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Editors

Statistical Tools for Finance and Insurance

Second Edition

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Preface to the second edition

The meltdown of financial assets in the fall of 2008 made the consequences of financial crisis clearly visible to the broad public. The rapid loss of value of asset backed securities, collateralized debt obligations and other structured products was caused by devaluation of complex financial products. We therefore found it important to revise our book and present up-to-date research in financial statistics and econometrics.

We have dropped several chapters, thoroughly revised other and added a lot of new material. In the Finance part, the revised chapter on stable laws (Chapter 1) seamlessly guides the Reader not only through the computationally intensive techniques for stable distributions, but also for tempered stable and generalized hyperbolic laws. This introductory chapter is now complemented by a new text on Expected Shortfall with fat-tailed and mixture distributions (Chapter 2). The book then continues with a new chapter on adaptive heteroscedastic time series modeling (Chapter 3), which smoothly introduces the Reader to Chapter 4 on stochastic volatility modeling with the Heston model. The quantitative analysis of new products like weather derivatives and variance swaps is conducted in two new chapters (5 and 6, respectively). Finally, two different powerful classification techniques - learning machines for bankruptcy forecasting and the distance matrix method for market structure analysis - are discussed in the following two chapters (7 and 8, respectively).

In the Insurance part, two classical chapters on building loss models (Chapter 9) and on ruin probabilities (Chapter 10) are followed by a new text on property and casualty insurance with GLMs (Chapter 11). We then turn to products linking the finance and insurance worlds. Pricing of catastrophe bonds is discussed in Chapter 12 and a new chapter introduces into the pricing and cost structures of equity linked retirement plans (Chapter 13).

The majority of chapters have quantlet codes in Matlab or R. These quantlets may be downloaded from the Springer.com page or from www.quantlet.de. Finally, we would like to thank Barbara Choros, Richard Song, and Weining Wang for their help in the text management.

Pavel Čížek, Wolfgang Karl Härdle, and Rafał Weron

Tilburg, Berlin, and Wroclaw, January 2011

Preface

This book is designed for students, researchers and practitioners who want to be introduced to modern statistical tools applied in finance and insurance. It is the result of a joint effort of the Center for Economic Research (CentER), Center for Applied Statistics and Economics (C.A.S.E.) and Hugo Steinhaus Center for Stochastic Methods (HSC). All three institutions brought in their specific profiles and created with this book a wide-angle view on and solutions to up-to-date practical problems.

The text is comprehensible for a graduate student in financial engineering as well as for an inexperienced newcomer to quantitative finance and insurance who wants to get a grip on advanced statistical tools applied in these fields. An experienced reader with a bright knowledge of financial and actuarial mathematics will probably skip some sections but will hopefully enjoy the various computational tools. Finally, a practitioner might be familiar with some of the methods. However, the statistical techniques related to modern financial products, like MBS or CAT bonds, will certainly attract him.

"Statistical Tools for Finance and Insurance" consists naturally of two main parts. Each part contains chapters with high focus on practical applications. The book starts with an introduction to *stable distributions*, which are the standard model for heavy tailed phenomena. Their numerical implementation is thoroughly discussed and applications to finance are given. The second chapter presents the ideas of *extreme value and copula analysis* as applied to multivariate financial data. This topic is extended in the subsequent chapter which deals with *tail dependence*, a concept describing the limiting proportion that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold. The fourth chapter reviews the market in *catastrophe insurance* risk, which emerged in order to facilitate the direct transfer of reinsurance risk associated with natural catastrophes from corporations, insurers, and reinsurers to capital market investors. The next contribution employs *functional data analysis* for the estimation of smooth implied volatility surfaces. These surfaces are a result of using an oversimplified market benchmark model – the Black-Scholes formula – to real data. An attractive approach to overcome this problem is discussed in chapter six, where *implied trinomial trees* are applied to modeling implied volatilities and the corresponding state-price densities. An alternative route to tackling the implied volatility smile has led researchers to develop stochastic volatility models. The relative simplicity and the direct link of model parameters to the market makes *Heston's model* very attractive to front office users. Its application to FX option markets is covered in chapter seven. The following chapter shows how the computational complexity of stochastic volatility models can be overcome with the help of the *Fast Fourier Transform*. In chapter nine the valuation of *Mortgage Backed Securities* is discussed. The optimal prepayment policy is obtained via optimal stopping techniques. It is followed by a very innovative topic of predicting corporate bankruptcy with *Support Vector Machines*. Chapter eleven presents a novel approach to *money-demand modeling* using fuzzy clustering techniques. The first part of the book closes with *productivity analysis* for cost and frontier estimation. The nonparametric Data Envelopment Analysis is applied to efficiency issues of insurance agencies.

The insurance part of the book starts with a chapter on *loss distributions*. The basic models for claim severities are introduced and their statistical properties are thoroughly explained. In chapter fourteen, the methods of simulating and visualizing the *risk process* are discussed. This topic is followed by an overview of the approaches to *approximating the ruin probability* of an insurer. Both finite and infinite time approximations are presented. Some of these methods are extended in chapters sixteen and seventeen, where classical and anomalous *diffusion approximations* to ruin probability are discussed and extended to cases when the risk process exhibits *good and bad periods*. The last three chapters are related to one of the most important aspects of the insurance business – *premium calculation*. Chapter eighteen introduces the basic concepts including the pure risk premium and various safety loadings under different loss distributions. Calculation of a joint premium for a portfolio of insurance policies in the individual and collective risk models is discussed as well. The inclusion of *deductibles* into premium calculation is the topic of the following contribution. The last chapter of the insurance part deals with setting the appropriate level of insurance premium within a broader context of business decisions, including risk transfer through *reinsurance* and the rate of return on capital required to ensure solvability.

Our e-book offers a complete PDF version of this text and the corresponding HTML files with links to algorithms and quantlets. The reader of this book may therefore easily reconfigure and recalculate all the presented examples and methods via the enclosed XploRe Quantlet Server (XQS), which is also available from www.xplore-stat.de and www.quantlet.com. A tutorial chapter explaining how to setup and use XQS can be found in the third and final part of the book.

We gratefully acknowledge the support of Deutsche Forschungsgemeinschaft (SFB 373 Quantifikation und Simulation Ökonomischer Prozesse, SFB 649 Ökonomisches Risiko) and Komitet Badań Naukowych (PBZ-KBN 016/P03/99 Mathematical models in analysis of financial instruments and markets in Poland). A book of this kind would not have been possible without the help of many friends, colleagues, and students. For the technical production of the e-book platform and quantlets we would like to thank Zdeněk Hlávka, Sigbert Klinke, Heiko Lehmann, Adam Misiorek, Piotr Uniejewski, Qingwei Wang, and Rodrigo Witzel. Special thanks for careful proofreading and supervision of the insurance part go to Krzysztof Burnecki.

Pavel Čížek, Wolfgang Härdle, and Rafał Weron

Tilburg, Berlin, and Wrocław, February 2005

Frequently used notation

$x \stackrel{\mathrm{def}}{=} \dots$	x is defined as ...
$[x]$	integer part of x
$x \approx y$	x is approximately equal to y
$A^\top$	transpose of matrix A
$(F \circ G)(x)$	$F\{G(x)\}$ for functions F and G
I	indicator function
$\mathbb{R}$	real numbers
$a_n, b_n, \dots$	sequences of real numbers of vectors
$\alpha_n = O(\beta_n)$	$\alpha_n/\beta_n \longrightarrow$ const. as $n \longrightarrow \infty$
$\alpha_n = o(\beta_n)$	$\alpha_n/\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$
$X \sim D$	the random variable X has a distribution D
$\mathrm{P}(A)$	probability of a set A
$\mathrm{E}(X)$	expected value of random variable X
$\mathrm{Var}(X)$	variance of random variable X
$\mathrm{Cov}(X, Y)$	covariance of two random variables X and Y
$\mathbb{N}(\mu, \Sigma)$	normal distribution with expectation μ and covariance matrix Σ; a similar notation is used if Σ is the correlation matrix
Φ	standard normal cumulative distribution function
φ	standard normal density function
χ^2_p	chi-squared distribution with p degrees of freedom
t_p	t-distribution (Student's) with p degrees of freedom
W_t	Wiener process
$\mathcal{F}_t$	the information set generated by all information available at time t
$A_n, B_n, \dots$	sequences of random variables
$A_n = O_p(B_n)$	$\forall \varepsilon > 0\ \exists M,\ \exists N$ such that $\mathrm{P}[\lvert A_n/B_n \rvert > M] < \varepsilon,\ \forall n > N$
$A_n = o_p(B_n)$	$\forall \varepsilon > 0\ :\ \lim_{n\to\infty} \mathrm{P}[\lvert A_n/B_n \rvert > \varepsilon] = 0$

Part I

Finance

1 Models for heavy-tailed asset returns

Szymon Borak, Adam Misiorek, and Rafał Weron

1.1 Introduction

Many of the concepts in theoretical and empirical finance developed over the past decades – including the classical portfolio theory, the Black-Scholes-Merton option pricing model or the RiskMetrics variance-covariance approach to Value at Risk (VaR) – rest upon the assumption that asset returns follow a normal distribution. But this assumption is not justified by empirical data! Rather, the empirical observations exhibit excess kurtosis, more colloquially known as *fat tails* or *heavy tails* (Guillaume et al., 1997; Rachev and Mittnik, 2000). The contrast with the Gaussian law can be striking, as in Figure 1.1 where we illustrate this phenomenon using a ten-year history of the Dow Jones Industrial Average (DJIA) index.

In the context of VaR calculations, the problem of the underestimation of risk by the Gaussian distribution has been dealt with by the regulators in an *ad hoc* way. The Basle Committee on Banking Supervision (1995) suggested that for the purpose of determining minimum capital reserves financial institutions use a 10-day VaR at the 99% confidence level multiplied by a safety factor $s \in [3, 4]$. Stahl (1997) and Danielsson, Hartmann and De Vries (1998) argue convincingly that the range of s is a result of the heavy-tailed nature of asset returns. Namely, if we assume that the distribution is symmetric and has finite variance σ^2 then from Chebyshev's inequality we have $\mathbb{P}(Loss \geq \epsilon) \leq \frac{1}{2}\sigma^2\epsilon^2$. Setting the right hand side to 1% yields an upper bound for $\text{VaR}_{99\%} \leq 7.07\sigma$. On the other hand, if we assume that returns are normally distributed we arrive at $\text{VaR}_{99\%} \leq 2.33\sigma$, which is roughly three times lower than the bound obtained for a heavy-tailed, finite variance distribution.

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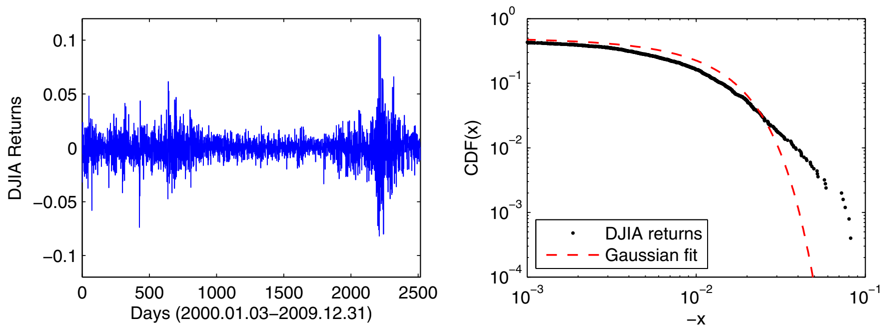

Figure 1.1: *Left panel*: Returns $\log(X_{t+1}/X_t)$ of the DJIA daily closing values X_t from the period January 3, 2000 – December 31, 2009. *Right panel*: Gaussian fit to the empirical cumulative distribution function (cdf) of the returns on a double logarithmic scale (only the left tail fit is displayed).

STFstab01

Being aware of the underestimation of risk by the Gaussian law we should consider using heavy-tailed alternatives. This chapter is intended as a guide to such models. In Section 1.2 we describe the historically oldest heavy-tailed model – the stable laws. Next, in Section 1.3 we briefly characterize their recent lighter-tailed generalizations, the so-called truncated and tempered stable distributions. In Section 1.4 we study the class of generalized hyperbolic laws, which – like tempered stable distributions – can be classified somewhere between infinite variance stable laws and the Gaussian distribution. Finally, in Section 1.5 we provide numerical examples.

1.2 Stable distributions

1.2.1 Definitions and basic properties

The theoretical rationale for modeling asset returns by the Gaussian distribution comes from the Central Limit Theorem (CLT), which states that the sum of a large number of independent, identically distributed (i.i.d.) variables – say, decisions of investors – from a finite-variance distribution will be (asymp-

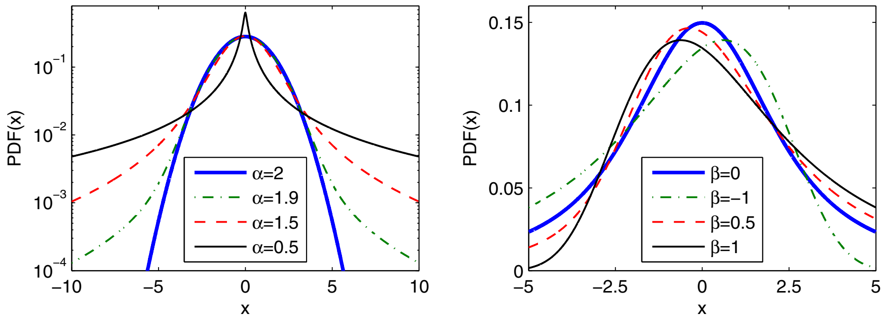

Figure 1.2: *Left panel*: A semi-logarithmic plot of symmetric ($\beta = \mu = 0$) stable densities for four values of α. Note, the distinct behavior of the Gaussian ($\alpha = 2$) distribution. *Right panel*: A plot of stable densities for $\alpha = 1.2$ and four values of β.

STFstab02

totically) normally distributed. Yet, this beautiful theoretical result has been notoriously contradicted by empirical findings. Possible reasons for the failure of the CLT in financial markets are (i) infinite-variance distributions of the variables, (ii) non-identical distributions of the variables, (iii) dependences between the variables or (iv) any combination of the three. If only the finite variance assumption is released we have a straightforward solution by virtue of the generalized CLT, which states that the limiting distribution of sums of such variables is stable (Nolan, 2010). This, together with the fact that stable distributions are leptokurtic and can accommodate fat tails and asymmetry, has led to their use as an alternative model for asset returns since the 1960s.

Stable laws – also called α-stable, stable Paretian or Lévy stable – were introduced by Paul Lévy in the 1920s. The name 'stable' reflects the fact that a sum of two independent random variables having a stable distribution with the same index α is again stable with index α. This invariance property holds also for Gaussian variables. In fact, the Gaussian distribution is stable with $\alpha = 2$.

For complete description the stable distribution requires four parameters. The index of stability $\alpha \in (0, 2]$, also called the tail index, tail exponent or characteristic exponent, determines the rate at which the tails of the distribution taper off, see the left panel in Figure 1.2. The skewness parameter $\beta \in [-1, 1]$ defines the asymmetry. When $\beta > 0$, the distribution is skewed to the right, i.e.

the right tail is thicker, see the right panel in Figure 1.2. When it is negative, it is skewed to the left. When $\beta = 0$, the distribution is symmetric about the mode (the peak) of the distribution. As α approaches 2, β loses its effect and the distribution approaches the Gaussian distribution regardless of β. The last two parameters, $\sigma > 0$ and $\mu \in \mathbb{R}$, are the usual scale and location parameters, respectively.

A far-reaching feature of the stable distribution is the fact that its probability density function (pdf) and cumulative distribution function (cdf) do not have closed form expressions, with the exception of three special cases. The best known of these is the Gaussian ($\alpha = 2$) law whose pdf is given by:

$$f_G(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}. \tag{1.1}$$

The other two are the lesser known Cauchy ($\alpha = 1$, $\beta = 0$) and Lévy ($\alpha = 0.5$, $\beta = 1$) laws. Consequently, the stable distribution can be most conveniently described by its characteristic function (cf) – the inverse Fourier transform of the pdf. The most popular parameterization of the characteristic function $\phi(t)$ of $X \sim S_\alpha(\sigma, \beta, \mu)$, i.e. a stable random variable with parameters α, σ, β and μ, is given by (Samorodnitsky and Taqqu, 1994; Weron, 1996):

$$\log \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 - i\beta \text{sign}(t) \tan\frac{\pi\alpha}{2}\} + i\mu t, & \alpha \neq 1, \\ \\ -\sigma|t|\{1 + i\beta \text{sign}(t) \frac{2}{\pi} \log|t|\} + i\mu t, & \alpha = 1. \end{cases} \tag{1.2}$$

Note, that the traditional scale parameter σ of the Gaussian distribution is not the same as σ in the above representation. A comparison of formulas (1.1) and (1.2) yields the relation: $\sigma_{\text{Gaussian}} = \sqrt{2}\sigma$.

For numerical purposes, it is often useful to use Nolan's (1997) parameterization:

$$\log \phi_0(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 + i\beta \text{sign}(t) \tan\frac{\pi\alpha}{2}[(\sigma|t|)^{1-\alpha} - 1]\} + i\mu_0 t, & \alpha \neq 1, \\ \\ -\sigma|t|\{1 + i\beta \text{sign}(t) \frac{2}{\pi} \log(\sigma|t|)\} + i\mu_0 t, & \alpha = 1, \end{cases} \tag{1.3}$$

which yields a cf (and hence the pdf and cdf) jointly continuous in all four parameters. The location parameters of the two representations (S and S^0) are related by $\mu = \mu_0 - \beta\sigma \tan\frac{\pi\alpha}{2}$ for $\alpha \neq 1$ and $\mu = \mu_0 - \beta\sigma\frac{2}{\pi}\log\sigma$ for $\alpha = 1$.

The 'fatness' of the tails of a stable distribution can be derived from the following property: the pth moment of a stable random variable is finite if and only if $p < \alpha$. Hence, when $\alpha > 1$ the mean of the distribution exists (and is equal to μ). On the other hand, when $\alpha < 2$ the variance is infinite and the tails exhibit a power-law behavior (i.e. they are asymptotically equivalent to a Pareto law). More precisely, using a CLT type argument it can be shown that (Janicki and Weron, 1994a; Samorodnitsky and Taqqu, 1994):

$$\begin{cases} \lim_{x\to\infty} x^{\alpha}\mathbb{P}(X > x) = C_{\alpha}(1+\beta)\sigma^{\alpha}, \\ \lim_{x\to\infty} x^{\alpha}\mathbb{P}(X < -x) = C_{\alpha}(1+\beta)\sigma^{\alpha}, \end{cases} \tag{1.4}$$

where $C_\alpha = \left(2\int_0^\infty x^{-\alpha}\sin(x)dx\right)^{-1} = \frac{1}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}$. The convergence to the power-law tail varies for different α's and is slower for larger values of the tail index. Moreover, the tails of stable cdfs exhibit a crossover from an approximate power decay with exponent $\alpha > 2$ to the true tail with exponent α. This phenomenon is more visible for large α's (Weron, 2001).

1.2.2 Computation of stable density and distribution functions

The lack of closed form formulas for most stable densities and distribution functions has far-reaching consequences. Numerical approximation or direct numerical integration have to be used instead of analytical formulas, leading to a drastic increase in computational time and loss of accuracy. Despite a few early attempts in the 1970s, efficient and general techniques have not been developed until late 1990s.

Mittnik, Doganoglu and Chenyao (1999) exploited the pdf–cf relationship and applied the fast Fourier transform (FFT). However, for data points falling between the equally spaced FFT grid nodes an interpolation technique has to be used. The authors suggested that linear interpolation suffices in most practical applications, see also Rachev and Mittnik (2000). Taking a larger number of grid points increases accuracy, however, at the expense of higher computational burden. Setting the number of grid points to $N = 2^{13}$ and the grid spacing to $h = 0.01$ allows to achieve comparable accuracy to the direct integration method (see below), at least for typically used values of $\alpha > 1.6$.

As for the computational speed, the FFT based approach is faster for large samples, whereas the direct integration method favors small data sets since it can be computed at any arbitrarily chosen point. Mittnik, Doganoglu and Chenyao (1999) report that for $N = 2^{13}$ the FFT based method is faster for samples exceeding 100 observations and slower for smaller data sets. We must stress, however, that the FFT based approach is not as universal as the direct integration method – it is efficient only for large alpha's and only as far as the pdf calculations are concerned. When computing the cdf the former method must numerically integrate the density, whereas the latter takes the same amount of time in both cases.

The direct integration method, proposed by Nolan (1997, 1999), consists of a numerical integration of Zolotarev's (1986) formulas for the density or the distribution function. Set $\zeta = -\beta \tan\frac{\pi\alpha}{2}$. Then the density $f(x;\alpha,\beta)$ of a standard stable random variable in representation S^0, i.e. $X \sim S^0_\alpha(1,\beta,0)$, can be expressed as (note, that Zolotarev (1986, Section 2.2) used another parametrization):

- when $\alpha \neq 1$ and $x \neq \zeta$:

$$f(x;\alpha,\beta) = \frac{\alpha(x-\zeta)^{\frac{1}{\alpha-1}}}{\pi \mid \alpha - 1 \mid} \int_{-\xi}^{\frac{\pi}{2}} V(\theta;\alpha,\beta) \exp\left\{-(x-\zeta)^{\frac{\alpha}{\alpha-1}} V(\theta;\alpha,\beta)\right\} d\theta, \tag{1.5}$$

  for $x > \zeta$ and $f(x;\alpha,\beta) = f(-x;\alpha,-\beta)$ for $x < \zeta$,

- when $\alpha \neq 1$ and $x = \zeta$:

$$f(x;\alpha,\beta) = \frac{\Gamma(1+\frac{1}{\alpha})\cos(\xi)}{\pi(1+\zeta^2)^{\frac{1}{2\alpha}}},$$

- when $\alpha = 1$:

$$f(x;1,\beta) = \begin{cases} \frac{1}{2|\beta|} e^{\frac{\pi x}{2\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\theta;1,\beta) \exp\left\{-e^{\frac{\pi x}{2\beta}} V(\theta;1,\beta)\right\} d\theta, & \beta \neq 0, \\ \frac{1}{\pi(1+x^2)}, & \beta = 0, \end{cases}$$

where

$$\xi = \begin{cases} \frac{1}{\alpha}\arctan(-\zeta), & \alpha \neq 1, \\ \frac{\pi}{2}, & \alpha = 1, \end{cases} \tag{1.6}$$

and

$$V(\theta;\alpha,\beta) = \begin{cases} (\cos\alpha\xi)^{\frac{1}{\alpha-1}} \left(\frac{\cos\theta}{\sin\alpha(\xi+\theta)}\right)^{\frac{\alpha}{\alpha-1}} \frac{\cos\{\alpha\xi+(\alpha-1)\theta\}}{\cos\theta}, & \alpha \neq 1, \\ \frac{2}{\pi}\left(\frac{\frac{\pi}{2}+\beta\theta}{\cos\theta}\right) \exp\left\{\frac{1}{\beta}(\frac{\pi}{2}+\beta\theta)\tan\theta\right\}, & \alpha = 1, \beta \neq 0. \end{cases}$$