VLADIMIR MAZ'YA

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SOBOLEV SPACES

WITH APPLICATIONS TO ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Sobolev Spaces

with Applications to Elliptic Partial Differential Equations

2nd, revised and augmented Edition

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To Tatyana

Preface

Sobolev spaces, i.e., the classes of functions with derivatives in L_p, occupy an outstanding place in analysis. During the last half-century a substantial contribution to the study of these spaces has been made; so now solutions to many important problems connected with them are known.

In the present monograph we consider various aspects of theory of Sobolev spaces in particular, the so-called embedding theorems. Such theorems, originally established by S.L. Sobolev in the 1930s, proved to be a useful tool in functional analysis and in the theory of linear and nonlinear partial differential equations.

A part of this book first appeared in German as three booklets of Teubner-Texte für Mathematik [552, 555]. In the Springer volume of "Sobolev Spaces" [556] published in 1985, the material was expanded and revised.

As the years passed the area became immensely vast and underwent important changes, so the main contents of the 1985 volume had the potential for further development, as shown by numerous references. Therefore, and since the volume became a bibliographical rarity, Springer-Verlag offered me the opportunity to prepare the second, updated edition of [556].

As in [556], the selection of topics was mainly influenced by my involvement in their study, so a considerable part of the text is a report on my work in the field. In comparison with [556], the present text is enhanced by more recent results. New comments and the significantly augmented list of references are intended to create a broader and modern view of the area. The book differs considerably from the monographs of other authors dealing with spaces of differentiable functions that were published in the last 50 years.

Each of the 18 chapters of the book is divided into sections and most of the sections consist of subsections. The sections and subsections are numbered by two and three numbers, respectively (3.1 is Sect. 1 in Chap. 3, 1.4.3 is Subsect. 3 in Sect. 4 in Chap. 1). Inside subsections we use an independent numbering of theorems, lemmas, propositions, corollaries, remarks, and so on. If a subsection contains only one theorem or lemma then this theorem or lemma has no number. In references to the material from another section or subsection we first indicate the number of this section or subsection. For example, Theorem 1.2.1/1 means Theorem 1 in Subsect. 1.2.1, (2.6.6) denotes formula (6) in Sect. 2.6.

The reader can obtain a general idea of the contents of the book from the Introduction. Most of the references to the literature are collected in the Comments. The list of notation is given at the end of the book.

The volume is addressed to students and researchers working in functional analysis and in the theory of partial differential operators. Prerequisites for reading this book are undergraduate courses in these subjects.

Acknowledgments

My cordial thanks are to S. Bobkov, Yu.D. Burago, A. Cianchi, E. Milman, S. Poborchi, P. Shvartsman, and I. Verbitsky who read parts of the book and supplied me with their comments.

I wish to express my deep gratitude to M. Nieves for great help in the technical preparation of the text.

The dedication of this book to its translator and my wife T.O. Shaposhnikova is a weak expression of my gratitude for her infinite patience, useful advice, and constant assistance.

Liverpool–Linköping Vladimir Maz'ya
January 2010

Contents

Introduction

In [711–713] Sobolev proved general integral inequalities for differentiable functions of several variables and applied them to a number of problems of mathematical physics. Sobolev considered the Banach space $W_p^l(\Omega)$ of functions in $L_p(\Omega)$, $p \geq 1$, with generalized derivatives of order l integrable with power p. In particular, using these theorems on the potential-type integrals as well as an integral representation of functions, Sobolev established the embedding of $W_p^l(\Omega)$ into $L_q(\Omega)$ or $C(\Omega)$ under certain conditions on the exponents p, l, and q.[1]

Later the Sobolev theorems were generalized and refined in various ways (Kondrashov, Il'in, Gagliardo, Nirenberg, et al.). In these studies the domains of functions possess the so-called cone property (each point of a domain is the vertex of a spherical cone with fixed height and angle which is situated inside the domain). Simple examples show that this condition is precise, e.g., if the boundary contains an outward "cusp" then a function in $W_p^1(\Omega)$ is not, in general, summable with power $pn/(n-p)$, $n > p$, contrary to the Sobolev inequality. On the other hand, looking at Fig. 1, the reader can easily see that the cone property is unnecessary for the embedding $W_p^1(\Omega) \subset L_{2p/(2-p)}(\Omega)$, $2 > p$. Indeed, by unifying Ω with its mirror image, we obtain a new domain with the cone property for which the above embedding holds by the Sobolev theorem. Consequently, the same is valid for the initial domain although it does not possess the cone property.

Now we note that, even before the Sobolev results, it was known that certain integral inequalities hold under fairly weak requirements on the domain. For instance, the Friedrichs inequality ([292], 1927)

$$\int_\Omega u^2 \, \mathrm{d}x \leq K\left(\int_\Omega (\operatorname{grad} u)^2 \, \mathrm{d}x + \int_{\partial\Omega} u^2 \, \mathrm{d}s\right)$$

[1] A sketch of a fairly rich prehistory of Sobolev spaces can be found in Naumann [624].

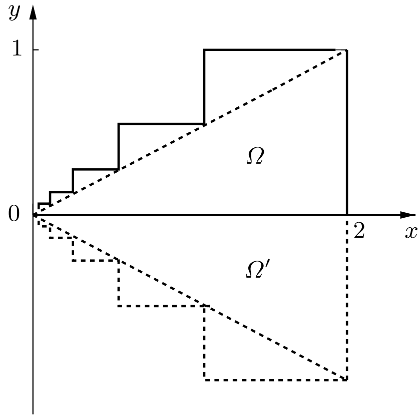

Fig. 1.

was established under the sole assumption that Ω is a bounded domain for which the Gauss–Green formula holds. In 1933, Nikodým [637] gave an example of a domain Ω such that the square integrability of the gradient does not imply the square integrability of the function defined in Ω. The monograph of Courant and Hilbert [216], Chap. 7, contains sufficient conditions for the validity of the Poincaré inequality

$$\int_\Omega u^2\,\mathrm{d}x \le K\int_\Omega (\operatorname{grad} u)^2\,\mathrm{d}x + \frac{1}{m_n\Omega}\left(\int_\Omega u\,\mathrm{d}x\right)^2$$

(see [663, p. 76] and [664, pp. 98–104]) and of the Rellich lemma [672] on the compactness in $L_2(\Omega)$ of the set bounded in the metric

$$\int_\Omega \left[(\operatorname{grad} u)^2 + u^2\right]\mathrm{d}x.$$

The previous historical remarks naturally suggest the problem of describing the properties of domains that are equivalent to various properties of embedding operators.

Starting to work on this problem in 1959 as a fourth-year undergraduate student, I discovered that Sobolev-type theorems for functions with gradients in $L_p(\Omega)$ are valid if and only if some isoperimetric and isocapacitary inequalities hold. Such necessary and sufficient conditions appeared in the early 1960s in my works [527–529, 531, 533, 534]. For $p = 1$ these conditions coincide with isoperimetric inequalities between the volume and the area of a part of the boundary of an arbitrary subset of the domain.

For $p > 1$, geometric functionals such as volume and area prove to be insufficient for an adequate description of the properties of domains. Here inequalities between the volume and the p-capacity or the p-conductivity arise.

Similar ideas were applied to complete characterizations of weight functions and measures in the norms involved in embedding theorems. Moreover, the method of proof of the criteria does not use specific properties of the Euclidean space. The arguments can be carried over to the case of Riemannian manifolds and even abstract metric spaces. A considerable part of the present book (Chaps. 2–9 and 11) is devoted to the development of this isoperimetric and isocapacitary ideology.

However, this theory does not exhaust the material of the book even conceptually. Without aiming at completeness, I mention that other areas of the study in the book are related to the following questions. How massive must a subset e of a domain Ω be in order that the inequality

$$\|u\|_{L_q(\Omega)} \leq C\|\nabla_l u\|_{L_p(\Omega)}$$

holds for all smooth functions vanishing on e? How does the class of domains admissible for integral inequalities depend upon additional requirements imposed upon the behavior of functions at the boundary? What are the conditions on domains and measures involved in the norms ensuring density of a space of differentiable functions in another one? We shall study the criteria of compactness of Sobolev-type embedding operators. Sometimes the best constants in functional inequalities will be discussed. The embedding and extension operators involving Birbaum–Orlicz spaces, the space BV of functions whose gradients are measures, and Besov and Bessel potential spaces of functions with fractional smoothness will also be dealt with.

The investigation of the above-mentioned and similar problems is not only of interest in its own right. By virtue of well-known general considerations it leads to conditions for the solvability of boundary value problems for elliptic equations and to theorems on the structure of the spectrum of the corresponding operators. Such applications are also included.

I describe briefly the contents of the book. More details can be found in the Introductions to the chapters.

Chapter 1 gives prerequisites to the theory. Along with classical facts this chapter contains certain new results. It addresses miscellaneous topics related to the theory of Sobolev spaces. Some of this material is of independent interest and some (Sects. 1.1–1.3) will be used in the sequel. The core of the chapter is a generalized version of Sobolev embedding theorems (Sect. 1.4). We also deal with various extension and approximation theorems (Sects. 1.5 and 1.7), and with maximal algebras in Sobolev spaces (Sect. 1.8). Section 1.6 is devoted to inequalities for functions vanishing on the boundary along with their derivatives up to some order.

The idea of the equivalence of isoperimetric and isocapacitary inequalities on the one hand and embedding theorems on the other hand is crucial for Chap. 2. Most of this chapter deals with the necessary and sufficient conditions for the validity of integral inequalities for gradients of functions that vanish at the boundary. Of special importance for applications are multidimensional inequalities of the Hardy–Sobolev type proved in Sect. 2.1. The basic results of Chap. 2 are applied to the spectral theory of the Schrödinger operator in Sect. 2.5.

Chapters 3 and 4 briefly address the so-called conductor and capacitary inequalities, which are stronger than inequalities of the Sobolev type and are valid for functions defined on quite general topological spaces.

The space $L_p^1(\Omega)$ of functions with gradients in $L_p(\Omega)$ is studied in Chaps. 5–8. Chapter 5 deals with the case $p = 1$. Here, the necessary and sufficient conditions for the validity of embedding theorems stated in terms of the classes $\mathscr{J}_\alpha$ characterized by isoperimetric inequalities are found. We also check whether some concrete domains belong to these classes. In Chaps. 6 and 7 we extend the presentation to the case $p > 1$. Here the criteria are formulated in terms of the p-conductivity. In Chap. 6 we discuss theorems on embeddings into $L_q(\Omega)$ and $L_q(\partial\Omega)$. Chapter 7 concerns embeddings into $L_\infty(\Omega) \cap C(\Omega)$. In particular, we present the necessary and sufficient conditions for the validity of the previously mentioned Friedrichs and Poincaré inequalities and of the Rellich compactness lemma. In Chap. 9 we study the essential norm and other noncompactness characteristics of the embedding operator $L_p^1(\Omega) \to L_q(\Omega)$.

Throughout the book and especially in Chaps. 5–8 we include numerous examples of domains that illustrate possible pathologies of embedding operators. For instance, in Sect. 1.1 we show that the square integrability of second derivatives and of the function do not imply the square integrability of the first derivatives. In Sect. 7.5 we consider the domain for which the embedding operator of $W_p^1(\Omega)$ into $L_\infty(\Omega) \cap C(\Omega)$ is continuous without being compact. This is impossible for domains with "good" boundaries. The results of Chaps. 5–7 show that not only the classes of domains determine the parameters p, q, and so on in embedding theorems, but that a feedback takes place. The criteria for the validity of integral inequalities are applied in Chap. 6 to the theory of elliptic boundary value problems. The exhaustive results on embedding operators can be restated as necessary and sufficient conditions for the unique solvability and for the discreteness of the spectrum of boundary value problems, in particular, of the Neumann problem.

Chapter 9, written together with Yu.D. Burago, is devoted to the study of the space $BV(\Omega)$ consisting of the functions whose gradients are vector charges. Here we present a necessary and sufficient condition for the existence of a bounded nonlinear extension operator $BV(\Omega) \to BV(\mathbb{R}^n)$. We find necessary and sufficient conditions for the validity of embedding theorems for the space $BV(\Omega)$, which are similar to those obtained for $L_1^1(\Omega)$ in Chap. 5. In some integral inequalities we obtain the best constants. The results of Sects. 9.5 and 9.6 on traces of functions in $BV(\Omega)$ make it possible to discuss boundary values of "bad" functions defined on "bad" domains. Along with the results due to Burago and the author in Chap. 9 we present the De Giorgi–Federer theorem on conditions for the validity of the Gauss–Green formula.

Chapters 2–9 mainly concern functions with *first* derivatives in L_p or in C^*. This restriction is essential since the proofs use the truncation of functions along their level surfaces. The next six chapters deal with functions that have derivatives of any integer, and sometimes of fractional, order.

In Chap. 10 we collect (sometimes without proofs) various properties of Bessel and Riesz potential spaces and of Besov spaces in $\mathbb{R}^n$. In Chap. 10 we also present a review of the results of the theory of (p, l)-capacities and of nonlinear potentials.

In Chap. 11 we investigate necessary and sufficient conditions for the validity of the trace inequality

$$\|u\|_{L_q(\mu)} \le C\|u\|_{S_p^l}, \quad u \in C_0^\infty\big(\mathbb{R}^n\big), \tag{0.0.1}$$

where $L_q(\mu)$ is the space with the norm $(\int |u|^q \,\mathrm{d}\mu)^{1/q}$, μ is a measure, and S_p^l is one of the spaces just mentioned. For $q \ge p$, (0.0.1) is equivalent to the isoperimetric inequality connecting the measure μ and the capacity generated by the space S_p^l. This result is of the same type as the theorems in Chaps. 2–9. It immediately follows from the capacitary inequality

$$\int_0^\infty \operatorname{cap}\big(\mathscr{N}_t; S_p^l\big)t^{p-1}\,\mathrm{d}t \le C\|u\|^p_{S_p^l},$$

where $\mathscr{N}_t = \{x : |u(x)| \ge t\}$. Inequalities of this type, initially found by the author for the spaces $L_p^1(\Omega)$ and $\mathring{L}_p^2(\mathbb{R}^n)$ [543], have proven to be useful in a number of problems of function theory and were intensively studied.

For $q > p \ge 1$ the criteria for the validity of (0.0.1), presented in Chap. 11 do not contain a capacity. In this case the measure of any ball is estimated by a certain function of the radius.

Chapter 12 is devoted to pointwise interpolation inequalities for derivatives of integer and fractional order.

Further, in Chap. 13 we introduce and study a certain kind of capacity. In comparison with the capacities defined in Chap. 10, here the class of admissible functions is restricted, they equal the unity in a neighborhood of a compactum. (In the case of the capacities in Chap. 10, the admissible functions majorize the unity on a compactum.) If the order l of the derivatives in the norm of the space equals 1, then the two capacities coincide. For $l \ne 1$ they are equivalent, which is proved in Sect. 13.3.

The capacity introduced in Chap. 13 is applied in subsequent chapters to prove various embedding theorems. An auxiliary inequality between the L_q-norm of a function on a cube and a certain Sobolev seminorm is studied in detail in Chap. 14. This inequality is used to justify criteria for the embedding of $\mathring{L}_p^l(\Omega)$ into different function spaces in Chap. 15. By $\mathring{L}_p^l(\Omega)$ we mean the completion of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla_l u\|_{L_p(\Omega)}$. It is known that this completion is not embedded, in general, into the distribution space $\mathscr{D}'$. In Chap. 15 we present the necessary and sufficient conditions for the embeddings of $\mathring{L}_p^l(\Omega)$ into $\mathscr{D}'$, $L_q(\Omega, \mathrm{loc})$, and $L_p(\Omega)$. For $p = 2$, these results can be interpreted as necessary and sufficient conditions for the solvability of the Dirichlet problem for the polyharmonic equation in unbounded domains provided the right-hand side is contained in $\mathscr{D}'$ or in $L_q(\Omega)$. In Chap. 16 we find criteria for the boundedness and the compactness of the embedding operator of the space $\mathring{L}_p^l(\Omega,\nu)$ into $W_q^r(\Omega)$, where ν is a measure and $\mathring{L}_p^l(\Omega,\nu)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\left(\int_\Omega |\nabla_l u|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}\nu\right)^{1/p}.$$

The topic of Chap. 17 is a necessary and sufficient condition for density of $C_0^\infty(\Omega)$ in a certain weighted Sobolev space which appears in applications. Finally, Chap. 18 contains variations on the theme of Molchanov's discreteness criterion for the spectrum of the Schrödinger operator as well as two-sided estimates for the first Dirichlet–Laplace eigenvalue.

Obviously, it is impossible to describe such a vast area as Sobolev spaces in one book. The treatment of various aspects of this theory can be found in the books by Sobolev [713]; R.A. Adams [23]; Nikolsky [639]; Besov, Il'in, and Nikolsky [94]; Gel'man and Maz'ya [305]; Gol'dshtein and Reshetnyak [316]; Jonsson and Wallin [408]; Ziemer [813]; Triebel [758–760]; D.R. Adams and Hedberg [15]; Maz'ya and Poborchi [576]; Burenkov [155]; Hebey [361]; Haroske, Runst, and Schmeisser [354]; Hajłasz [342]; Saloff-Coste [687]; Attouch, Buttazzo, and Michaille [54]; Tartar [744]; Haroske and Triebel [355]; Leoni [486]; Maz'ya and Shaposhnikova [588]; Maz'ya [565]; and A. Laptev (Ed.) [479].

1

Basic Properties of Sobolev Spaces

The plan of this chapter is as follows. Sections 1.1 and 1.2 contain the prerequisites on Sobolev spaces and other function analytic facts to be used in the book. In Sect. 1.3 a complete study of the one-dimensional Hardy inequality with two weights is presented. The case of a weight of unrestricted sign on the left-hand side is also included here, following Maz'ya and Verbitsky [593]. Section 1.4 contains theorems on necessary and sufficient conditions for the L_q integrability with respect to an arbitrary measure of functions in $W_p^l(\Omega)$. These results are due to D.R. Adams, $p > 1$, [2, 3] and the author, $p = 1$, [551]. Here, as in Sobolev's papers, it is assumed that the domain is "good," for instance, it possesses the cone property. In general, in requirements on a domain in Chap. 1 we follow the "all or nothing" principle. However, this rule is violated in Sect. 1.5 which concerns the class preserving extension of functions in Sobolev spaces. In particular, we consider an example of a domain for which the extension operator exists and which is not a quasicircle.

In Sect. 1.6 an integral representation of functions in $W_p^l(\Omega)$ that vanish on $\partial\Omega$ along with all their derivatives up to order $k-1$, $2k \geq l$, is obtained. This representation entails the embedding theorems of the Sobolev type for any bounded domain Ω. In the case $2k < l$ it is shown by example that some requirements on $\partial\Omega$ are necessary. Section 1.7 is devoted to an approximation of Sobolev functions by bounded ones. Here we reveal a difference between the cases $l = 1$ and $l > 1$. The chapter finishes with a discussion in Sect. 1.8 of the maximal subalgebra of $W_p^l(\Omega)$ with respect to multiplication.

1.1 The Spaces $L_p^l(\Omega)$, $V_p^l(\Omega)$ and $W_p^l(\Omega)$

1.1.1 Notation

Let Ω be an open subset of n-dimensional Euclidean space $\mathbb{R}^n = \{x\}$. Connected open sets Ω will be called domains. The notations $\partial\Omega$ and $\bar{\Omega}$ stand for the boundary and the closure of Ω, respectively. Let $C^\infty(\Omega)$ denote the space

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of infinitely differentiable functions on Ω; by $C^\infty(\bar{\Omega})$ we mean the space of restrictions to Ω of functions in $C^\infty(\mathbb{R}^n)$.

In what follows $\mathscr{D}(\Omega)$ or $C_0^\infty(\Omega)$ is the space of functions in $C^\infty(\mathbb{R}^n)$ with compact supports in Ω. The classes $C^k(\Omega)$, $C^k(\bar{\Omega})$, and $C_0^k(\Omega)$ of functions with continuous derivatives of order k and the classes $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}(\bar{\Omega})$, and $C_0^{k,\alpha}(\Omega)$ of functions for which the derivatives of order k satisfy a Hölder condition with exponent $\alpha \in (0,1]$ are defined in an analogous way.

Let $\mathscr{D}'(\Omega)$ be the space of distributions dual to $\mathscr{D}(\Omega)$ (cf. L. Schwartz [695], Gel'fand and Shilov [304]). Let $L_p(\Omega)$, $1 \le p < \infty$, denote the space of Lebesgue measurable functions, defined on Ω, for which

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p \,\mathrm{d}x\right)^{1/p} < \infty.$$

We use the notation $L_\infty(\Omega)$ for the space of essentially bounded Lebesgue measurable functions, i.e., uniformly bounded up to a set of measure zero. As a norm of f in $L_\infty(\Omega)$ one can take its essential supremum, i.e.,

$$\|f\|_{L_\infty(\Omega)} = \inf\bigl\{c > 0 : |f(x)| \le c \text{ for almost all } x \in \Omega\bigr\}.$$

By $L_p(\Omega, \mathrm{loc})$ we mean the space of functions locally integrable with power p in Ω. The space $L_p(\Omega, \mathrm{loc})$ can be naturally equipped with a countable system of seminorms $\|u\|_{L_p(\omega_k)}$, where $\{\omega_k\}_{k\ge1}$ is a sequence of domains with compact closures $\bar{\omega}_k$, $\bar{\omega}_k \subset \omega_{k+1} \subset \Omega$, and $\bigcup_k \omega_k = \Omega$. Then $L_p(\Omega, \mathrm{loc})$ becomes a complete metrizable space.

If $\Omega = \mathbb{R}^n$ we shall often omit Ω in notations of spaces and norms. Integration without indication of limits extends over $\mathbb{R}^n$. Further, let $\operatorname{supp} f$ be the support of a function f and let $\operatorname{dist}(F, E)$ denote the distance between the sets F and E. Let $B(x, \varrho)$ or $B_\varrho(x)$ denote the open ball with center x and radius ϱ, $B_\varrho = B_\varrho(0)$. We shall use the notation m_n for n-dimensional Lebesgue measure in $\mathbb{R}^n$ and v_n for $m_n(B_1)$.

Let c, c_1, $c_2, \dots,$ denote positive constants that depend only on "dimensionless" parameters n, p, l, and the like. We call the quantities a and b equivalent and write $a \sim b$ if $c_1 a \le b \le c_2 a$. If α is a multi-index $(\alpha_1, \dots, \alpha_n)$, then, as usual, $|\alpha| = \sum_j \alpha_j$, $\alpha! = \alpha_1!, \dots, \alpha_n!$, $D^\alpha = D_{x_1}^{\alpha_1}, \dots, D_{x_n}^{\alpha_n}$, where $D_{x_i} = \partial/\partial x_i$, $x^\alpha = x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$. The inequality $\beta \ge \alpha$ means that $\beta_i \ge \alpha_i$ for $i = 1, \dots, n$. Finally, $\nabla_l = \{D^\alpha\}$, where $|\alpha| = l$ and $\nabla = \nabla_1$.

1.1.2 Local Properties of Elements in the Space $L_p^l(\Omega)$

Let $L_p^l(\Omega)$ denote the space of distributions on Ω with derivatives of order l in the space $L_p(\Omega)$. We equip $L_p^l(\Omega)$ with the seminorm

$$\|\nabla_l u\|_{L_p(\Omega)} = \left(\int_\Omega \left(\sum_{|\alpha|=l} \bigl|D^\alpha u(x)\bigr|^2\right)^{p/2}\right)^{1/p}.$$