

UseR!

Pavel Čížek
Wolfgang Karl Härdle
Rafał Weron
Editors

Statistical Tools for Finance and Insurance

Second Edition



 Springer

Čížek • Härdle • Weron

(Eds.)

Statistical Tools for Finance and Insurance

Pavel Čížek • Wolfgang Karl Härdle • Rafał Weron
(Eds.)

Statistical Tools for Finance and Insurance

Second Edition

 Springer

Editors

Pavel Čížek
Tilburg University
Dept. of Econometrics & OR
P.O. Box 90153
5000 LE Tilburg, Netherlands
P.Cizek@uvt.nl

Rafał Weron
Wrocław University of Technology
Institute of Organization and Management
Wyb. Wyspiańskiego 27
50-370 Wrocław, Poland
Rafal.Weron@pwr.wroc.pl

Wolfgang Karl Härdle
Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. Centre for Applied Statistics and Economics
School of Business and Economics
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin, Germany
haerdle@wiwi.hu-berlin.de

The majority of chapters have quantlet codes in Matlab or R. These quantlets may be downloaded from <http://extras.springer.com> directly or via a link on <http://springer.com/978-3-642-18061-3> and from www.quantlet.de.

ISBN 978-3-642-18061-3 e-ISBN 978-3-642-18062-0
DOI 10.1007/978-3-642-18062-0
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2011922138

© Springer-Verlag Berlin Heidelberg 2005, 2011

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: WMXDesign GmbH

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Contents

Contributors	9
Preface to the second edition	11
Preface	13
Frequently used notation	17
I Finance	19
1 Models for heavy-tailed asset returns	21
<i>Szymon Borak, Adam Misiorek, and Rafał Weron</i>	
1.1 Introduction	21
1.2 Stable distributions	22
1.2.1 Definitions and basic properties	22
1.2.2 Computation of stable density and distribution functions	25
1.2.3 Simulation of stable variables	28
1.2.4 Estimation of parameters	29
1.3 Truncated and tempered stable distributions	34
1.4 Generalized hyperbolic distributions	36
1.4.1 Definitions and basic properties	36
1.4.2 Simulation of generalized hyperbolic variables	40
1.4.3 Estimation of parameters	42
1.5 Empirical evidence	44
2 Expected shortfall	57
<i>Simon A. Broda and Marc S. Paolella</i>	
2.1 Introduction	57
2.2 Expected shortfall for several asymmetric, fat-tailed distributions	58
2.2.1 Expected shortfall: definitions and basic results	58
2.2.2 Student's t and extensions	60

2.2.3	ES for the stable Paretian distribution	65
2.2.4	Generalized hyperbolic and its special cases	67
2.3	Mixture distributions	70
2.3.1	Introduction	70
2.3.2	Expected shortfall for normal mixture distributions	71
2.3.3	Symmetric stable mixture	72
2.3.4	Student's t mixtures	73
2.4	Comparison study	73
2.5	Lower partial moments	76
2.6	Expected shortfall for sums	82
2.6.1	Saddlepoint approximation for density and distribution	83
2.6.2	Saddlepoint approximation for expected shortfall	84
2.6.3	Application to sums of skew normal	85
2.6.4	Application to sums of proper generalized hyperbolic	87
2.6.5	Application to sums of normal inverse Gaussian	90
2.6.6	Application to portfolio returns	92
3	Modelling conditional heteroscedasticity in nonstationary series	101
	<i>Pavel Čížek</i>	
3.1	Introduction	101
3.2	Parametric conditional heteroscedasticity models	103
3.2.1	Quasi-maximum likelihood estimation	104
3.2.2	Estimation results	105
3.3	Time-varying coefficient models	108
3.3.1	Time-varying ARCH models	109
3.3.2	Estimation results	111
3.4	Pointwise adaptive estimation	114
3.4.1	Search for the longest interval of homogeneity	116
3.4.2	Choice of critical values	118
3.4.3	Estimation results	119
3.5	Adaptive weights smoothing	123
3.5.1	The AWS algorithm	124
3.5.2	Estimation results	127
3.6	Conclusion	127
4	FX smile in the Heston model	133
	<i>Agnieszka Janek, Tino Kluge, Rafał Weron, and Uwe Wystup</i>	
4.1	Introduction	133
4.2	The model	134

4.3	Option pricing	136
4.3.1	European vanilla FX option prices and Greeks	138
4.3.2	Computational issues	140
4.3.3	Behavior of the variance process and the Feller condition	142
4.3.4	Option pricing by Fourier inversion	144
4.4	Calibration	149
4.4.1	Qualitative effects of changing the parameters	149
4.4.2	The calibration scheme	150
4.4.3	Sample calibration results	152
4.5	Beyond the Heston model	155
4.5.1	Time-dependent parameters	155
4.5.2	Jump-diffusion models	158
5	Pricing of Asian temperature risk	163
	<i>Fred Espen Benth, Wolfgang Karl Härdle, and Brenda Lopez Cabrera</i>	
5.1	The temperature derivative market	165
5.2	Temperature dynamics	167
5.3	Temperature futures pricing	170
5.3.1	CAT futures and options	171
5.3.2	CDD futures and options	173
5.3.3	Infering the market price of temperature risk	175
5.4	Asian temperature derivatives	177
5.4.1	Asian temperature dynamics	177
5.4.2	Pricing Asian futures	188
6	Variance swaps	201
	<i>Wolfgang Karl Härdle and Elena Silyakova</i>	
6.1	Introduction	201
6.2	Volatility trading with variance swaps	202
6.3	Replication and hedging of variance swaps	203
6.4	Constructing a replication portfolio in practice	209
6.5	3G volatility products	211
6.5.1	Corridor and conditional variance swaps	213
6.5.2	Gamma swaps	214
6.6	Equity correlation (dispersion) trading with variance swaps	216
6.6.1	Idea of dispersion trading	216
6.7	Implementation of the dispersion strategy on DAX index	219

7	Learning machines supporting bankruptcy prediction	225
	<i>Wolfgang Karl Härdle, Linda Hoffmann, and Rouslan Moro</i>	
7.1	Bankruptcy analysis	226
7.2	Importance of risk classification and Basel II	237
7.3	Description of data	238
7.4	Calculations	239
7.5	Computational results	240
7.6	Conclusions	245
8	Distance matrix method for network structure analysis	251
	<i>Janusz Miśkiewicz</i>	
8.1	Introduction	251
8.2	Correlation distance measures	252
8.2.1	Manhattan distance	253
8.2.2	Ultrametric distance	253
8.2.3	Noise influence on the time series distance	254
8.2.4	Manhattan distance noise influence	255
8.2.5	Ultrametric distance noise influence	257
8.2.6	Entropy distance	262
8.3	Distance matrices analysis	263
8.4	Examples	265
8.4.1	Structure of stock markets	265
8.4.2	Dynamics of the network	268
8.5	Summary	279
II	Insurance	291
9	Building loss models	293
	<i>Krzysztof Burnecki, Joanna Janczura, and Rafał Weron</i>	
9.1	Introduction	293
9.2	Claim arrival processes	294
9.2.1	Homogeneous Poisson process (HPP)	295
9.2.2	Non-homogeneous Poisson process (NHPP)	297
9.2.3	Mixed Poisson process	300
9.2.4	Renewal process	301
9.3	Loss distributions	302
9.3.1	Empirical distribution function	303
9.3.2	Exponential distribution	304
9.3.3	Mixture of exponential distributions	305

9.3.4	Gamma distribution	307
9.3.5	Log-Normal distribution	309
9.3.6	Pareto distribution	311
9.3.7	Burr distribution	313
9.3.8	Weibull distribution	314
9.4	Statistical validation techniques	315
9.4.1	Mean excess function	315
9.4.2	Tests based on the empirical distribution function	318
9.5	Applications	321
9.5.1	Calibration of loss distributions	321
9.5.2	Simulation of risk processes	324
10	Ruin probability in finite time	329
	<i>Krzysztof Burnecki and Marek Teuerle</i>	
10.1	Introduction	329
10.1.1	Light- and heavy-tailed distributions	331
10.2	Exact ruin probabilities in finite time	333
10.2.1	Exponential claim amounts	334
10.3	Approximations of the ruin probability in finite time	334
10.3.1	Monte Carlo method	335
10.3.2	Segerdahl normal approximation	335
10.3.3	Diffusion approximation by Brownian motion	337
10.3.4	Corrected diffusion approximation	338
10.3.5	Diffusion approximation by α -stable Lévy motion	338
10.3.6	Finite time De Vylder approximation	340
10.4	Numerical comparison of the finite time approximations	342
11	Property and casualty insurance pricing with GLMs	349
	<i>Jan Iwanik</i>	
11.1	Introduction	349
11.2	Insurance data used in statistical modeling	350
11.3	The structure of generalized linear models	351
11.3.1	Exponential family of distributions	352
11.3.2	The variance and link functions	353
11.3.3	The iterative algorithm	353
11.4	Modeling claim frequency	354
11.4.1	Pre-modeling steps	355
11.4.2	The Poisson model	355
11.4.3	A numerical example	356

11.5	Modeling claim severity	356
11.5.1	Data preparation	357
11.5.2	A numerical example	358
11.6	Some practical modeling issues	360
11.6.1	Non-numeric variables and banding	360
11.6.2	Functional form of the independent variables	360
11.7	Diagnosing frequency and severity models	361
11.7.1	Expected value as a function of variance	361
11.7.2	Deviance residuals	361
11.7.3	Statistical significance of the coefficients	363
11.7.4	Uniformity over time	364
11.7.5	Selecting the final models	365
11.8	Finalizing the pricing models	366
12	Pricing of catastrophe bonds	371
	<i>Krzysztof Burnecki, Grzegorz Kukla, and David Taylor</i>	
12.1	Introduction	371
12.1.1	The emergence of CAT bonds	372
12.1.2	Insurance securitization	374
12.1.3	CAT bond pricing methodology	375
12.2	Compound doubly stochastic Poisson pricing model	377
12.3	Calibration of the pricing model	379
12.4	Dynamics of the CAT bond price	381
13	Return distributions of equity-linked retirement plans	393
	<i>Nils Detering, Andreas Weber, and Uwe Wystup</i>	
13.1	Introduction	393
13.2	The displaced double-exponential jump diffusion model	395
13.2.1	Model equation	395
13.2.2	Drift adjustment	398
13.2.3	Moments, variance and volatility	398
13.3	Parameter estimation	399
13.3.1	Estimating parameters from financial data	399
13.4	Interest rate curve	401
13.5	Products	401
13.5.1	Classical insurance strategy	401
13.5.2	Constant proportion portfolio insurance	402
13.5.3	Stop loss strategy	404
13.6	Payments to the contract and simulation horizon	405
13.7	Cost structures	406

13.8 Results without costs 407
13.9 Impact of costs 409
13.10 Impact of jumps 411
13.11 Summary 412

Index **415**

Contributors

Fred Espen Benth Center of Mathematics for Applications, University of Oslo

Szymon Borak Center for Applied Statistics and Economics, Humboldt-Universität zu Berlin

Simon Broda Department of Quantitative Economics, Amsterdam School of Economics

Krzysztof Burnecki Hugo Steinhaus Center for Stochastic Methods, Wrocław University of Technology

Brenda Lopez Cabrera Center for Applied Statistics and Economics, Humboldt Universität zu Berlin

Pavel Čížek Center for Economic Research, Tilburg University

Nils Detering MathFinance AG, Waldems, Germany

Wolfgang Karl Härdle Center for Applied Statistics and Economics, Humboldt Universität zu Berlin and National Central University, Jhongli, Taiwan

Linda Hoffmann Center for Applied Statistics and Economics, Humboldt Universität zu Berlin

Jan Iwanik RBS Insurance, London

Agnieszka Janek Institute of Mathematics and Computer Science, Wrocław University of Technology

Joanna Janczura Hugo Steinhaus Center for Stochastic Methods, Wrocław University of Technology

Tino Kluge MathFinance AG, Waldems, Germany

Grzegorz Kukla Towarzystwo Ubezpieczeniowe EUROPA S.A., Wrocław

Adam Misiorek Santander Consumer Bank S.A., Wrocław

Janusz Miśkiewicz Institute of Theoretical Physics, University of Wrocław

Rouslan Moro Brunel University, London

Marc Paoella Swiss Banking Institute, University of Zurich

Dorothea Schäfer Deutsches Institut für Wirtschaftsforschung e.V., Berlin

Elena Silyakova Center for Applied Statistics and Economics, Humboldt Universität zu Berlin

David Taylor School of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg

Marek Teuerle Institute of Mathematics and Computer Science, Wrocław University of Technology

Andreas Weber MathFinance AG, Waldems, Germany

Rafał Weron Institute of Organization and Management, Wrocław University of Technology

Agnieszka Wyłomańska Hugo Steinhaus Center for Stochastic Methods, Wrocław University of Technology

Uwe Wystup MathFinance AG, Waldems, Germany

Preface to the second edition

The meltdown of financial assets in the fall of 2008 made the consequences of financial crisis clearly visible to the broad public. The rapid loss of value of asset backed securities, collateralized debt obligations and other structured products was caused by devaluation of complex financial products. We therefore found it important to revise our book and present up-to-date research in financial statistics and econometrics.

We have dropped several chapters, thoroughly revised other and added a lot of new material. In the Finance part, the revised chapter on stable laws (Chapter 1) seamlessly guides the Reader not only through the computationally intensive techniques for stable distributions, but also for tempered stable and generalized hyperbolic laws. This introductory chapter is now complemented by a new text on Expected Shortfall with fat-tailed and mixture distributions (Chapter 2). The book then continues with a new chapter on adaptive heteroscedastic time series modeling (Chapter 3), which smoothly introduces the Reader to Chapter 4 on stochastic volatility modeling with the Heston model. The quantitative analysis of new products like weather derivatives and variance swaps is conducted in two new chapters (5 and 6, respectively). Finally, two different powerful classification techniques - learning machines for bankruptcy forecasting and the distance matrix method for market structure analysis - are discussed in the following two chapters (7 and 8, respectively).

In the Insurance part, two classical chapters on building loss models (Chapter 9) and on ruin probabilities (Chapter 10) are followed by a new text on property and casualty insurance with GLMs (Chapter 11). We then turn to products linking the finance and insurance worlds. Pricing of catastrophe bonds is discussed in Chapter 12 and a new chapter introduces into the pricing and cost structures of equity linked retirement plans (Chapter 13).

The majority of chapters have quantlet codes in Matlab or R. These quantlets may be downloaded from the Springer.com page or from www.quantlet.de. Finally, we would like to thank Barbara Choros, Richard Song, and Weining Wang for their help in the text management.

Pavel Čížek, Wolfgang Karl Härdle, and Rafał Weron

Tilburg, Berlin, and Wrocław, January 2011

Preface

This book is designed for students, researchers and practitioners who want to be introduced to modern statistical tools applied in finance and insurance. It is the result of a joint effort of the Center for Economic Research (CentER), Center for Applied Statistics and Economics (C.A.S.E.) and Hugo Steinhaus Center for Stochastic Methods (HSC). All three institutions brought in their specific profiles and created with this book a wide-angle view on and solutions to up-to-date practical problems.

The text is comprehensible for a graduate student in financial engineering as well as for an inexperienced newcomer to quantitative finance and insurance who wants to get a grip on advanced statistical tools applied in these fields. An experienced reader with a bright knowledge of financial and actuarial mathematics will probably skip some sections but will hopefully enjoy the various computational tools. Finally, a practitioner might be familiar with some of the methods. However, the statistical techniques related to modern financial products, like MBS or CAT bonds, will certainly attract him.

“Statistical Tools for Finance and Insurance” consists naturally of two main parts. Each part contains chapters with high focus on practical applications. The book starts with an introduction to *stable distributions*, which are the standard model for heavy tailed phenomena. Their numerical implementation is thoroughly discussed and applications to finance are given. The second chapter presents the ideas of *extreme value and copula analysis* as applied to multivariate financial data. This topic is extended in the subsequent chapter which deals with *tail dependence*, a concept describing the limiting proportion that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold. The fourth chapter reviews the market in *catastrophe insurance* risk, which emerged in order to facilitate the direct transfer of reinsurance risk associated with natural catastrophes from corporations, insurers, and reinsurers to capital market investors. The next contribution employs *functional data analysis* for the estimation of smooth implied volatility surfaces. These surfaces are a result of using an oversimplified market benchmark model – the Black-Scholes formula – to real data. An attractive approach to

overcome this problem is discussed in chapter six, where *implied trinomial trees* are applied to modeling implied volatilities and the corresponding state-price densities. An alternative route to tackling the implied volatility smile has led researchers to develop stochastic volatility models. The relative simplicity and the direct link of model parameters to the market makes *Heston's model* very attractive to front office users. Its application to FX option markets is covered in chapter seven. The following chapter shows how the computational complexity of stochastic volatility models can be overcome with the help of the *Fast Fourier Transform*. In chapter nine the valuation of *Mortgage Backed Securities* is discussed. The optimal prepayment policy is obtained via optimal stopping techniques. It is followed by a very innovative topic of predicting corporate bankruptcy with *Support Vector Machines*. Chapter eleven presents a novel approach to *money-demand modeling* using fuzzy clustering techniques. The first part of the book closes with *productivity analysis* for cost and frontier estimation. The nonparametric Data Envelopment Analysis is applied to efficiency issues of insurance agencies.

The insurance part of the book starts with a chapter on *loss distributions*. The basic models for claim severities are introduced and their statistical properties are thoroughly explained. In chapter fourteen, the methods of simulating and visualizing the *risk process* are discussed. This topic is followed by an overview of the approaches to *approximating the ruin probability* of an insurer. Both finite and infinite time approximations are presented. Some of these methods are extended in chapters sixteen and seventeen, where classical and anomalous *diffusion approximations* to ruin probability are discussed and extended to cases when the risk process exhibits *good and bad periods*. The last three chapters are related to one of the most important aspects of the insurance business – *premium calculation*. Chapter eighteen introduces the basic concepts including the pure risk premium and various safety loadings under different loss distributions. Calculation of a joint premium for a portfolio of insurance policies in the individual and collective risk models is discussed as well. The inclusion of *deductibles* into premium calculation is the topic of the following contribution. The last chapter of the insurance part deals with setting the appropriate level of insurance premium within a broader context of business decisions, including risk transfer through *reinsurance* and the rate of return on capital required to ensure solvability.

Our e-book offers a complete PDF version of this text and the corresponding HTML files with links to algorithms and quantlets. The reader of this book may therefore easily reconfigure and recalculate all the presented examples and methods via the enclosed XploRe Quantlet Server (XQS), which is also

available from www.xplore-stat.de and www.quantlet.com. A tutorial chapter explaining how to setup and use XQS can be found in the third and final part of the book.

We gratefully acknowledge the support of Deutsche Forschungsgemeinschaft (SFB 373 Quantifikation und Simulation Ökonomischer Prozesse, SFB 649 Ökonomisches Risiko) and Komitet Badań Naukowych (PBZ-KBN 016/P03/99 Mathematical models in analysis of financial instruments and markets in Poland). A book of this kind would not have been possible without the help of many friends, colleagues, and students. For the technical production of the e-book platform and quantlets we would like to thank Zdeněk Hlávka, Sigbert Klinke, Heiko Lehmann, Adam Misiorek, Piotr Uniejewski, Qingwei Wang, and Rodrigo Witzel. Special thanks for careful proofreading and supervision of the insurance part go to Krzysztof Burnecki.

Pavel Čížek, Wolfgang Härdle, and Rafał Weron
Tilburg, Berlin, and Wrocław, February 2005

Frequently used notation

$x \stackrel{\text{def}}{=} \dots$	x is defined as ...
$[x]$	integer part of x
$x \approx y$	x is approximately equal to y
A^\top	transpose of matrix A
$(F \circ G)(x)$	$F\{G(x)\}$ for functions F and G
I	indicator function
\mathbb{R}	real numbers
a_n, b_n, \dots	sequences of real numbers of vectors
$\alpha_n = O(\beta_n)$	$\alpha_n/\beta_n \rightarrow \text{const. as } n \rightarrow \infty$
$\alpha_n = o(\beta_n)$	$\alpha_n/\beta_n \rightarrow 0 \text{ as } n \rightarrow \infty$
$X \sim D$	the random variable X has a distribution D
$P(A)$	probability of a set A
$E(X)$	expected value of random variable X
$\text{Var}(X)$	variance of random variable X
$\text{Cov}(X, Y)$	covariance of two random variables X and Y
$\mathbb{N}(\mu, \Sigma)$	normal distribution with expectation μ and covariance matrix Σ ; a similar notation is used if Σ is the correlation matrix
Φ	standard normal cumulative distribution function
φ	standard normal density function
χ_p^2	chi-squared distribution with p degrees of freedom
t_p	t -distribution (Student's) with p degrees of freedom
W_t	Wiener process
\mathcal{F}_t	the information set generated by all information available at time t
A_n, B_n, \dots	sequences of random variables
$A_n = O_p(B_n)$	$\forall \varepsilon > 0 \exists M, \exists N$ such that $P[A_n/B_n > M] < \varepsilon, \forall n > N$
$A_n = o_p(B_n)$	$\forall \varepsilon > 0 : \lim_{n \rightarrow \infty} P[A_n/B_n > \varepsilon] = 0$

Part I

Finance

1 Models for heavy-tailed asset returns

Szymon Borak, Adam Misiorek, and Rafał Weron

1.1 Introduction

Many of the concepts in theoretical and empirical finance developed over the past decades – including the classical portfolio theory, the Black-Scholes-Merton option pricing model or the RiskMetrics variance-covariance approach to Value at Risk (VaR) – rest upon the assumption that asset returns follow a normal distribution. But this assumption is not justified by empirical data! Rather, the empirical observations exhibit excess kurtosis, more colloquially known as *fat tails* or *heavy tails* (Guillaume et al., 1997; Rachev and Mittnik, 2000). The contrast with the Gaussian law can be striking, as in Figure 1.1 where we illustrate this phenomenon using a ten-year history of the Dow Jones Industrial Average (DJIA) index.

In the context of VaR calculations, the problem of the underestimation of risk by the Gaussian distribution has been dealt with by the regulators in an *ad hoc* way. The Basle Committee on Banking Supervision (1995) suggested that for the purpose of determining minimum capital reserves financial institutions use a 10-day VaR at the 99% confidence level multiplied by a safety factor $s \in [3, 4]$. Stahl (1997) and Danielsson, Hartmann and De Vries (1998) argue convincingly that the range of s is a result of the heavy-tailed nature of asset returns. Namely, if we assume that the distribution is symmetric and has finite variance σ^2 then from Chebyshev's inequality we have $\mathbb{P}(Loss \geq \epsilon) \leq \frac{1}{2}\sigma^2\epsilon^2$. Setting the right hand side to 1% yields an upper bound for $VaR_{99\%} \leq 7.07\sigma$. On the other hand, if we assume that returns are normally distributed we arrive at $VaR_{99\%} \leq 2.33\sigma$, which is roughly three times lower than the bound obtained for a heavy-tailed, finite variance distribution.

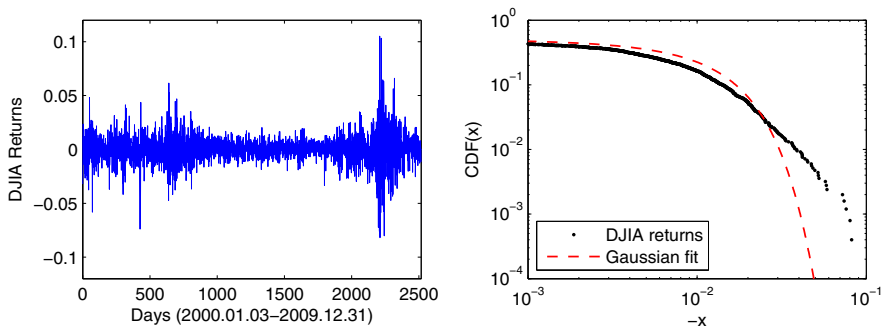


Figure 1.1: *Left panel:* Returns $\log(X_{t+1}/X_t)$ of the DJIA daily closing values X_t from the period January 3, 2000 – December 31, 2009. *Right panel:* Gaussian fit to the empirical cumulative distribution function (cdf) of the returns on a double logarithmic scale (only the left tail fit is displayed).



Being aware of the underestimation of risk by the Gaussian law we should consider using heavy-tailed alternatives. This chapter is intended as a guide to such models. In Section 1.2 we describe the historically oldest heavy-tailed model – the stable laws. Next, in Section 1.3 we briefly characterize their recent lighter-tailed generalizations, the so-called truncated and tempered stable distributions. In Section 1.4 we study the class of generalized hyperbolic laws, which – like tempered stable distributions – can be classified somewhere between infinite variance stable laws and the Gaussian distribution. Finally, in Section 1.5 we provide numerical examples.

1.2 Stable distributions

1.2.1 Definitions and basic properties

The theoretical rationale for modeling asset returns by the Gaussian distribution comes from the Central Limit Theorem (CLT), which states that the sum of a large number of independent, identically distributed (i.i.d.) variables – say, decisions of investors – from a finite-variance distribution will be (asymptotically)

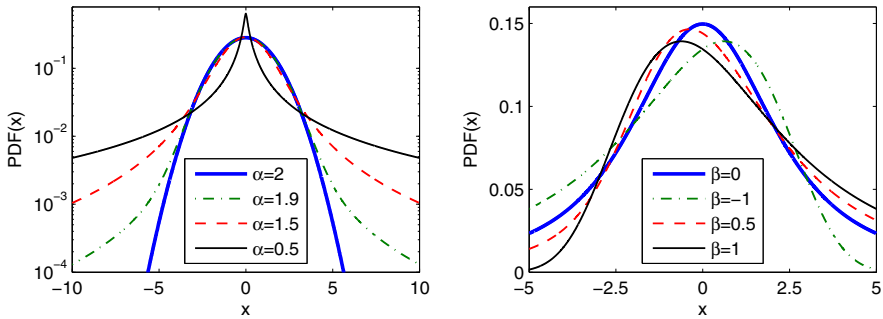


Figure 1.2: *Left panel:* A semi-logarithmic plot of symmetric ($\beta = \mu = 0$) stable densities for four values of α . Note, the distinct behavior of the Gaussian ($\alpha = 2$) distribution. *Right panel:* A plot of stable densities for $\alpha = 1.2$ and four values of β .



totically) normally distributed. Yet, this beautiful theoretical result has been notoriously contradicted by empirical findings. Possible reasons for the failure of the CLT in financial markets are (i) infinite-variance distributions of the variables, (ii) non-identical distributions of the variables, (iii) dependences between the variables or (iv) any combination of the three. If only the finite variance assumption is released we have a straightforward solution by virtue of the generalized CLT, which states that the limiting distribution of sums of such variables is stable (Nolan, 2010). This, together with the fact that stable distributions are leptokurtic and can accommodate fat tails and asymmetry, has led to their use as an alternative model for asset returns since the 1960s.

Stable laws – also called α -stable, stable Paretian or Lévy stable – were introduced by Paul Lévy in the 1920s. The name ‘stable’ reflects the fact that a sum of two independent random variables having a stable distribution with the same index α is again stable with index α . This invariance property holds also for Gaussian variables. In fact, the Gaussian distribution is stable with $\alpha = 2$.

For complete description the stable distribution requires four parameters. The index of stability $\alpha \in (0, 2]$, also called the tail index, tail exponent or characteristic exponent, determines the rate at which the tails of the distribution taper off, see the left panel in Figure 1.2. The skewness parameter $\beta \in [-1, 1]$ defines the asymmetry. When $\beta > 0$, the distribution is skewed to the right, i.e.

the right tail is thicker, see the right panel in Figure 1.2. When it is negative, it is skewed to the left. When $\beta = 0$, the distribution is symmetric about the mode (the peak) of the distribution. As α approaches 2, β loses its effect and the distribution approaches the Gaussian distribution regardless of β . The last two parameters, $\sigma > 0$ and $\mu \in \mathbb{R}$, are the usual scale and location parameters, respectively.

A far-reaching feature of the stable distribution is the fact that its probability density function (pdf) and cumulative distribution function (cdf) do not have closed form expressions, with the exception of three special cases. The best known of these is the Gaussian ($\alpha = 2$) law whose pdf is given by:

$$f_G(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}. \quad (1.1)$$

The other two are the lesser known Cauchy ($\alpha = 1, \beta = 0$) and Lévy ($\alpha = 0.5, \beta = 1$) laws. Consequently, the stable distribution can be most conveniently described by its characteristic function (cf) – the inverse Fourier transform of the pdf. The most popular parameterization of the characteristic function $\phi(t)$ of $X \sim S_\alpha(\sigma, \beta, \mu)$, i.e. a stable random variable with parameters α, σ, β and μ , is given by (Samorodnitsky and Taqqu, 1994; Weron, 1996):

$$\log \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 - i\beta \text{sign}(t) \tan \frac{\pi\alpha}{2}\} + i\mu t, & \alpha \neq 1, \\ -\sigma |t| \{1 + i\beta \text{sign}(t) \frac{2}{\pi} \log |t|\} + i\mu t, & \alpha = 1. \end{cases} \quad (1.2)$$

Note, that the traditional scale parameter σ of the Gaussian distribution is not the same as σ in the above representation. A comparison of formulas (1.1) and (1.2) yields the relation: $\sigma_{\text{Gaussian}} = \sqrt{2}\sigma$.

For numerical purposes, it is often useful to use Nolan's (1997) parameterization:

$$\log \phi_0(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 + i\beta \text{sign}(t) \tan \frac{\pi\alpha}{2} [(\sigma|t|)^{1-\alpha} - 1]\} + i\mu_0 t, & \alpha \neq 1, \\ -\sigma |t| \{1 + i\beta \text{sign}(t) \frac{2}{\pi} \log(\sigma|t|)\} + i\mu_0 t, & \alpha = 1, \end{cases} \quad (1.3)$$

which yields a cf (and hence the pdf and cdf) jointly continuous in all four parameters. The location parameters of the two representations (S and S^0) are related by $\mu = \mu_0 - \beta\sigma \tan \frac{\pi\alpha}{2}$ for $\alpha \neq 1$ and $\mu = \mu_0 - \beta\sigma \frac{2}{\pi} \log \sigma$ for $\alpha = 1$.

The ‘fatness’ of the tails of a stable distribution can be derived from the following property: the p th moment of a stable random variable is finite if and only if $p < \alpha$. Hence, when $\alpha > 1$ the mean of the distribution exists (and is equal to μ). On the other hand, when $\alpha < 2$ the variance is infinite and the tails exhibit a power-law behavior (i.e. they are asymptotically equivalent to a Pareto law). More precisely, using a CLT type argument it can be shown that (Janicki and Weron, 1994a; Samorodnitsky and Taqqu, 1994):

$$\begin{cases} \lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(X > x) = C_\alpha(1 + \beta)\sigma^\alpha, \\ \lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(X < -x) = C_\alpha(1 + \beta)\sigma^\alpha, \end{cases} \quad (1.4)$$

where $C_\alpha = (2 \int_0^\infty x^{-\alpha} \sin(x) dx)^{-1} = \frac{1}{\pi} \Gamma(\alpha) \sin \frac{\pi\alpha}{2}$. The convergence to the power-law tail varies for different α 's and is slower for larger values of the tail index. Moreover, the tails of stable cdfs exhibit a crossover from an approximate power decay with exponent $\alpha > 2$ to the true tail with exponent α . This phenomenon is more visible for large α 's (Weron, 2001).

1.2.2 Computation of stable density and distribution functions

The lack of closed form formulas for most stable densities and distribution functions has far-reaching consequences. Numerical approximation or direct numerical integration have to be used instead of analytical formulas, leading to a drastic increase in computational time and loss of accuracy. Despite a few early attempts in the 1970s, efficient and general techniques have not been developed until late 1990s.

Mittnik, Doganoglu and Chenyao (1999) exploited the pdf–cf relationship and applied the fast Fourier transform (FFT). However, for data points falling between the equally spaced FFT grid nodes an interpolation technique has to be used. The authors suggested that linear interpolation suffices in most practical applications, see also Rachev and Mittnik (2000). Taking a larger number of grid points increases accuracy, however, at the expense of higher computational burden. Setting the number of grid points to $N = 2^{13}$ and the grid spacing to $h = 0.01$ allows to achieve comparable accuracy to the direct integration method (see below), at least for typically used values of $\alpha > 1.6$.

As for the computational speed, the FFT based approach is faster for large samples, whereas the direct integration method favors small data sets since it can be computed at any arbitrarily chosen point. Mittnik, Doganoglu and Chenyao (1999) report that for $N = 2^{13}$ the FFT based method is faster

for samples exceeding 100 observations and slower for smaller data sets. We must stress, however, that the FFT based approach is not as universal as the direct integration method – it is efficient only for large alpha's and only as far as the pdf calculations are concerned. When computing the cdf the former method must numerically integrate the density, whereas the latter takes the same amount of time in both cases.

The direct integration method, proposed by Nolan (1997, 1999), consists of a numerical integration of Zolotarev's (1986) formulas for the density or the distribution function. Set $\zeta = -\beta \tan \frac{\pi\alpha}{2}$. Then the density $f(x; \alpha, \beta)$ of a standard stable random variable in representation S^0 , i.e. $X \sim S^0_\alpha(1, \beta, 0)$, can be expressed as (note, that Zolotarev (1986, Section 2.2) used another parametrization):

- when $\alpha \neq 1$ and $x \neq \zeta$:

$$f(x; \alpha, \beta) = \frac{\alpha(x - \zeta)^{\frac{1}{\alpha-1}}}{\pi |\alpha - 1|} \int_{-\xi}^{\frac{\pi}{2}} V(\theta; \alpha, \beta) \exp \left\{ -(x - \zeta)^{\frac{\alpha}{\alpha-1}} V(\theta; \alpha, \beta) \right\} d\theta, \quad (1.5)$$

for $x > \zeta$ and $f(x; \alpha, \beta) = f(-x; \alpha, -\beta)$ for $x < \zeta$,

- when $\alpha \neq 1$ and $x = \zeta$:

$$f(x; \alpha, \beta) = \frac{\Gamma(1 + \frac{1}{\alpha}) \cos(\xi)}{\pi(1 + \zeta^2)^{\frac{1}{2\alpha}}},$$

- when $\alpha = 1$:

$$f(x; 1, \beta) = \begin{cases} \frac{1}{2|\beta|} e^{\frac{\pi x}{2\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\theta; 1, \beta) \exp \left\{ -e^{\frac{\pi x}{2\beta}} V(\theta; 1, \beta) \right\} d\theta, & \beta \neq 0, \\ \frac{1}{\pi(1+x^2)}, & \beta = 0, \end{cases}$$

where

$$\xi = \begin{cases} \frac{1}{\alpha} \arctan(-\zeta), & \alpha \neq 1, \\ \frac{\pi}{2}, & \alpha = 1, \end{cases} \quad (1.6)$$

and

$$V(\theta; \alpha, \beta) = \begin{cases} (\cos \alpha \xi)^{\frac{1}{\alpha-1}} \left(\frac{\cos \theta}{\sin \alpha(\xi+\theta)} \right)^{\frac{\alpha}{\alpha-1}} \frac{\cos\{\alpha\xi+(\alpha-1)\theta\}}{\cos \theta}, & \alpha \neq 1, \\ \frac{2}{\pi} \left(\frac{\frac{\pi}{2} + \beta\theta}{\cos \theta} \right) \exp \left\{ \frac{1}{\beta} \left(\frac{\pi}{2} + \beta\theta \right) \tan \theta \right\}, & \alpha = 1, \beta \neq 0. \end{cases}$$