An Oscillator with Quadratic and Cubic Nonlinearities *17*
1.5.1 Successive Transformations *17*
1.5.2 The Method of Multiple Scales *19*
1.5.3 A Single Transformation *21*
1.6 A General System with Quadratic and Cubic Nonlinearities *22*
1.7 The van der Pol Oscillator *24*
1.7.1 The Method of Normal Forms *25*
1.7.2 The Method of Multiple Scales *26*
1.8 Exercises *27*

2 Systems of First-Order Equations *31*
2.1 Introduction *31*
2.2 A Two-Dimensional System with Diagonal Linear Part *34*
2.3 A Two-Dimensional System with a Nonsemisimple Linear Form *39*
2.4 An n-Dimensional System with Diagonal Linear Part *40*
2.5 A Two-Dimensional System with Purely Imaginary Eigenvalues *42*
2.5.1 The Method of Normal Forms *43*
2.5.2 The Method of Multiple Scales *47*
2.6 A Two-Dimensional System with Zero Eigenvalues *48*
2.7 A Three-Dimensional System with Zero and Two Purely Imaginary Eigenvalues *52*
2.8 The Mathieu Equation *54*
2.9 Exercises *57*

3 **Maps** *61*
3.1 Linear Maps *61*
3.1.1 Case of Distinct Eigenvalues *62*
3.1.2 Case of Repeated Eigenvalues *64*
3.2 Nonlinear Maps *66*
3.3 Center-Manifold Reduction *72*
3.4 Local Bifurcations *76*
3.4.1 Fold or Tangent or Saddle-Node Bifurcation *76*
3.4.2 Transcritical Bifurcation *79*
3.4.3 Pitchfork Bifurcation *80*
3.4.4 Flip or Period-Doubling Bifurcation *81*
3.4.5 Hopf or Neimark–Sacker Bifurcation *85*
3.5 Exercises *91*

4 **Bifurcations of Continuous Systems** *97*
4.1 Linear Systems *97*
4.1.1 Case of Distinct Eigenvalues *98*
4.1.2 Case of Repeated Eigenvalues *99*
4.2 Fixed Points of Nonlinear Systems *100*
4.2.1 Stability of Fixed Points *100*
4.2.2 Classification of Fixed Points *101*
4.2.3 Hartman–Grobman and Shoshitaishvili Theorems *102*
4.3 Center-Manifold Reduction *103*
4.4 Local Bifurcations of Fixed Points *107*
4.4.1 Saddle-Node Bifurcation *108*
4.4.2 Nonbifurcation Point *110*
4.4.3 Transcritical Bifurcation *111*
4.4.4 Pitchfork Bifurcation *113*
4.4.5 Hopf Bifurcations *114*
4.5 Normal Forms of Static Bifurcations *117*
4.5.1 The Method of Multiple Scales *117*
4.5.2 Center-Manifold Reduction *126*
4.5.3 A Projection Method *132*
4.6 Normal Form of Hopf Bifurcation *137*
4.6.1 The Method of Multiple Scales *138*
4.6.2 Center-Manifold Reduction *141*
4.6.3 Projection Method *144*
4.7 Exercises *146*

5 **Forced Oscillations of the Duffing Oscillator** *161*
5.1 Primary Resonance *161*
5.2 Subharmonic Resonance of Order One-Third *164*
5.3 Superharmonic Resonance of Order Three *167*
5.4 An Alternate Approach *169*
5.4.1 Subharmonic Case *171*
5.4.2 Superharmonic Case *172*

5.5 Exercises *172*

6 Forced Oscillations of SDOF Systems *175*
6.1 Introduction *175*
6.2 Primary Resonance *176*
6.3 Subharmonic Resonance of Order One-Half *178*
6.4 Superharmonic Resonance of Order Two *180*
6.5 Subharmonic Resonance of Order One-Third *182*

7 Parametrically Excited Systems *187*
7.1 The Mathieu Equation *187*
7.1.1 Fundamental Parametric Resonance *188*
7.1.2 Principal Parametric Resonance *190*
7.2 Multiple-Degree-of-Freedom Systems *191*
7.2.1 The Case of Ω Near $\omega_2 + \omega_1$ *194*
7.2.2 The Case of Ω Near $\omega_2 - \omega_1$ *194*
7.2.3 The Case of Ω Near $\omega_2 + \omega_1$ and $\omega_3 - \omega_2$ *194*
7.2.4 The Case of Ω Near $2\omega_3$ and $\omega_2 + \omega_1$ *195*
7.3 Linear Systems Having Repeated Frequencies *195*
7.3.1 The Case of Ω Near $2\omega_1$ *198*
7.3.2 The Case of Ω Near $\omega_3 + \omega_1$ *199*
7.3.3 The Case of Ω Near $\omega_3 - \omega_1$ *200*
7.3.4 The Case of Ω Near ω_1 *200*
7.4 Gyroscopic Systems *205*
7.4.1 The Case of Ω Near $2\omega_1$ *208*
7.4.2 The Case of Ω Near $\omega_2 - \omega_1$ *208*
7.5 A Nonlinear Single-Degree-of-Freedom System *208*
7.5.1 The Case of Ω Away from 2ω *209*
7.5.2 The Case of Ω Near 2ω *211*
7.6 Exercises *212*

8 MDOF Systems with Quadratic Nonlinearities *217*
8.1 Nongyroscopic Systems *217*
8.1.1 Two-to-One Autoparametric Resonance *220*
8.1.2 Combination Autoparametric Resonance *222*
8.1.3 Simultaneous Two-to-One Autoparametric Resonances *223*
8.1.4 Primary Resonances *223*
8.2 Gyroscopic Systems *225*
8.2.1 Primary Resonances *226*
8.2.2 Secondary Resonances *227*
8.3 Two Linearly Coupled Oscillators *229*
8.4 Exercises *232*

9 TDOF Systems with Cubic Nonlinearities *235*
9.1 Nongyroscopic Systems *235*
9.1.1 The Case of No Internal Resonances *236*
9.1.2 Three-to-One Autoparametric Resonance *238*

9.1.3 One-to-One Internal Resonance *239*
9.1.4 Primary Resonances *239*
9.1.5 A Nonsemisimple One-to-One Internal Resonance *240*
9.1.6 A Parametrically Excited System with a Nonsemisimple Linear Structure *244*
9.2 Gyroscopic Systems *249*
9.2.1 Primary Resonances *250*
9.2.2 Secondary Resonances in the Absence of Internal Resonances *251*
9.2.3 Three-to-One Internal Resonance *255*

10 Systems with Quadratic and Cubic Nonlinearities *257*
10.1 Introduction *257*
10.2 The Case of No Internal Resonance *262*
10.3 The Case of Three-to-One Internal Resonance *263*
10.4 The Case of One-to-One Internal Resonance *264*
10.5 The Case of Two-to-One Internal Resonance *266*
10.6 Method of Multiple Scales *267*
10.6.1 Second-Order Form *268*
10.6.2 State-Space Form *271*
10.6.3 Complex-Valued Form *274*
10.7 Generalized Method of Averaging *276*
10.8 A Nonsemisimple One-to-One Internal Resonance *279*
10.8.1 The Method of Normal Forms *279*
10.8.2 The Method of Multiple Scales *283*
10.9 Exercises *285*

11 Retarded Systems *287*
11.1 A Scalar Equation *287*
11.1.1 The Method of Multiple Scales *289*
11.1.2 Center-Manifold Reduction *291*
11.2 A Single-Degree-of-Freedom System *295*
11.2.1 The Method of Multiple Scales *296*
11.2.2 Center-Manifold Reduction *299*
11.3 A Three-Dimensional System *304*
11.3.1 The Method of Multiple Scales *306*
11.3.2 Center-Manifold Reduction *308*
11.4 Crane Control with Time-Delayed Feedback *311*
11.5 Exercises *313*

References *315*

Further Reading *319*

Index *325*

Preface

This book gives an introductory treatment of the method of normal forms. This technique has its application in many branches of engineering, physics, and applied mathematics. Approximation techniques such as these are important for people working with dynamical problems and are a valuable tool they should have in their tool box.

The exposition is largely by means of examples. The readers need not understand the physical bases of the examples used to describe the techniques. However, it is assumed that they have a knowledge of basic calculus as well as the elementary properties of ordinary differential equations. For most of the examples, the results obtained with the method of normal forms are shown to be equivalent to those obtained with other perturbation methods, such as the methods of multiple scales and averaging. As such, new sections are added treating some of the examples with these methods. Moreover, exercises are added to most chapters.

Because the normal forms of maps and differential equations are very useful in bifurcation analysis, I added in this edition three chapters dealing with the normal forms and bifurcations of maps, continuous systems, and retarded systems. The normal forms of continuous systems are constructed using the method of multiple scales, a combination of center-manifold reduction and the method of normal forms, and the new method of projection, which is developed first in this edition. Also, the normal forms of retarded systems are constructed using center-manifold reduction and the method of multiple scales. In the center-manifold reduction, we represent the retarded equations as operator differential equations, decompose the solution space of their linearized form into stable and center subspaces, define an inner product, determine the adjoint of the operator equations, calculate the center manifold, carry out details of the projection using the adjoint of the center subspace, and finally calculate the normal form on the center manifold.

I am very much indebted to my late parents, Hasan and Khadrah, who in spite of their lack of formal education insisted that all their sons obtain the highest degrees. If it were not for their incredible foresight on the value of an education even under the most severe conditions, I would not have finished secondary school. This book and its second edition would not have been written without the patience and continuous encouragement of my wife, Samirah.

Blacksburg, VA, December 2010 *Ali Hasan Nayfeh*

Introduction

The method of normal forms dates back to the days of Euler, Delaunay, Poincaré, Dulac, and Birkhoff. Moreover, the concept of using coordinate transformations to simplify mathematical problems involving algebraic, ordinary differential, partial differential, integral, and integro-differential equations has been used for a long time, as illustrated by the following examples.

As a first example, we consider Bessel's equation of order one-half; that is,

$$x^2 u'' + x u' + \left(x^2 - \tfrac{1}{4}\right) u = 0$$

Using the transformation $x^{-1/2} v(x)$, we transform this equation into the simple equation

$$v'' + v = 0$$

whose solution is

$$v = c_1 \cos x + c_2 \sin x$$

where c_1 and c_2 are arbitrary constants. Hence, Bessel's function of order one-half $J_{1/2}(x)$ is given by

$$J_{1/2}(x) = x^{-1/2} \left(c_1 \cos x + c_2 \sin x\right)$$

As a second example, we consider the vibrations of an n degree-of-freedom system governed by the following set of n coupled, linear equations of motion:

$$\ddot{\boldsymbol{x}} + K\boldsymbol{x} = 0$$

where $\boldsymbol{x}$ is a column vector of length n and K is an $n \times n$ constant symmetric matrix. Using the transformation $\boldsymbol{x} = P\boldsymbol{v}$, we obtain

$$\ddot{\boldsymbol{v}} + P^{-1} K P \boldsymbol{v} = 0$$

Assuming the eigenvalues $\lambda_1, \lambda_2, \ldots,$ and λ_n of K to be distinct and choosing the columns of P to be the orthonormal eigenvectors of K, we find that $P^{-1}KP$ is a

The Method of Normal Forms, Second Edition. Ali Hasan Nayfeh

diagonal matrix Λ with entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Hence, the system of equations can be written as

$$\ddot{\boldsymbol{v}} + \Lambda \boldsymbol{v} = 0$$

or in the decoupled form

$$\ddot{v}_i + \lambda_i v_i = 0 \quad \text{and} \quad i = 1, 2, \ldots, n$$

which is called the normal-modal form of

$$\ddot{\boldsymbol{x}} + K\boldsymbol{x} = 0$$

As a third example, we consider the system

$$\begin{aligned} \dot{a} &= -\mu a - \tfrac{1}{2} a \sin 2\beta \\ a\dot{\beta} &= -\tfrac{1}{2}\sigma a - \tfrac{1}{2} a \cos 2\beta \end{aligned}$$

where μ and σ are constants, which describes the time variation of the amplitude and phase of a parametrically excited linear oscillator in the case of a principal parametric resonance (Nayfeh and Mook, 1979). This system is nonlinear and its solution is not apparent. However, using the nonlinear transformation $x = a\cos\beta$ and $y = a\sin\beta$, we transform the nonlinear system into the following linear system:

$$\begin{aligned} \dot{x} &= -\mu x + \tfrac{1}{2}(\sigma - 1)\, y \\ \dot{y} &= -\mu y - \tfrac{1}{2}(\sigma + 1)\, x \end{aligned}$$

whose closed-form solution is readily obtainable.

As a fourth example, we consider the nonlinear system

$$\begin{aligned} \dot{x} &= y + (ax + by)\left(x^2 + y^2\right) \\ \dot{y} &= -x + (ay - bx)\left(x^2 + y^2\right) \end{aligned}$$

where a and b are constants, which describes the motion near a Hopf bifurcation point (Marsden and McCracken, 1976; Wiggins, 1990), as described in Section 4.4.5. Again the solution of this system is not apparent. However, using the nonlinear transformation $x = r\cos\beta$ and $y = -r\sin\beta$, we transform the system into

$$\begin{aligned} \dot{r} &= ar^3 \\ \dot{\beta} &= 1 + br^2 \end{aligned}$$

whose closed-form solution is readily obtainable.

As a fifth example, we consider the linear partial differential equation

$$u_{tt} - c^2 u_{xx} = f'(x - ct)\, f''(x - ct)$$

where f is a known twice differential function, the prime denotes the derivative with respect to the argument $(x - ct)$, and subscripts denote partial derivatives. The general solution of this equation can be readily obtained if we express the independent variables x and t in terms of the characteristics

$$\xi = x - ct \quad \text{and} \quad \eta = x + ct$$

Thus, this partial differential equation is transformed into

$$-4c^2 u_{\xi\eta} = f'(\xi) f''(\xi)$$

whose general solution is

$$u = -\frac{1}{8c^2} f'^2(\xi)\eta + g(\xi) + h(\eta)$$

where $g(\xi)$ and $h(\eta)$ are general functions of ξ and η.

As a sixth example, we consider the nonlinear partial differential equation

$$u_t + u u_x = \nu u_{xx}$$

where ν is a constant, which is known as Burger's equation (Whitham, 1974). Replacing u with ψ_x and integrating the result once yields

$$\psi_t + \tfrac{1}{2}\psi_x^2 = \nu \psi_{xx}$$

Then, using the nonlinear transformation $\psi = -2\nu \ln(\phi)$, Hopf (1950) and Cole (1951) transformed the nonlinear equation into the linear heat transfer equation

$$\phi_t = \nu \phi_{xx}$$

which can be solved much more easily than the original nonlinear equation.

As a seventh example, we consider the steady, incompressible, high-Reynolds number flow over a flat plate aligned with the oncoming uniform stream. The boundary layer approximation to the stream function $\psi(x, y)$ is governed by Van Dyke (1964)

$$\begin{aligned}
&\psi_{yyy} + \psi_x \psi_{yy} - \psi_y \psi_{xy} = 0 \\
&\psi(x, 0) = 0 \\
&\psi_y(x, 0) = 0 \quad \text{and} \quad 0 < x < \infty \\
&\psi_y(x, \infty) = 1
\end{aligned}$$

This nonlinear partial differential equation can be reduced to an ordinary differential equation by using the similarity transformation

$$\psi(x, y) = \sqrt{2x}\, f(\eta)\,, \quad \eta = y/\sqrt{2x}$$

With this transformation, the boundary layer problem becomes

$$f''' + f f'' = 0\,, \quad f(0) = 0\,, \quad f'(0) = 0\,, \quad f'(\infty) = 1$$

which is the Blasius problem.

In the preceding examples, transformations were introduced to transform a difficult problem into a more readily solvable problem. Next, we consider cases in which a transformation is used to transform the problem into a new "approximate" problem for which the exact solution can be readily obtained. Specifically, we consider the Liouville equation

$$y'' + \lambda^2 q(x) y = 0 \quad \text{when} \quad \lambda \gg 1$$

where λ is a constant, $q(x)$ is a known function, and the prime denotes the derivative with respect to x. To determine an approximate solution of this equation when $\lambda \gg 1$, we transform both of the dependent and independent variables as

$$z = \phi(x) \quad \text{and} \quad v(z) = \psi(x) y(x)$$

With this transformation, the Liouville equation becomes

$$\frac{d^2 v}{dz^2} + \frac{1}{\phi'^2}\left(\phi'' - \frac{2\phi'\psi'}{\psi}\right)\frac{dv}{dz} + \left(\frac{\lambda^2 q}{\phi'^2} - \frac{\psi''}{\psi\phi'^2} + \frac{2\psi'^2}{\psi^2\phi'^2}\right) v = 0$$

We choose ϕ and ψ so that the dominant part of the transformed equation has the simplest possible form and, at the same time, has solutions that have qualitatively the same behavior as the solutions of the original equations. In other words, we have to insist on the transformation being regular everywhere in the interval of interest. To this end, we force the coefficient of dv/dz to be zero; that is,

$$\phi'' - \frac{2\phi'\psi'}{\psi} = 0$$

Hence, $\psi = \sqrt{\phi'}$. In order that the transformation be regular, ψ must be regular and have no zeros in the interval of interest. Then, because $\psi = \sqrt{\phi'}$, ϕ' must be regular and have no zeros in the interval of interest. Consequently, we set

$$\lambda^2 q = \phi'^2 \xi(z)$$

so that the transformed equation becomes

$$\frac{d^2 v}{dz^2} + \xi(z) v = -\delta v$$

and choose the simplest possible function $\xi(z)$ that yields a nonsingular transformation. In order that ϕ' be regular and have no zeros in the interval of interest, $\xi(z)$ must have the same number, type, and order of singularities and zeros as q.

For example, when $q > 0$ everywhere in the interval of interest, the solutions of the original equation are oscillatory, and hence ϕ and ψ must be chosen so that the dominant part of the transformed equation is

$$\frac{d^2 v}{dz^2} + v = 0$$

which is the simplest possible equation with oscillatory solutions. When $q < 0$ everywhere in the interval of interest, one of the solutions of the original equation grows exponentially with x and the other decays exponentially with x. Hence, ϕ and ψ must be chosen so that the dominant part of the transformed equation is

$$\frac{d^2 v}{dz^2} - v = 0$$

which is the simplest possible equation with exponentially growing and decaying solutions.

When q changes sign once in the interval of interest, the solutions of the original equation are oscillatory on one side of the sign change and exponentially growing and decaying on the other side. For example, if $q = 1 - x^3$, the solutions of the original equation are oscillatory for $x < 1$ and exponential for $x > 1$. Hence, ϕ and ψ must be chosen so that the dominant part of the transformed equation has solutions whose behavior changes from oscillatory to exponentially growing and decaying at a given point. The simplest possible equation with these properties is the Airy equation

$$\frac{d^2 v}{dz^2} - zv = 0$$

When $z > 0$ the solutions of this equation are growing and decaying with z, and when $z < 0$ they are oscillatory. In other words, if $q(x)$ is regular and has only a simple zero (simple turning point) such as $1 - x^3$, then $\xi(z)$ must be chosen to be regular and have only a simple zero. The simplest possible function that satisfies these requirements is $\xi(z) = z$. If $q(x)$ is regular and has only a double zero at a point in the interval of interest (i.e., turning point of order 2), $\xi(z)$ must be chosen to be regular and have only a double zero. The simplest possible function satisfying these requirements is $\xi(z) = z^2$. If $q(x)$ is regular and has only a zero of order n (i.e., turning point of order n), $\xi(z)$ must be chosen to be z^n. If $q(x)$ has two zeros at $x = a$ and b, where $b > 1$, of order m and n, then one uses

$$\xi(z) = z^m (1 - z)^n$$

In analyzing oscillations of a weakly nonlinear system, the method of variation of parameters is usually used to transform the equations governing these oscillations into the standard form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; \epsilon) = \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} \boldsymbol{f}_m(\boldsymbol{x})$$

where

$$\boldsymbol{f}_m(\boldsymbol{x}) = \left. \frac{\partial^m \boldsymbol{f}}{\partial \epsilon^m} \right|_{\epsilon=0}$$

Here $\boldsymbol{x}$ and $\boldsymbol{f}$ are vectors with N components. The vector $\boldsymbol{x}$ may represent, for example, the amplitudes and phases of the system. If we denote the components of the vector $\boldsymbol{f}_m$ by f_{mn}, then a component x_k of the vector $\boldsymbol{x}$ is said to be a rapidly rotating phase if $\boldsymbol{f}_{0k} \neq 0$.

To analyze this standard system, we introduce a near-identity transformation

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y};\epsilon) = \boldsymbol{y} + \epsilon \boldsymbol{X}_1(\boldsymbol{y}) + \epsilon^2 \boldsymbol{X}_2(\boldsymbol{y}) + \cdots$$

from $\boldsymbol{x}$ to $\boldsymbol{y}$ such that the system is transformed into

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y};\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \boldsymbol{g}_n(\boldsymbol{y})$$

where the $\boldsymbol{g}_n$ contain long-period terms only. Using the generalized method of averaging (Nayfeh, 1973), one determines the $\boldsymbol{X}_n$ and $\boldsymbol{g}_n$ by substituting the transformation into the standard system and separating the short- and long-period terms assuming that the $\boldsymbol{X}_n$ contain short-period terms only.

Alternatively, we can define the transformation $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y};\epsilon)$ as the solution of the N differential equations

$$\frac{d\boldsymbol{x}}{d\epsilon} = \boldsymbol{W}(\boldsymbol{x};\epsilon)\,, \quad \boldsymbol{x}(\epsilon = 0) = \boldsymbol{y}$$

The vector $\boldsymbol{W}$ is called the generating vector. This equation generates the so-called Lie transforms (Kamel, 1970), which are invertible because they are close to the identity. It seems at first that we are going in circles because we are proposing to simplify the original system of differential equations by solving a system of N differential equations. This is not the case, because we are interested in the solution of the original system for large t, whereas we need the solution of the transformation for small ϵ, which is a significant simplification.

These examples clearly show that linear and nonlinear coordinate transformations can be used to simplify linear and nonlinear problems. A powerful method for systematically constructing these transformations is the method of normal forms. The basic idea underlying the method of normal forms is the use of “local” coordinate transformations to “simplify” the equations describing the dynamics of the system under consideration. In other words, with the method of normal forms, one seeks a near-identity coordinate transformation in which the dynamical system takes the “simplest” or so-called normal form. The transformations are generated in a neighborhood of a known solution, such as a fixed point (constant, stationary, or equilibrium solution) or a periodic orbit (limit cycle) of a system. In this text, the normalization is usually carried out with respect to a perturbation parameter.

1
SDOF Autonomous Systems

1.1
Introduction

In this chapter, we describe the method of normal forms using single-degree-of-freedom (SDOF) autonomous systems that can be modeled by the following second-order nonlinear ordinary differential equation:

$$\ddot{u} + \omega^2 u = f\,(u, \dot{u}) \tag{1.1}$$

where $f(u, \dot{u})$ can be developed in a power series in terms of u and $\dot{u}$. In what follows, we will refer to $\dot{u} + \omega^2 u = 0$ as the *unperturbed system* and (1.1) as the *perturbed system*. We assume that (1.1) has an equilibrium at $u = 0$ and $\dot{u} = 0$. Equation 1.1 can be cast as a system of two first-order equations by letting

$$x_1 = u \quad \text{and} \quad x_2 = \dot{u} \tag{1.2}$$

The result is

$$\dot{x}_1 = x_2 \tag{1.3}$$

$$\dot{x}_2 = -\omega^2 x_1 + f(x_1, x_2) \tag{1.4}$$

It is clear that the unperturbed system

$$\dot{x}_1 = x_2 \quad \text{and} \quad \dot{x}_2 = -\omega^2 x_1$$

has a simple pair of purely imaginary eigenvalues $\pm i\omega$.

The main idea underlying the method of normal forms is to introduce a near-identify transformation

$$x_1 = y_1 + h_1(y_1, y_2) \tag{1.5a}$$

$$x_2 = y_2 + h_2(y_1, y_2) \tag{1.5b}$$

from (x_1, x_2) to (y_1, y_2) into (1.3) and (1.4) to produce the simplest possible equations (the so-called normal form). We call the transformation (1.5) near-identity

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because $x_1(t) - y_1(t)$ and $x_2(t) - y_2(t)$ are small; that is, $o(x_1(t), x_2(t))$. This procedure is also called *normalization*. To this end, we substitute (1.5) into (1.3) and (1.4) and obtain

$$\dot{y}_1 = y_2 + h_2 - \frac{\partial h_1}{\partial y_1}\dot{y}_1 - \frac{\partial h_1}{\partial y_2}\dot{y}_2 \tag{1.6a}$$

$$\dot{y}_2 = -\omega^2 y_1 - \omega^2 h_1 + f\,(y_1 + h_1, y_2 + h_2) - \frac{\partial h_2}{\partial y_1}\dot{y}_1 - \frac{\partial h_2}{\partial y_2}\dot{y}_2 \tag{1.6b}$$

Then, we choose h_1 and h_2 such that (1.6) assume their simplest form. This task is accomplished in steps. If one decomposes $f(x_1, x_2)$ as

$$f(x_1, x_2) = \sum_{n=1}^{N} f_n(x_1, x_2) \tag{1.7}$$

where f_n is a polynomial of degree n in x_1 and x_2, then one chooses h_1 and h_2 to simplify the terms resulting from the lowest-order polynomial $f_m(x_1, x_2)$, where $m \geq 2$, in $f(x_1, x_2)$. In the next step, one chooses a second near-identity transformation to simplify the polynomial terms of degree $m + 1$, and so on.

It turns out that, because the unperturbed system (1.3) and (1.4) represents an oscillator, the governing equations can conveniently be expressed as a single complex-valued equation. To this end, we follow steps similar to those used in the method of variation of parameters (Nayfeh, 1981). When $f \equiv 0$, the solution of (1.1) can be expressed as

$$u = Be^{i\omega t} + \bar{B}e^{-i\omega t} \tag{1.8}$$

where B is a constant and $\bar{B}$ is the complex conjugate of B. Hence,

$$\dot{u} = i\omega\left(Be^{i\omega t} - \bar{B}e^{-i\omega t}\right) \tag{1.9}$$

When $f \neq 0$, we continue to represent the solution of (1.1) as in (1.8) subject to the constraint (1.9) but with time-varying rather than constant B. Next, we replace $Be^{i\omega t}$ with $\zeta(t)$ and rewrite (1.8) and (1.9) as

$$u = \zeta(t) + \bar{\zeta}(t) \quad \text{and} \quad \dot{u} = i\omega\left[\zeta(t) - \bar{\zeta}(t)\right] \tag{1.10}$$

Hence, solving for ζ and $\bar{\zeta}$, we obtain

$$\zeta = \frac{1}{2}\left(u - \frac{i}{\omega}\dot{u}\right) \quad \text{and} \quad \bar{\zeta} = \frac{1}{2}\left(u + \frac{i}{\omega}\dot{u}\right) \tag{1.11}$$

Differentiating (1.11) with respect to t yields

$$\dot{\zeta} = \frac{1}{2}\left(\dot{u} - \frac{i}{\omega}\ddot{u}\right) = \frac{1}{2}\left(\dot{u} + i\omega u - \frac{i}{\omega}f\right) \tag{1.12}$$

on account of (1.1). Hence,

$$\dot{\zeta} = \frac{1}{2}i\omega\left(u - \frac{i}{\omega}\dot{u}\right) - \frac{i}{2\omega}f(u, \dot{u}) \tag{1.13}$$

which, upon using (1.10), becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i}{2\omega} f\left[\zeta + \bar{\zeta}, i\omega\left(\zeta - \bar{\zeta}\right)\right] \tag{1.14}$$

Next, we consider different polynomial forms for *f*.

1.2
Duffing Equation

The Duffing equation is

$$\ddot{u} + \omega^2 u = \alpha u^3$$

so that, in this case, $f = \alpha u^3$ and (1.14) becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{1.15}$$

We introduce a near-identity transformation from ζ to η in the form

$$\zeta = \eta + h\left(\eta, \bar{\eta}\right) \tag{1.16}$$

and obtain

$$\dot{\eta} = i\omega\eta + i\omega h - \frac{\partial h}{\partial \eta}\dot{\eta} - \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} - \frac{i\alpha}{2\omega}\left(\eta + h + \bar{\eta} + \bar{h}\right)^3 \tag{1.17}$$

Because the nonlinearity is cubic, we assume that h is third order in η and $\bar{\eta}$; that is,

$$h = \Lambda_1\eta^3 + \Lambda_2\eta^2\bar{\eta} + \Lambda_3\eta\bar{\eta}^2 + \Lambda_4\bar{\eta}^3 \tag{1.18}$$

and choose the Λ_i so that (1.17) takes the simplest possible (normal) form.

In the first step, we eliminate $\dot{\eta}$ and $\dot{\bar{\eta}}$ from the right-hand side of (1.17). This task is accomplished by iteration. To the first approximation, it follows from (1.17) that

$$\dot{\eta} = i\omega\eta \quad \text{and} \quad \dot{\bar{\eta}} = -i\omega\bar{\eta} \tag{1.19}$$

Next, we replace $\dot{\eta}$ and $\dot{\bar{\eta}}$ on the right-hand side of (1.17) using (1.19), use (1.18), keep up to third-order terms, and obtain

$$\begin{aligned}\dot{\eta} = {} & i\omega\eta - i\omega\left(2\Lambda_1 + \frac{\alpha}{2\omega^2}\right)\eta^3 - \frac{3i\alpha}{2\omega}\eta^2\bar{\eta} + i\omega\left(2\Lambda_3 - \frac{3\alpha}{2\omega^2}\right)\eta\bar{\eta}^2 \\ & + i\omega\left(4\Lambda_4 - \frac{\alpha}{2\omega^2}\right)\bar{\eta}^3\end{aligned} \tag{1.20}$$

Next, we choose Λ_1, Λ_3, and Λ_4 to eliminate the terms involving η^3, $\eta\bar{\eta}^2$, and $\bar{\eta}^3$; that is,

$$\Lambda_1 = -\frac{\alpha}{4\omega^2}, \quad \Lambda_3 = \frac{3\alpha}{4\omega^2}, \quad \Lambda_4 = \frac{\alpha}{8\omega^2} \tag{1.21}$$

However, because Λ_2 does not appear in (1.20), the term involving $\eta^2\bar{\eta}$ cannot be eliminated; it is called a *resonance term*. Consequently, to the second approximation, the simplest possible form for $\dot{\eta}$ is

$$\dot{\eta} = i\omega\eta - \frac{3i\alpha}{2\omega}\eta^2\bar{\eta} \tag{1.22}$$

To show that $\eta^2\bar{\eta}$ is a resonance term, we find a solution for (1.22) by iteration. To the first approximation, $\eta = Ae^{i\omega t}$, where A is a constant. Then, (1.22) becomes

$$\dot{\eta} = i\omega\eta - \frac{3i\alpha}{2\omega}A^2\bar{A}e^{i\omega t}$$

whose solution can be written as

$$\eta = Ae^{i\omega t} - \frac{3\alpha}{2\omega}A^2\bar{A}te^{i\omega t} \tag{1.23a}$$

It is clear that this expansion, which is also a straightforward expansion, is nonuniform for large t because of the presence of a secular term created by $\eta^2\bar{\eta}$. Alternatively, we can demonstrate that the term $\zeta^2\bar{\zeta}$ is a *resonance term* in the original equation (1.15). To the first approximation, we neglect the nonlinear term in (1.15) and find that $\zeta = Ae^{i\omega t}$. Then, to the second approximation, (1.15) becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i\alpha}{2\omega}\left(A^3e^{3i\omega t} + 3A^2\bar{A}e^{i\omega t} + 3A\bar{A}^2e^{-i\omega t} + \bar{A}^3e^{-3i\omega t}\right)$$

whose solution can be written as

$$\zeta = Ae^{i\omega t} - \frac{\alpha}{4\omega^2}A^3e^{3i\omega t} - \frac{3i\alpha}{2\omega}A^2\bar{A}te^{i\omega t} + \frac{3\alpha}{4\omega^2}A\bar{A}^2e^{-i\omega t} + \frac{\alpha}{8\omega^2}\bar{A}^3e^{-3i\omega t} \tag{1.23b}$$

It is clear that this expansion is nonuniform because of the presence of a secular term created by $\zeta^2\bar{\zeta}$. The other three terms proportional to $A^3e^{3i\omega t}$, $A\bar{A}^2e^{-i\omega t}$, and $\bar{A}^3e^{-3i\omega t}$ created by ζ^3, $\zeta\bar{\zeta}^2$, and $\bar{\zeta}^3$ do not produce secular terms and hence they are *nonresonance*. Consequently, one can choose a near-identity transformation to eliminate them.

As a second alternative, starting with the original equation (1.15), we break the nonlinear part $f(\zeta, \bar{\zeta})$ into two parts as

$$f(\zeta, \bar{\zeta}) = f_1(\zeta, \bar{\zeta}) + f_2(\zeta, \bar{\zeta})$$

where

$$e^{-i\omega t}f_1\left(e^{i\omega t}, e^{-i\omega t}\right)$$

is *time invariant*, whereas

$$e^{-i\omega t} f_2\left(e^{i\omega t}, e^{-i\omega t}\right)$$

is not time invariant. In the present case,

$$f = \left(\zeta + \bar{\zeta}\right)^3, \quad f_1 = 3\zeta^2\bar{\zeta}, \quad f_2 = \zeta^3 + 3\zeta\bar{\zeta}^2 + \bar{\zeta}^3$$

Thus,

$$e^{-i\omega t} f_1\left(e^{i\omega t}, e^{-i\omega t}\right) = e^{-i\omega t}\left(3e^{2i\omega t}e^{-i\omega t}\right) = 3$$

which is time invariant, whereas

$$e^{-i\omega t} f_2\left(e^{i\omega t}, e^{-i\omega t}\right) = e^{2i\omega t} + 3e^{-2i\omega t} + e^{-4i\omega t}$$

which does not contain any time-invariant terms.

Substituting (1.16) and (1.18) into (1.10), using (1.21), and setting $\Lambda_2 = 0$ because it is arbitrary yields

$$u = \eta + \bar{\eta} - \frac{\alpha}{8\omega^2}\left(\eta^3 + \bar{\eta}^3\right) + \frac{3\alpha}{4\omega^2}\left(\eta\bar{\eta}^2 + \eta^2\bar{\eta}\right) \tag{1.24}$$

where η is given by (1.22). Next, we separate the fast from the slow variations in η by introducing the transformation

$$\eta = A(t)e^{i\omega t}$$

where ω is the natural frequency of the system and A is a function of time, into (1.22) and (1.24) and obtain

$$\dot{A} = -\frac{3i\alpha}{2\omega}A^2\bar{A} \tag{1.25}$$

$$u = Ae^{i\omega t} + \bar{A}e^{-i\omega t} - \frac{\alpha}{8\omega^2}\left(A^3e^{3i\omega t} + \bar{A}^3e^{-3i\omega t}\right) + \frac{3\alpha}{4\omega^2}\left(A^2\bar{A}e^{i\omega t} + \bar{A}^2Ae^{-i\omega t}\right) + \cdots \tag{1.26}$$

Expressing A in the polar form

$$A = \tfrac{1}{2}ae^{i\beta} \tag{1.27}$$

where a and β are functions of t, we rewrite (1.26) as

$$u = \left(a + \frac{3\alpha}{16\omega^2}a^3\right)\cos(\omega t + \beta) - \frac{\alpha a^3}{32\omega^2}\cos(3\omega t + 3\beta) + \cdots \tag{1.28}$$

Substituting (1.27) into (1.25) and separating real and imaginary parts, we have

$$\dot{a} = 0 \tag{1.29}$$

$$a\dot{\beta} = -\frac{3\alpha}{8\omega}a^3 \tag{1.30}$$

In determining the normal form (1.22), we had to use an ordering scheme to indicate the relative magnitudes of the different terms in (1.15). We based the ordering scheme on the fact that ζ and $\bar{\zeta}$ are small and hence ζ^3, $\zeta^2\bar{\zeta}$, $\zeta\bar{\zeta}^2$, and $\bar{\zeta}^3$ are much smaller than ζ and $\bar{\zeta}$. In other words, we based the ordering scheme on the degree of the terms. This worked well in this example, but there are many physical systems where the ordering does not follow from the degree of the polynomial but from the presence of certain parameters in their models. We consider such an example in the next section.

Next, we treat (1.15) by using the method of multiple scales. To this end, we introduce a small nondimensional parameter ϵ as a bookkeeping device and rewrite (1.15) as

$$\dot{\zeta} = i\omega\zeta - \frac{i\epsilon\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{1.31}$$

Then, we seek an approximate solution of (1.31) in the form

$$\zeta(t;\epsilon) = \zeta_0(T_0, T_1) + \epsilon\zeta_1(T_0, T_1) + \cdots \tag{1.32}$$

where $T_n = \epsilon^n t$ and

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \cdots = D_0 + \epsilon D_1 + \cdots \tag{1.33}$$

Substituting (1.32) and (1.33) into (1.31) and equating coefficients of like powers of ϵ yields

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.34}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 - \frac{i\alpha}{2\omega}\left(\zeta_0 + \bar{\zeta}_0\right)^3 \tag{1.35}$$

The solution of (1.34) can be expressed as

$$\zeta_0 = A(T_1)e^{i\omega T_0} \tag{1.36}$$

Then, (1.35) becomes

$$\begin{aligned} D_0\zeta_1 - i\omega\zeta_1 = -A'e^{i\omega T_0} - \frac{i\alpha}{2\omega}&\left(A^3e^{3i\omega T_0} + 3A^2\bar{A}e^{i\omega T_0}\right. \\ &\left. + 3A\bar{A}^2e^{-i\omega T_0} + \bar{A}^3e^{-3i\omega T_0}\right) \end{aligned} \tag{1.37}$$

Eliminating the terms that lead to secular terms from (1.37), we have

$$A' = -\frac{3i\alpha}{2\omega}A^2\bar{A} \tag{1.38}$$

Then, a particular solution of (1.37) can be expressed as

$$\zeta_1 = -\frac{\alpha}{4\omega^2} A^3 e^{3i\omega T_0} + \frac{3\alpha}{4\omega^2} A\bar{A}^2 e^{-i\omega T_0} + \frac{\alpha}{8\omega^2} \bar{A}^3 e^{-3i\omega T_0} \tag{1.39}$$

Substituting (1.36) and (1.39) into (1.10), we obtain

$$\begin{aligned} u &= Ae^{i\omega t} + \bar{A}e^{-i\omega t} - \frac{\epsilon\alpha}{8\omega^2}\left(A^3 e^{3i\omega t} + \bar{A}^3 e^{-3i\omega t}\right) \\ &+ \frac{3\epsilon\alpha}{4\omega^2}\left(A^2\bar{A}e^{i\omega t} + A\bar{A}^2 e^{-i\omega t}\right) + \cdots \end{aligned} \tag{1.40}$$

Equations 1.38–1.40 are in full agreement with (1.25) and (1.26) obtained with the method of normal forms because $T_1 = \epsilon t$ and ϵ can be set equal to unity.

1.3 Rayleigh Equation

The Rayleigh equation is

$$\ddot{u} + \omega^2 u = \epsilon\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right) \tag{1.41}$$

where ϵ is a small, positive nondimensional parameter. Here

$$f = \epsilon\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right)$$

and (1.14) becomes

$$\dot{\zeta} = i\omega\zeta + \tfrac{1}{2}\epsilon\left[\zeta - \bar{\zeta} + \tfrac{1}{3}\omega^2\left(\zeta - \bar{\zeta}\right)^3\right] \tag{1.42}$$

In this example, the ordering is not based on the degree of the polynomial, but on the small nondimensional parameter ϵ. Normalization is carried out in terms of the small parameter ϵ. In fact, the perturbation contains linear as well as cubic terms.

Using the transformation (1.16), we rewrite (1.42) as

$$\begin{aligned} \dot{\eta} &= i\omega\eta + i\omega h - \frac{\partial h}{\partial \eta}\dot{\eta} - \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} + \frac{1}{2}\epsilon\left[\eta - \bar{\eta} + h - \bar{h}\right. \\ &\left. + \frac{1}{3}\omega^2\left(\eta - \bar{\eta} + h - \bar{h}\right)^3\right] \end{aligned} \tag{1.43}$$

Because the perturbation in (1.43) involves linear and cubic terms, we express h in the form

$$h = \epsilon\left(\Delta_1\eta + \Delta_2\bar{\eta} + \Lambda_1\eta^3 + \Lambda_2\eta^2\bar{\eta} + \Lambda_3\eta\bar{\eta}^2 + \Lambda_4\bar{\eta}^3\right) \tag{1.44}$$

Moreover, to the first approximation, $\dot{\eta}$ and $\dot{\bar{\eta}}$ are given by (1.19). Then, substituting (1.19) and (1.44) into the right-hand side of (1.43) and keeping terms up to $O(\epsilon)$, we obtain

$$\dot{\eta} = i\omega\eta + 2i\epsilon\omega\left(\Delta_2 + \frac{i}{4\omega}\right)\bar{\eta} + \frac{1}{2}\epsilon\eta - i\epsilon\omega\left(2\Lambda_1 + \frac{1}{6}i\omega\right)\eta^3$$
$$- \frac{1}{2}\epsilon\omega^2\eta^2\bar{\eta} + i\epsilon\omega\left(2\Lambda_3 - \frac{1}{2}i\omega\right)\eta\bar{\eta}^2 + i\epsilon\omega\left(4\Lambda_4 + \frac{1}{6}i\omega\right)\bar{\eta}^3 \tag{1.45}$$

We note that (1.45) is independent of Δ_1 and Λ_2 and hence they are arbitrary. Moreover, the terms proportional to $\epsilon\eta$ and $\epsilon\eta^2\bar{\eta}$ are resonance terms and hence cannot be eliminated from (1.45). Next, we choose $\Delta_2, \Lambda_1, \Lambda_3$, and Λ_4 to eliminate the terms involving $\bar{\eta}, \eta^3, \eta\bar{\eta}^2$, and $\bar{\eta}^3$, thereby producing the simplest possible equation for η. Thus, we have

$$\Delta_2 = -\frac{i}{4\omega}\,, \quad \Lambda_1 = -\frac{1}{12}i\omega\,, \quad \Lambda_3 = \frac{1}{4}i\omega\,, \quad \Lambda_4 = -\frac{1}{24}i\omega \tag{1.46}$$

With this choice, (1.45) takes the normal form

$$\dot{\eta} = i\omega\eta + \tfrac{1}{2}\epsilon\eta - \tfrac{1}{2}\epsilon\omega^2\eta^2\bar{\eta} \tag{1.47}$$

Again, in this case, we could have identified the resonance terms in (1.42) by one of the procedures described in Section 1.2. Because the solution of the unperturbed problem is proportional to $e^{i\omega t}$, the resonance terms in

$$f(\zeta, \bar{\zeta}) = \zeta - \bar{\zeta} + \tfrac{1}{3}\omega^2(\zeta - \bar{\zeta})^3$$

are the terms proportional to $e^{i\omega t}$ or the time-invariant terms in

$$e^{-i\omega t} f\left[e^{i\omega t} - e^{-i\omega t}, i\omega\left(e^{i\omega t} - e^{-i\omega t}\right)\right]$$

A simple calculation shows that the term $1/2\epsilon(\zeta - \omega^2\zeta^2\bar{\zeta})$ is the only resonance term. Hence, keeping only the resonance terms in (1.42), we have

$$\dot{\zeta} = i\omega\zeta + \tfrac{1}{2}\epsilon\left(\zeta - \omega^2\zeta^2\bar{\zeta}\right) + \cdots$$

which is formally equivalent to (1.47).

Next, we treat (1.42) with the method of multiple scales. To this end, we substitute (1.32) and (1.33) into (1.42), equate coefficients of equal powers of ϵ, and obtain

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.48}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 + \tfrac{1}{2}\left[\zeta_0 - \bar{\zeta}_0 + \tfrac{1}{3}\omega^2\left(\zeta_0 - \bar{\zeta}_0\right)^3\right] \tag{1.49}$$

The solution of (1.48) can be expressed as

$$\zeta_0 = A(T_1)e^{i\omega T_0} \tag{1.50}$$

Then, (1.49) becomes

$$\begin{aligned} D_0\zeta_1 - i\omega\zeta_1 = &-A'e^{i\omega T_0} + \tfrac{1}{2}Ae^{i\omega T_0} - \tfrac{1}{2}\bar{A}e^{-i\omega T_0} + \tfrac{1}{6}\omega^2 A^3 e^{3i\omega T_0} \\ &- \tfrac{1}{2}\omega^2 A^2\bar{A}e^{i\omega T_0} + \tfrac{1}{2}\omega^2 A\bar{A}^2 e^{-i\omega T_0} - \tfrac{1}{6}\omega^2\bar{A}^3 e^{-3i\omega T_0} \end{aligned} \tag{1.51}$$

Eliminating the terms that lead to secular terms from (1.51), we have

$$A' = \tfrac{1}{2}A - \tfrac{1}{2}\omega^2 A^2\bar{A} \tag{1.52}$$

Letting $\eta = Ae^{i\omega t}$ in (1.47), we obtain (1.52) because $T_1 = \epsilon t$.

1.4 Duffing–Rayleigh–van der Pol Equation

The Duffing, Rayleigh, and van der Pol equations are special cases of

$$\ddot{u} + \omega^2 u = \epsilon\left(\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right) \tag{1.53}$$

so that

$$f = \epsilon\left(\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right)$$

and (1.14) becomes

$$\begin{aligned} \dot{\zeta} = i\omega\zeta - \frac{i\epsilon}{2\omega}\Big[&i\mu\omega\left(\zeta - \bar{\zeta}\right) + \alpha_1\left(\zeta + \bar{\zeta}\right)^3 + i\omega\alpha_2\left(\zeta + \bar{\zeta}\right)^2\left(\zeta - \bar{\zeta}\right) \\ &-\omega^2\alpha_3\left(\zeta + \bar{\zeta}\right)\left(\zeta - \bar{\zeta}\right)^2 - i\omega^3\alpha_4\left(\zeta - \bar{\zeta}\right)^3\Big] \end{aligned} \tag{1.54}$$

Using the transformation (1.16), where $h = O(\epsilon)$, we rewrite (1.54) as

$$\begin{aligned} \dot{\eta} = {} & i\omega\eta + i\omega h - \frac{\partial h}{\partial\eta}\dot{\eta} - \frac{\partial h}{\partial\bar{\eta}}\dot{\bar{\eta}} \\ & - \frac{i\epsilon}{2\omega}\left[i\mu\omega\left(\eta - \bar{\eta}\right) + i\omega\alpha_2\left(\eta + \bar{\eta}\right)^2\left(\eta - \bar{\eta}\right)\right. \\ & \left. +\alpha_1\left(\eta + \bar{\eta}\right)^3 - \omega^2\alpha_3\left(\eta + \bar{\eta}\right)\left(\eta - \bar{\eta}\right)^2 - i\omega^3\alpha_4\left(\eta - \bar{\eta}\right)^3\right] \end{aligned} \tag{1.55}$$

where terms of $O(\epsilon^2)$ and higher have been neglected.

Again, because the perturbation contains linear as well as third-order terms, h has the form (1.44). Moreover, to the first approximation, $\dot{\eta}$ and $\dot{\bar{\eta}}$ are given by

(1.19). Hence, substituting (1.19) and (1.44) into (1.55) yields

$$\begin{aligned}\dot{\eta} &= i\omega\eta + 2i\epsilon\omega\left(\Delta_2 + \frac{i\mu}{4\omega}\right)\bar{\eta} \\ &- \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\eta^2\bar{\eta} \\ &+ \frac{1}{2}\epsilon\mu\eta + i\epsilon\omega\left[-2\Lambda_1 - \frac{1}{2\omega^2}\left(\alpha_1 + i\omega\alpha_2 - \omega^2\alpha_3 - i\omega^3\alpha_4\right)\right]\eta^3 \\ &+ i\epsilon\omega\left[2\Lambda_3 - \frac{1}{2\omega^2}\left(3\alpha_1 - i\omega\alpha_2 + \omega^2\alpha_3 - 3i\omega^3\alpha_4\right)\right]\eta\bar{\eta}^2 \\ &+ i\epsilon\omega\left[4\Lambda_4 - \frac{1}{2\omega^2}\left(\alpha_1 - i\omega\alpha_2 - \omega^2\alpha_3 + i\omega^3\alpha_4\right)\right]\bar{\eta}^3 \end{aligned} \tag{1.56}$$

We note that Δ_1 and Λ_2 do not appear in (1.56) and hence they are arbitrary and the terms η and $\eta^2\bar{\eta}$ are resonance terms. To produce the simplest form for (1.56), we choose $\Delta_2, \Lambda_1, \Lambda_3$, and Λ_4 to eliminate the terms involving $\bar{\eta}, \eta^3, \eta\bar{\eta}^2$, and $\bar{\eta}^3$; that is,

$$\Delta_2 = -\frac{i\mu}{4\omega} \tag{1.57}$$

$$\Lambda_1 = -\frac{1}{4\omega^2}\left(\alpha_1 + i\omega\alpha_2 - \omega^2\alpha_3 - i\omega^3\alpha_4\right) \tag{1.58}$$

$$\Lambda_3 = \frac{1}{4\omega^2}\left(3\alpha_1 - i\omega\alpha_2 + \omega^2\alpha_3 - 3i\omega^3\alpha_4\right) \tag{1.59}$$

$$\Lambda_4 = \frac{1}{8\omega^2}\left(\alpha_1 - i\omega\alpha_2 - \omega^2\alpha_3 + i\omega^3\alpha_4\right) \tag{1.60}$$

With these choices, (1.56) assumes the simple form

$$\dot{\eta} = i\omega\eta + \frac{1}{2}\epsilon\mu\eta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\eta^2\bar{\eta} \tag{1.61}$$

Again, we did not have to go through the lengthy algebra to arrive at the normal form (1.61). Because the solution of the unperturbed problem (1.54) is proportional to $e^{i\omega t}$, we could have replaced ζ with $e^{i\omega t}$ in the perturbation and identified the terms proportional to $e^{i\omega t}$. In this case, they are

$$\frac{1}{2}\epsilon\mu\zeta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\zeta^2\bar{\zeta}$$

Hence, keeping only the resonance terms in (1.54), we obtain the normal form

$$\dot{\zeta} = i\omega\zeta + \frac{1}{2}\epsilon\mu\zeta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\zeta^2\bar{\zeta}$$

which is formally equivalent to (1.61).