QUANTUM DYNAMICS FOR CLASSICAL SYSTEMS

WITH APPLICATIONS OF THE NUMBER OPERATOR

FABIO BAGARELLO

WILEY

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Science never solves a problem
without creating ten more.

George Bernard Shaw

The most exciting phrase to hear in science,
the one who heralds new discoveries,
is not *Eureka* but *That's funny*...

Isaac Asimov

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PREFACE

In 2005 or so, I started wondering whether that particular part of quantum mechanics that theoretical physicists call *second quantization* could be used in the analysis of some particular, somehow *discrete*, classical system. In particular, I started considering stock markets, or a reasonably simplified version of these, since at that time this was a very fashionable topic: econophysics was in its early years, and the general feeling was that there was still room, and need, for many other contributions from physicists and mathematicians. I got the idea that the analysis of the huge amount of information going around a real market was only part of what was interesting to do. I was much more interested in considering the viewpoint of the single trader, who is more likely interested in having some control of his own portfolio. Therefore, I constructed a model of a simplified market, just to see if this *strange* approach could be interesting for such a hypothetical trader, and I suddenly realized that "yes, it might make sense to carry on in this line of research, but, wow, it is hard to have such a paper accepted in a good journal." However, after a few weeks, I also realized that this topic seemed to be interesting not only for me, but also for a large community of scientists, and that this community was increasing very fast, producing more and more contributions on the *ArXiv*. People started citing my first paper, and I was contacted by people interested in what I was doing and who wanted to discuss my point of view. This pushed me in the direction of considering more sophisticated models for stock markets, using my knowledge of quantum mechanics for systems with infinite degrees of freedom in this other, and apparently completely different, field. I thought that this was essentially the end of the story: quantum versus economics. Unexpectedly, a few years ago during a conference in Acireale where I gave a talk on my *quantum stock markets*, I had a discussion with an old friend of mine, Franco Oliveri, and he suggested using the same general strategy in a different classical context. I remember that in our first discussion, we were thinking of foxes and chicken, a topic that was not very exciting for me. After a while, we realized that what we were discussing could also have been used to describe something completely different: a love story. And that was the beginning of a long story that still continues. Since then, we have constructed several models for different classical systems, playing with our understanding of these systems and looking for some phenomenological description. It turned out that these models quite often produce nontrivial and, in my opinion, quite interesting features that are not fully explored yet. Moreover, what is also very intriguing to me is that the same general framework can be used in many different contexts, giving rise to a sort of unifying setting.

This book might be considered as a first attempt to summarize what I have done so far in this field. My idea was to make the book *reasonably simple* and self-contained. This is because I expect that some not necessarily mathematical-minded readers might be intrigued by the title, and I do not want to lose these readers. However, a complex system can be made easy only up to a certain extent, and this is exactly what I have tried to do in these notes. Even the love story I will consider in Chapter 3, which from the purely dynamic point of view is surely the simplest situation, is not simple at all. This is not a big surprise, as almost every lover knows very well from personal experience. I should also clarify that it is not my main interest to discuss the *psychological aspects* behind a love story, a migration process, or the choices of traders in a stock market. I am not even interested in giving any abstract, or too general, description of these systems. Here I want to be quantitative! I want to deduce formulas from general ideas, and I want to see what these formulas imply for the system I have in mind, and if they have some predictive power. However, this ultimate goal implies some effort to the reader, who is required to create his own background on quantum mechanics (if needed) by reading Chapter 2. Dear reader, if you can understand Chapter 2, you can understand the rest! On the other hand, if Chapter 2 is too technical for you, do not worry: you could still try to read the book, simply jumping over this chapter. Of course, if you are not a physicist, you will lose a lot. But you can still get the feeling of what is going on. It is up to you! I really hope you enjoy reading this book!

FABIO BAGARELLO

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It is always a pleasure to thank old friends of mine such as Camillo Trapani and Franco Oliveri for their very precious help in so many different situations that I cannot even remember myself. This could be seen as (the first) evidence of the death of my neurons. But, do not worry! I have still enough neurons left in my brain to remember that if something in functional analysis is not particularly clear, Camillo is the right one! And I also have a post-it on my desk that says "Any numerical problem? Call Franco!" What is more funny is that they still answer my questions, even when they are very, very busy! Franco is also responsible, in part, for what is discussed in these notes, and I also thank him for this scientific collaboration and for his enthusiasm.

It is also a pleasure to thank the various editors and referees whom I have *met* along these years, including the ones who referred this book (before it became a book!). Most of them have contributed significantly to the growth of my research, with many useful, and sometimes unexpected, suggestions. Particular thanks goes to Wiley, for the enthusiasm shown for my manuscript.

I dedicate this book to my beloved parents Giovanna and Benedetto, to my brother Vincenzo, and, *dulcis in fundo,* to Federico, Giovanna, and Grazyna. When I look at them I often ask myself the same question, a question for which I have no answer, yet: how can they resist with so much mathematics and physics going around?

CHAPTER 1

WHY A QUANTUM TOOL IN CLASSICAL CONTEXTS?

Of course, there is no other way to begin this book. In our experience, this is the first question that a referee usually raises when he receives a paper of ours. Hence this is the question that we try to answer in this chapter, to motivate our analysis.

Taking a look at the index, we see that the applications discussed here cover a large variety of problems, from love affairs to migration, from competition between species to stock markets. First of all, we have to stress that we are not claiming that, for instance, a love affair has a quantum nature! (Even though, as every lover knows very well, each love story is surely characterized by a strong stochastic ingredient that could be analyzed, for instance, using tools from probability theory. It is not surprising, then, that one could try to use quantum mechanics as well, in view of its probabilistic interpretation.) Therefore, we are not going to discuss any *quantum love affair*. Rather, we just claim that some quantum tools, and the number operator in particular, can be used, *cum grano salis*, in the analysis of several dynamical systems in which the variables are seen as operator-valued functions. The interesting fact is that the results we deduce using these tools describe very well the dynamics of the system we are considering. This is shown to be true for love affairs first, but this

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same conclusion apparently holds in other, completely different, contexts (migrations, stock markets, competition between species, etc.).

However, to answer in more detail the question raised in the title, we need a long introduction, and this is the main content of this chapter. We begin with a few useful facts on (anti-)commutation rules, which are used to motivate our answer. Then, we describe briefly other appearances of quantum mechanics in the description of classical systems, proposed by several authors in recent years. We conclude the chapter with the plan of the book.

1.1 A FIRST VIEW OF (ANTI-)COMMUTATION RULES

In many fields of quantum mechanics of systems with few or many degrees of freedom, the use of annihilation or creation operators, and of their related number operators, has been proved to be very useful. The first explicit application of the so-called canonical commutation relations (CCR) is usually found at the first level degree in physics while studying the one-dimensional quantum harmonic oscillator (Merzbacher, 1970). CCR are used to produce a purely algebraic procedure that helps in finding the eigenvalues and the eigenvectors of the energy operator $H = \frac{1}{2}\left(p^2 + x^2\right)$ of the oscillator, expressed here in convenient units. This procedure is much simpler than the one that returns the explicit solution of the Schrödinger equation in configuration space (i.e., in terms of the position variable x). The quantum nature of the system is reflected by the fact that p and x do not commute. Indeed, they satisfy the following rule: $[x, p] := xp - px = i\, 1\!\!1$, $1\!\!1$ being the identity operator in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ where the oscillator lives. This means that x and p are not classical functions of time but, rather, operators acting on $\mathcal{H}$. As life is usually not easy, and science is even harder, x and p are unbounded operators. This produces a number of extra difficulties, mainly on the mathematical side, which we try to avoid as much as possible in these notes, but which is necessary at least to mention and to have in mind.[1] Introducing $a := \frac{1}{\sqrt{2}}(x + ip)$, its adjoint $a^\dagger := \frac{1}{\sqrt{2}}(x - ip)$, and $N := a^\dagger a$, we can rewrite the Hamiltonian as $H = N + \frac{1}{2}\,1\!\!1$, and, if φ_0 is a vector of $\mathcal{H}$, which is annihilated by a, $a\varphi_0 = 0$, then, calling $\varphi_n := \frac{1}{\sqrt{n!}}(a^\dagger)^n\varphi_0$, $n = 0, 1, 2, \ldots$, we have $H\varphi_n = (n + 1/2)\varphi_n$. In the literature, φ_0 is called

[1] Along this book, we add some remarks concerning the unboundedness of some operators used in the description of the system under investigation.

the vacuum or *the ground state* of the harmonic oscillator. Then we find, avoiding the hard explicit solution of the Schrödinger differential equation

$$i\frac{\partial\Psi(x,t)}{\partial t} = \frac{1}{2}\left(-\frac{\partial^2\Psi(x,t)}{\partial x^2} + x^2\Psi(x,t)\right),$$

the set of eigenvalues ($E_n = n + 1/2$) and eigenvectors (φ_n) of H. In the derivation of these results, the crucial ingredient is the following commutation rule: $[a, a^\dagger] = \mathbb{1}$, easily deduced from $[x, p] = i\,\mathbb{1}$ and from the definitions of a and $a^\dagger$, which have the useful consequence $[N, {a^\dagger}^n] = n{a^\dagger}^n$, $n = 0, 1, 2, \ldots$. Using this result, $H\varphi_n = E_n\varphi_n$ follows immediately.

Standard quantum mechanical literature states a simple extension of $[a, a^\dagger] = \mathbb{1}$ is found soon after the one-dimensional harmonic oscillator, while moving to higher dimensional systems. In this case, the CCR look like $[a_l, a_n^\dagger] = \mathbb{1}\,\delta_{l,n}$, $l, n = 1, 2, 3, \ldots L$: we have L independent modes. It might be interesting to remind that Lth dimensional oscillators are usually the key ingredients to set up, both at a classical and at a quantum level, many perturbation schemes that are quite useful whenever the dynamics of the system cannot be deduced exactly. Some quantum perturbation approaches are quickly reviewed in Chapter 2 and used all along the book.

Studying quantum field theory, one is usually forced to consider mainly two different kinds of particles, which obey very different commutation rules and, as a consequence of the spin-statistic theorem, different statistics: the *bosons* and the *fermions*. Bosons are particles with integer spin, such as the photons. Fermions are particles with half-integer spin, such as the electrons. This difference in the value of the spin has an important consequence: fermions satisfy the Pauli exclusion principle, whereas bosons do not. This is reflected, first of all, by the wave function that describes any set of identical fermions, which has to be antisymmetric with respect to the change of their variables, or by the wave function for the bosons, which has to be symmetric. Hence, if two indistinguishable fermions have exactly the same quantum numbers (e.g., they occupy the same position in space and they have the same energy), their wave function collapses to 0: such a configuration cannot occur! This is the Pauli exclusion principle, which, of course, does not hold for the bosons. In *second quantization*, (Roman, 1965), the bosons are created by the operators $a_l^\dagger$ and annihilated by their conjugate, a_l. Together, they satisfy the CCR above. Analogously, fermions are annihilated and created by similar operators, b_k and $b_k^\dagger$, but these satisfy a different rule, the so-called anticommutation relation (CAR): $\{b_l, b_k^\dagger\} = b_l\, b_k^\dagger + b_k^\dagger\, b_l = \mathbb{1}\,\delta_{l,k}$, with $\{b_l, b_k\} = \{b_l^\dagger, b_k^\dagger\} = 0$, $l, k = 1, 2, 3, \ldots$. The main difference between these two commutation rules is easily understood. While the operator a_l^2 is different from 0, the square of b_l is automatically 0, together with all its higher powers. This is again an evidence of the Pauli principle: if we try to construct a system with two fermions with the same quantum numbers (labeled by l) using the language of second quantization, we should act twice with $b_l^\dagger$ on the vacuum φ_0, that is, on the vector annihilated by all the b_ls. But, as ${b_l^\dagger}^2 = 0$, the resulting vector is 0: such a state has probability 0 to occur and, as a consequence, the Pauli principle is preserved!

In Chapter 2, we show, among other things, that the eigenvalues of $N_l^{(a)} := a_l^\dagger a_l$ are $0, 1, 2, \ldots$, whereas those of $N_l^{(b)} := b_l^\dagger b_l$ are simply 0, 1. This is related to the fact that the *fermionic* and the *bosonic* Hilbert spaces differ as the first one is finite dimensional, whereas the second is infinite dimensional. Needless to say, this produces severe differences from a technical point of view. In particular, operators acting on a (finite modes) fermionic Hilbert space are automatically bounded, whereas those acting on a bosonic Hilbert space are quite often unbounded.

1.2 OUR POINT OF VIEW

In many classical problems, the relevant quantities we are interested in change discontinuously. For instance, if you consider a certain population $\mathcal{P}$, and its time evolution, the number of people forming $\mathcal{P}$ cannot change arbitrarily: if, at $t_0 = 0$, $\mathcal{P}$ consists of N_0 elements, at $t_1 = t_0 + \Delta t$, $\mathcal{P}$ may only consist of N_1 elements, with $N_1 - N_0 \in \mathbb{Z}$. The same happens if our system consists of two (or more) different populations, $\mathcal{P}_1$ and $\mathcal{P}_2$ (e.g., preys and predators or two migrating species): again, the total number of their elements can only take integer values.

Analogously, if we consider what in these notes is called *a simplified stock market* (SSM), that is, a group of people (the *traders*) with some money and a certain number of shares of different kind, which are exchanged between the traders who pay some cash for that, it is again clear that natural numbers play a crucial role: in our SSM, a trader may have only a natural number of shares (30, 5000, or 10^6, but not 0.75 shares), and a natural number of units of cash (there is nothing < 1 cent of euro, for instance). Hence, if two traders buy or sell a share, the number of shares in their portfolios increases or decreases by one unit, and the amount of money they possess also changes by an integer multiple of the unit of cash.

In the first part of these notes, we also consider some quantities that change continuously but that can also still be measured, quite naturally, using discrete values: this is the case, for instance, of the love affair between Alice and Bob described in Chapter 3: in some old papers, see

Strogatz (1988) and Rinaldi (1998a,b) for instance, the mutual affection between the two actors of the love affair is described by means of two continuous functions. However, it is not hard to imagine how a similar description could be given in terms of discrete quantities: this is what we have done, for instance, in Bagarello and Oliveri (2010, 2011): Bob's affection for Alice is measured by a discrete index, $n_{\rm B}$, which, when it increases during a time interval $[t_i, t_f]$, from, say, a value 1 to the value 2, describes the fact that Bob's love for Alice increases during that particular time interval. Analogously, Alice's affection for Bob can be naturally measured by a second discrete index, $n_{\rm A}$, which, when its value decreases from, say, 1 to 0, describes the fact that Alice's love for Bob simply disappears.

These are just a few examples, all described in detail in these notes, showing how the use of discrete quantities is natural and can be used in the description of several systems, in very different situations. Of course, at first sight, this may look as a simple discretization of a continuous problem, for which several procedures have been proposed along the years. However, this is not our point of view. We adopt here a rather different philosophy, which can be summarized as follows: the discrete quantities used in the description of the system $\mathcal{S}$ under analysis are closely related to the eigenvalues of some self-adjoint operator. Moreover, these operators can be quite often approximated with effective, finite dimensional, self-adjoint matrices, whose dimensions are somehow fixed by the initial conditions; see, for instance, Chapter 3. Then the natural question is the following: how can we deduce the dynamical behavior of $\mathcal{S}$? This is, of course, the hard part of the job! Along all our work, we have chosen to use a Heisenberg-like dynamics, or its Schrödinger counterpart, which we believe is a good choice for the following reasons:

1. It is a natural choice when operators are involved. This is exactly the choice used in quantum mechanics, where the Heisenberg representation is adopted in the description of the dynamics of any closed microscopic system.
2. It is quite easy to write down an energy-like operator, the Hamiltonian H of the system $\mathcal{S}$, which drives the dynamics of the system. This is due to the fact that, following the same ideas adopted in particle physics, the Hamiltonian contains in itself the phenomena it has to describe. This aspect is clarified first in Sections 3.3 and 3.4 in a concrete situation, while in Chapter 6, we discuss the role and the construction of H in more detail and in a very general condition. Among the other criteria, the explicit definition of H will also be suggested by the existence of some conserved quantities of

$\mathcal{S}$: if X is an operator, which is expected to be preserved during the time evolution of $\mathcal{S}$, for instance, the total amount of cash in a closed SSM, then, because of the definition of the Heisenberg dynamics, H must commute with X: $[H, X] = 0$. This gives some extra hints on how to define H explicitly, and then H can be used to find the time evolution of any observable A of $\mathcal{S}$ using the standard prescription: $A(t) = e^{iHt}A(0)e^{-iHt}$, $A(0)$ being the value of A at $t = 0$. We refer to Chapter 2 for many more details on the dynamics of $\mathcal{S}$.

3. It produces results which, at least for those systems considered in the first part of these notes, look quite reasonable; that is, they are exactly those results, which one could expect to find as they reproduce what we observe in real life. This is a good indication, or at least gives us some hope, that the dynamics deduced for the systems discussed in Part II of the book, that is, for SSMs, reflect a reasonable time evolution for those systems.

This list shows that we have two technical and one a posteriori reason to use an energy-like operator H to compute the dynamics of $\mathcal{S}$. This is not, of course, the end of the story, but, in our opinion, it is already a very good starting point.

1.3 DO NOT WORRY ABOUT HEISENBERG!

People with a quantum mechanical background know very well that, whenever incompatible (i.e., not commuting) observables appear in the description of a given physical system $\mathcal{S}$, some uncertainty results follow. Hence, one may wonder how our quantum-like description could be compatible with the classical nature of $\mathcal{S}$, whose observable quantities are not expected to be affected by any error, except, at most, by the error due to the experimental settings. This problem, actually, does not exist at all in the applications considered in these notes as all the observables we are interested in form a commuting subset of a larger nonabelian algebra. Therefore, they can be diagonalized simultaneously and a common orthonormal (o.n.) basis of the Hilbert space $\mathcal{H}$ used in the description of $\mathcal{S}$, made of eigenstates of these observables, can be indeed obtained, as we see several times in Chapters 3–9. This means that, in the complete description of $\mathcal{S}$, all the results that are deduced using our approach are not affected by any uncertainty because all the relevant self-adjoint operators whose eigenvalues are relevant to us are compatible, that is, mutually commuting.

It should also be mentioned that, in some specific applications, the impossibility of observing simultaneously two (apparently) classical quantities has been taken as a strong indication of the relevance of a quantum-like structure in the description of that process, showing, in particular, the importance of noncommuting operators. This is what was proposed, for instance, in Segal and Segal (1998), which is based on the natural assumption that a trader in a market cannot know, at the same time, the price of a certain share and its forward time derivative. The reason is clear: if the trader has access to both these information with absolute precision, then he is surely able to earn as much as he wants! For this reason, Segal and Segal proposed to use two noncommuting operators to describe the price and its time derivative. Going back to the title of this section, although in this book we are happy to not deal with the uncertainty principle, in other approaches this is actually seen as the main motivation to use a quantum or noncommutative approach for a macroscopic system. For this reason, also in view of possible future applications, we describe in Section 2.4 a possible mathematical derivation of a rather general inequality for noncommuting operators, which, fixing the operators in a suitable way, gives back the Heisenberg uncertainty relation.

1.4 OTHER APPEARANCES OF QUANTUM MECHANICS IN CLASSICAL PROBLEMS

Going back to the crucial aspect of this book, which is surely the mixture of quantum and classical words, we want to stress again that this is surely not the first place in which such a mixture is extensively adopted. On the contrary, in the past few years, a growing interest in classical applications of quantum ideas appeared in the literature, showing that more and more people believe that there is not a really big difference between these two worlds or that, at least, some mathematical tool originally introduced in quantum mechanics may also play a significant role in the analysis of classical systems. These kinds of mixtures can be found in very different fields such as economics (Aerts et al., 2012; Segal and Segal, 1998; Schaden, 2002; Baaquie, 2004; Accardi and Boukas, 2006; Al, 2007; Choustova, 2007; Ishio and Haven, 2009; Khrennikov, 2010; Romero et al., 2011; Pedram, 2012), biology (Engel et al., 2007; Arndt et al., 2009; Pusuluk and Deliduman; Martin-Delgado; Panitchayangkoon et al., 2011; Ritz et al. 2004), sociology, and psychology (Shi, 2005; Jimenez and Moya, 2005; Busemeyer et al., 2006; Khrennikov, 2006; Aerts et al., 2009, 2010; Yukalov and Sornette, 2009a,b; Aerts, 2010; Mensky, 2010; Makowski and Piotrowski, 2011; Yukalov and Sornette), and also in more