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Preface

This book is devoted to the study of value distribution of functions which are meromorphic on the complex plane or in an angular domain with vertex at the origin. We characterize such meromorphic functions in terms of distribution of some of their value points. The study, together with certain related topics, is known as theory of value distribution of meromorphic functions. The theory is too vast to be justified within a single work. Therefore we selected and organized the content based on their significant importance to our understanding and interests in this book. I gladly acknowledge my indebtedness in particular to the books of M. Tsuji, A. A. Goldberg and I. V. Ostrovskii, Yang L. and the papers of A. Eremenko.

An outline of the book is provided below. The introduction of the Nevanlinna characteristic to the study of meromorphic functions is a new starting symbol of the theory of value distribution. The Nevanlinna characteristic is powerful, and its thought has been used to produce various characteristics such as the Nevanlinna characteristic and Tsuji characteristic for an angular domain. And from geometric point of view, namely the Ahlfors theory of covering surfaces, the Ahlfors-Shimizu characteristic have also been introduced. These characteristics are real-valued functions defined on the positive real axis. Therefore, in the first chapter, we collect the basic results about positive real functions that are often used in the study of meromorphic function theory. Some of these results are distributed in other books, some in published papers, and some have been newly established in order to serve our specific objectives in the book.

In the present book, we discuss value distribution not only in the complex plane, but also in an angular domain. Therefore, we introduce, in the second chapter, various characteristics of a meromorphic function: The Nevanlinna characteristic for a disk, the Nevanlinna characteristic for an angle, the Tsuji characteristic and the Ahlfors-Shimizu characteristic for an angle. Although they were distributed in another books, we collected all of them, and more importantly, we carefully compared them with one another to reveal their relations that enabled us to produce new results and applications. We establish the first and second fundamental theorems for the various characteristics and the corresponding integrated counting functions, and provide an estimate of the error term related to the Nevanlinna characteristic for an angle in terms of the Nevanlinna characteristic in a larger angle. We discuss in an angle the growth order of a meromorphic function and exponent of convergence of its a-points by means of the Ahlfors-Shimizu characteristic. We establish unique theorems in an angular domain with the help of the Tsuji characteristic, which is a new topic, because this has never been touched before while only the case of the whole complex plane was discussed.

After providing a brief overview of the characteristics in Chapter 2, we carefully investigate, in the third chapter, a new singular direction of a meromorphic function called T direction, which is different from the Julia, Borel and Nevanlinna directions. A singular direction is characterized essentially with the help of a property that in any angle containing it, the function assumes abundantly any value possibly except at most two values. The word "abundantly" is expressed by "infinitely often" for the Julia directions and by the growth order of the function for the Borel directions. The definition of T directions is to compare the integrated counting function in an angle to the characteristic and so it does not depend on the growth order, which is different from the Borel directions. So we can naturally consider T directions of meromorphic functions with zero order or infinite order. The second fundamental theorem of Nevanlinna is considered as the background of T directions. The following inequality

$$\limsup_{r\to\infty}\frac{N(r,\mathbb{C},f=a)}{T(r,f)}>0$$

always holds for all but at most two values of a. For a T direction, we consider the above inequality in any angle containing it instead of the whole complex plane. First we discuss the existence of T directions including the case of small functions in our consideration, next do relationship with the Borel directions, then common T directions of the function and its derivatives including the Hayman T directions. The singular directions of meromorphic solutions of linear differential equations possess some special properties, which are carefully studied and finally, we survey the results on the uniqueness and singular directions of an algebroid function.

The book includes discussion of argument distribution as well as modulo distribution and their relations. In the fourth chapter, we reveal relations between the numbers of deficient values and T directions. The results established there are new and unpublished elsewhere. The essential idea for discussion of this topic comes from observation that if the function assumes two values a and b at few points and is in close proximity to a complex number $c\neq a,b$ at enough points in a bounded domain, then it is close to c in the whole domain possibly outside a small set and that if the function is analytic, in view of the two constant theorem for the harmonic measure, we can use the modulo of the function on some part of the boundary of the domain to control the function modulo inside the domain. In the final section, we make a survey on this topic.

In the fifth chapter, we discuss the growth of the meromorphic functions that have two radially distributed values and a Nevanlinna deficient value. We first consider the growth of the meromorphic functions without any restriction imposed on their order and then those with the finite lower order. We attain our purpose in terms of the Nevanlinna characteristic for an angle, as Goldberg and Ostrovskii did, but our starting point is to establish an estimate of the Nevanlinna characteristic for a disk centered at the origin in terms of $B_{\alpha,\beta}(r,f)$ under an observation of the Nevanlinna deficient value, and then $B_{\alpha,\beta}(r,f)$ is estimated by two $C_{\alpha,\beta}(r,*)$ which may deal with the derivatives with help of fundamental inequalities for the Nevanlinna characteristic for an angle, and finally, $C_{\alpha,\beta}(r,*)$ are replaced by the integrated counting functions $N(r,\Omega,*)$ in terms of the relations between them. Thus the Nevanlinna characteristic for a disk can be estimated by two $N(r,\Omega,*)$ and we reduce the study of this subject to estimation of $B_{\alpha,\beta}$ in terms of $C_{\alpha,\beta}$. However, this comes from the study of fundamental inequality for the Nevanlinna characteristic for an angle. As we know, most of the fundamental inequalities for a disk can be validly and easily transferred to the case of an angle and therefore, we give out a simple and elementary approach to the discussions of this subject. When the function is of the finite lower order, we use the Baernstein spread relation to discuss the estimation of the Nevanlinna characteristic for a disk in terms of $B_{\alpha,\beta}(r,f)$ and hence we can attain deeper results for this subject.

In the sixth chapter, we collect and develop results about singularities of the inverse of a meromorphic function. A transcendental meromorphic function is equipped with a parabolic simply connected Riemann surface. The boundary points of the Riemann surface correspond to transcendental singularities of the inverse of the function, that is, asymptotic values of the function, and vice versa. We discuss relationships between the number of direct singularities and the growth (lower) order. The isolated transcendental singularity is logarithmic, and hence we observe that an asymptotic value over which the singularity is not logarithmic is a limit of other singular values. For a meromorphic function of finite order, such an asymptotic value is a limit point of critical values , which is the Bergweiler-Eremenko's result. We show Eremenko's construction of a transcendental meromorphic function with the finite given order which has every value on the extended complex plane as its asymptotic value, and next discuss the fixed points of bounded-type meromorphic functions, that is, meromorphic functions whose singular value set are bounded, from which we obverse that meromorphic functions possess special characters if their singular values are suitably restricted.

The final chapter is mainly devoted to the Eremenko's proof of the famous F. Nevanlinna conjecture on meromorphic functions with maximum total sum of Nevanlinna deficiencies. The conjecture was proved first by David Drasin, but his proof is very complicated. A. Eremenko used the potential theory to give a simple proof to the conjecture, from which we see the power of the potential theory in the study of value distribution of meromorphic functions. The theory to study subharmonic functions is the potential theory. The defence of two subharmonic functions is called δ-subharmonic. The logarithm of modulo of a meromorphic function is a δ-subharmonic function. Therefore, some problems about value distribution of meromorphic functions can be transferred to those about subharmonic functions. And the limit functions of a sequence of subharmonic functions produced by the subharmonic function in question are easier to be characterized than the meromorphic functions. The property or behavior of the limit functions can be used to describe the meromorphic functions. This is one of the approaches in which the potential theory are used to discuss problems about meromorphic functions.

For the benefit of readers, and for our intent to introduce and develop the potential theory in value distributions, we introduce and gather the basic knowledge about the potential theory including the normality of subharmonic function family in the sense of $\mathscr{L}_{\mathrm{loc}}$ and the Nevanlinna theory of subharmonic functions which consist of works of Anderson, Baernstein, Eremenko, Sodin, and others. The works of these mathematicians are very special and very important, and in our opinion, represent one aspect of value distribution theory which is worth further investigating and developing.

The first draft of this book was finished at the end of 2006, and main content of the book, except the seventh chapter was lectured in the summer course for postgraduated students held at Jiang Xi Normal University in the summer of 2007. I am indebted to Professor Yi Caifeng for her organizing the summer school, to Professor He Yuzhan for his comments and offering me some important materials, and to Professor Ye Zhuang for his support of this book. I would like to send many thanks to others including my students who pointed out some mistakes or some tough statements in the original draft when they read. This book has been partially supported by the National Natural Science Foundation of China.

Jianhua Zheng
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Contents

Chapter 1
Preliminaries of Real Functions

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Abstract: The various characteristics of meromorphic functions are main tool in the study of value distribution of meromorphic functions this book will introduce. They are real-valued functions defined on the positive real axis. In this chapter, we discuss certain properties of such real functions for application in later chapters. We begin with the order and the lower order of such functions which include the proximate order and the type function. We discuss the existence of the Pólya peak sequence. Also, we identify a sequence of positive numbers with some of the Pólya peak properties. We mainly introduce a result of Edrei and Fuchs for the regularity, thereby, improving the lemma of Borel and quasi-invariance of inequalities of two real functions under differentiation and integration. Finally, we exhibit the Green formula and collect several integral inequalities.

Key words: Real functions, Proximate order, Pólya peak, Regularity, Quasi-invariance

1.1 Functions of a Real Variable

In investigation of theory of meromorphic functions, we often meet the study of some properties of functions of a real variable, because various characteristics of meromorphic functions are such functions. Therefore, in this section, we collect the main properties of such functions which will be frequently used in the sequel.

1.1.1 The Order and Lower Order of a Real Function

Let $T(r)$ be a non-negative continuous function on $[r_0,\infty)$ for some $r_0 \geqslant 0$ and define $\log^+ x = \log\max\{1,x\}$. For $T(r)$, we define its lower order μ and order λ in turn as follows:

$$\mu = \mu(T) = \liminf_{r\to\infty} \frac{\log^+ T(r)}{\log r}$$

and

$$\lambda = \lambda(T) = \limsup_{r\to\infty} \frac{\log^+ T(r)}{\log r}.$$

We concentrate mainly on the function $T(r)$ which tends to infinity as r does. The order of a positive increasing continuous function can be characterized in term of an integral value.

Lemma 1.1.1. *Let $T(r)$ be a continuous, non-decreasing and positive function on $[r_0,\infty)$. Then for each $\rho < \lambda(T)$, we have*

$$\int^{\infty} \frac{T(t)}{t^{\rho+1}} \mathrm{d}t = \infty;$$

Conversely, if the above equation holds for certain ρ, then $\lambda(T) \geqslant \rho$.

Proof. Suppose that the integral is finite, and then for all $r \geqslant r_0$,

$$K > \int_r^{2r} \frac{T(t)}{t^{\rho+1}} \mathrm{d}t \geqslant \frac{T(r)}{(2r)^{\rho+1}} r = 2^{-\rho-1} T(r) r^{-\rho},$$

where $K = \int_{r_0}^{\infty} \frac{T(t)}{t^{\rho+1}} \mathrm{d}t$. This immediately deduces $\lambda(T) \leqslant \rho$ and the former half part of the lemma follows.

If $\lambda(T) < \rho$, then for each s with $\lambda(T) < s < \rho$, we have $T(r) < r^s$ for all sufficiently large r. Thus $T(r)r^{-\rho-1} < r^{-(\rho-s)-1}$, which yields the integral $\int^{\infty} \frac{T(t)}{t^{\rho+1}} \mathrm{d}t$ is convergent.

This completes the proof of Lemma 1.1.1. □

A continuous function may be too complicated to grasp, and thus sometime it is necessary to modify it by preserving, roughly speaking, only the values of r at which $T(r)$ can be approximately written into r^{λ}. The precise statement is as under

Theorem 1.1.1. *Let $T(r)$ be a continuous and positive function for $r \geqslant r_0 > 0$ and tend to infinity as $r \to \infty$ with $\lambda = \lambda(T) < \infty$. Then, there exists a function $\lambda(r)$ with the following properties:*

(1) $\lambda(r)$ *is a monotone and piecewise continuous differentiable function for $r \geqslant r_0$ with* $\lim_{r\to\infty} \lambda(r) = \lambda$;

(2) $\lim_{r\to\infty} \lambda'(r) r \log r = 0$;

(3) $\limsup_{r\to\infty} \frac{T(r)}{r^{\lambda(r)}} = 1$;

(4) *for each positive number d,*

$$\lim_{r\to\infty} \frac{U(dr)}{U(r)} = d^{\lambda}, \qquad U(r) = r^{\lambda(r)}. \tag{1.1.1}$$

We shall call the function $\lambda(r)$ the proximate order of $T(r)$ and the function $U(r)$ the type function of $T(r)$. It is obvious that the proximate order and the type function of a real function are not unique. As $\lambda > 0$, $U(r) = \mathrm{e}^{\lambda(r)\log r}$ is increasing for all larger r. A simple calculation implies that a monotone increasing function $T(r)$ satisfying (1.1.1) must have $\mu(T) = \lambda(T) = \lambda$. The formula (1.1.1) is the key point of Theorem 1.1.1 and it makes sense essentially for the limit being finite. This explains the necessity for the condition that a function $T(r)$ in question is of finite order. However, in the case of infinite order, we have the following

Theorem 1.1.2. *Let $T(r)$ be a continuous and positive function for $r \geqslant r_0 > 0$ and tend to infinity as $r \to \infty$ with $\lambda = \lambda(T) = \infty$. Assume that $\omega(r)$ is a positive, continuous and non-increasing function with $\int_1^\infty \frac{\omega(t)}{t}\mathrm{d}t < +\infty$.*

Then, there exists a function $\lambda(r)$ with the following properties

(1) *$\lambda(r)$ is non-decreasing and continuous and tends to infinity as $r \to \infty$;*

(2) $\limsup\limits_{r\to\infty} \frac{T(r)}{r^{\lambda(r)}} = 1$;

(3) *Set $U(r) = r^{\lambda(r)}$ and*

$$\lim_{r\to\infty} \frac{U(r+\omega(U(r)))}{U(r)} = 1. \tag{1.1.2}$$

The proofs of Theorem 1.1.1 and Theorem 1.1.2 can be found in Chuang [2].

The following result will be used often in the next chapters.

Lemma 1.1.2. *Let $T(r)$ be a non-negative and non-decreasing function in $0 < r < \infty$. If*

$$\liminf_{r\to\infty} \frac{T(dr)}{T(r)} \geqslant c > 1$$

for some $d > 1$, then

$$\int_1^r \frac{T(t)}{t}\mathrm{d}t \leqslant \frac{2c\log d}{c-1} T(r) + O(1);$$

If

$$\liminf_{r\to\infty} \frac{T(dr)}{T(r)} > d^\omega$$

for some $d > 1$ and $\omega > 0$, then

$$\int_1^r \frac{T(t)}{t^{\omega+1}}\mathrm{d}t \leqslant K\frac{T(r)}{r^\omega} + O(1),$$

where K is a positive constant.

Proof. Write $s = \frac{c+1}{2}$ and we can find a natural number N such that for $r \geqslant r_0 = d^N$, we have $T(d^{-1}r) < s^{-1}T(r)$. Then for each $r \geqslant r_0 = d^N$, we have $n \geqslant N$ such that $d^n \leqslant r < d^{n+1}$, and let us estimate the following integral

$$\begin{aligned}\int_{r_0}^{r}\frac{T(t)}{t}\mathrm{d}t &= \sum_{k=N}^{n-1}\int_{d^k}^{d^{k+1}}\frac{T(t)}{t}\mathrm{d}t+\int_{d^n}^{r}\frac{T(t)}{t}\mathrm{d}t\\ &\leqslant \sum_{k=N}^{n-1}T(d^{k+1})\log d+T(r)\log d\\ &= T(d^n)\log d\sum_{k=N}^{n-1}\frac{T(d^{k+1})}{T(d^n)}+T(r)\log d\\ &< T(d^n)\log d\sum_{k=0}^{\infty}s^{-k}+T(r)\log d\\ &\leqslant \frac{2c\log d}{c-1}T(r).\end{aligned}$$

This yields the first desired inequality.

Now, we come to the proof of the second part of Lemma 1.1.2. Under the given assumption, for $r\geqslant r_0=d^N$ and some $\varepsilon>0$ we have $T(d^{-1}r)<(d+\varepsilon)^{-\omega}T(r)$. Thus, it follows that

$$\begin{aligned}\int_{r_0}^{r}\frac{T(t)}{t^{\omega+1}}\mathrm{d}t &= \sum_{k=N}^{n-1}\int_{d^k}^{d^{k+1}}\frac{T(t)}{t^{\omega+1}}\mathrm{d}t+\int_{d^n}^{r}\frac{T(t)}{t^{\omega+1}}\mathrm{d}t\\ &\leqslant \sum_{k=N}^{n-1}T(d^{k+1})\frac{1}{\omega}\left(\frac{1}{d^{k\omega}}-\frac{1}{d^{(k+1)\omega}}\right)+T(r)\frac{1}{\omega}\left(\frac{1}{d^{n\omega}}-\frac{1}{r^{\omega}}\right)\\ &< \frac{1}{\omega}T(d^n)\sum_{k=N}^{n-1}(d+\varepsilon)^{-\omega(n-k-1)}\left(\frac{1}{d^{k\omega}}-\frac{1}{d^{(k+1)\omega}}\right)+\frac{1}{\omega}\frac{T(r)}{d^{n\omega}}\\ &< \frac{d^{\omega}-1}{\omega}\frac{T(d^n)}{(d+\varepsilon)^{n\omega}}\frac{\left(\frac{d+\varepsilon}{d}\right)^{(n+1)\omega}-1}{\left(\frac{d+\varepsilon}{d}\right)^{\omega}-1}+\frac{1}{\omega}\frac{T(r)}{d^{n\omega}}\\ &\leqslant K_0\frac{T(r)}{d^{n\omega}}<K_0d^{\omega}\frac{T(r)}{r^{\omega}},\end{aligned}$$

where $K_0=\frac{d^{\omega}-1}{\omega}\frac{(d+\varepsilon)^{\omega}}{(d+\varepsilon)^{\omega}-d^{\omega}}+\frac{1}{\omega}$.

This completes the proof of Lemma 1.1.2. □

1.1.2 The Pólya Peak Sequence of a Real Function

In this subsection, we consider the Pólya peak for a $T(r)$, which was first introduced by Edrei [6].

Definition 1.1.1. *A sequence of positive numbers $\{r_n\}$ is called a sequence of Pólya peaks of order β for $T(r)$ (outside a set E) provided that there exist four sequences $\{r'_n\}$, $\{r''_n\}$, $\{\varepsilon_n\}$ and $\{\varepsilon'_n\}$ such that*

(1) $r_n\notin E,\ r'_n\to\infty,\ \frac{r_n}{r'_n}\to\infty,\ \frac{r''_n}{r_n}\to\infty,\ \varepsilon_n\to 0,\ \varepsilon'_n\to 0\ (n\to\infty)$;

(2) $\liminf\limits_{n\to\infty} \frac{\log T(r_n)}{\log r_n} \geqslant \beta$;

(3) $T(t) < (1+\varepsilon_n)\left(\frac{t}{r_n}\right)^{\beta} T(r_n)$, $t \in [r'_n, r''_n]$;

(4) $T(t)/t^{\beta-\varepsilon'_n} \leqslant KT(r_n)/r_n^{\beta-\varepsilon'_n}$, $1 \leqslant t \leqslant r''_n$ *and for a positive constant* K.

Actually, it is easy to see that (2) follows from (4). It is obvious that any subsequence of a Pólya peak sequence is still a sequence of the Pólya peak. Please note that the above definition of the Pólya peaks has some differences from that in other literatures where a sequence of Pólya peak is only required to satisfy (1) and (3) listed in Definition 1.1.1. The sequence $\{r_n\}$ is called a sequence of relaxed Pólya peaks of order β for a constant $C > 1$, provided that (1), (2) and (4) in Definition 1.1.1 hold and (3) does for C in place of "$(1+\varepsilon_n)$". It is easily seen that for a sequence $\{r_n\}$ of Pólya peak and $d \geqslant 1$, $\{dr_n\}$ must be a sequence of the relaxed Pólya peak.

The following is a modifying version of well-known result which can be found in Section 8.1 of Yang [12].

Theorem 1.1.3. *Let $T(r)$ be a non-negative and non-decreasing continuous function in $0 < r < \infty$ with $0 \leqslant \mu(T) < \infty$ and $0 < \lambda(T) \leqslant \infty$. Then for arbitrary finite and positive number β satisfying $\mu \leqslant \beta \leqslant \lambda$ and a set F with finite logarithmic measure, i.e., $\int_F t^{-1}\mathrm{d}t < \infty$, there exists a sequence of the Pólya peaks of order β for $T(r)$ outside F.*

Proof. We choose a sequence of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$. By induction, we seek the desired Pólya peak sequence $\{r_n\}$. Suppose we have r_{n-1} and want to find r_n.

First of all consider the case when $\beta = \lambda(T) < \infty$. It is easy to see that for n,

$$\limsup_{t\to\infty} \frac{T(t)}{t^{\beta-\varepsilon_n}} = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{T(t)}{t^{\beta+\varepsilon_n}} = 0.$$

Therefore, we can find a real number $u > \max\{n\varepsilon_n^{-1}, r_{n-1}\}$ such that

$$T(u)u^{-\beta+\varepsilon_n} = \max_{1\leqslant t\leqslant u}\{T(t)t^{-\beta+\varepsilon_n}\}$$

and a $v \geqslant u$ such that

$$T(v)v^{-\beta-\varepsilon_n} = \max_{t\geqslant u}\{T(t)t^{-\beta-\varepsilon_n}\}.$$

We choose r_n with $u \leqslant r_n \leqslant v$ such that

$$T(r_n)r_n^{-\beta+\varepsilon_n} = \max_{u\leqslant t\leqslant v}\{T(t)t^{-\beta+\varepsilon_n}\} \geqslant T(u)u^{-\beta+\varepsilon_n}.$$

Thus for $t \leqslant v$, we have

$$T(r_n)r_n^{-\beta+\varepsilon_n} \geqslant T(t)t^{-\beta+\varepsilon_n} \tag{1.1.3}$$

and for $t \geqslant r_n$

$$T(t)t^{-\beta-\varepsilon_n} \leqslant T(v)v^{-\beta-\varepsilon_n} \leqslant T(r_n)r_n^{-\beta+\varepsilon_n}v^{-2\varepsilon_n} \leqslant T(r_n)r_n^{-\beta-\varepsilon_n}$$

and, therefore, for $r_n \leqslant t \leqslant r_n/\varepsilon_n$,

$$\begin{aligned} T(t)t^{-\beta+\varepsilon_n} &= T(t)t^{-\beta-\varepsilon_n}t^{2\varepsilon_n} \leqslant T(r_n)r_n^{-\beta-\varepsilon_n}t^{2\varepsilon_n} \\ &= T(r_n)r_n^{-\beta+\varepsilon_n}\left(\frac{t}{r_n}\right)^{2\varepsilon_n} \\ &\leqslant \left(\frac{1}{\varepsilon_n}\right)^{2\varepsilon_n} T(r_n)r_n^{-\beta+\varepsilon_n}. \end{aligned} \tag{1.1.4}$$

Combining (1.1.3) and (1.1.4) deduces that r_n satisfies (4) for $r_n'' = r_n/\varepsilon_n$. This also immediately yields

$$T(t) \leqslant \mathrm{e}^{-2\varepsilon_n \log \varepsilon_n}\left(\frac{t}{r_n}\right)^{\beta} T(r_n) \ \text{ for } \ \varepsilon_n r_n \leqslant t \leqslant \varepsilon_n^{-1} r_n. \tag{1.1.5}$$

Now let us consider the case when $\mu \leqslant \beta < \lambda$. Assume without any loss of generalities that $\varepsilon_n < \lambda - \beta$. Then

$$\limsup_{t\to\infty} \frac{T(t)}{t^{\beta+\varepsilon_n}} = \infty \text{ and } \liminf_{t\to\infty} \frac{T(t)}{t^{\beta+\varepsilon_n/2}} = 0.$$

Application of a theorem of Edrei [6] deduces the existence of r_n with $r_n > \max\{r_{n-1}, \varepsilon_n^{-\frac{2\beta+\varepsilon_n}{\varepsilon_n}}\}$ such that

$$T(t) \leqslant \left(\frac{t}{r_n}\right)^{\beta+\varepsilon_n} T(r_n)$$

for $1 \leqslant t \leqslant r_n^{\frac{\beta+\varepsilon_n}{\beta+\varepsilon_n/2}}$. This immediately implies (1.1.5) and r_n satisfies (4), because for $1 \leqslant t \leqslant \varepsilon_n^{-1} r_n (< r_n^{\frac{\beta+\varepsilon_n}{\beta+\varepsilon_n/2}})$,

$$\left(\frac{t}{r_n}\right)^{2\varepsilon_n} \leqslant \mathrm{e}^{2\varepsilon_n|\log \varepsilon_n|}$$

and the quantity on the right side is bounded and tends to 1.

Thus, we have gotten a sequence $\{r_n\}$ satisfying (1.1.5) and (4) in Definition 1.1.1.

Put $d_n = 1 + 1/n$ and $V = \cup_{n=1}^{\infty}[r_n, d_n r_n]$. V has the infinite logarithmic measure and, therefore, there exist a subsequence of $\{[r_n, d_n r_n]\}$, each member of which contains at least a point outside F. Without any loss of generalities we can assume for each n a $\hat{r}_n \in [r_n, d_n r_n] \setminus F$. Then for $\hat{\varepsilon}_n \hat{r}_n \leqslant t \leqslant \hat{r}_n/\hat{\varepsilon}_n$ with $\hat{\varepsilon}_n = d_n \varepsilon_n$, we have

$$T(t) \leqslant \left(\frac{t}{r_n}\right)^{\beta+\varepsilon_n} T(r_n) \leqslant (d_n)^{\beta+\varepsilon_n} \left(\frac{t}{\hat{r}_n}\right)^{\beta+\varepsilon_n} T(\hat{r}_n)$$
$$\leqslant (d_n)^{\beta+\varepsilon_n} \left(\frac{1}{\hat{\varepsilon}_n}\right)^{2\varepsilon_n} \left(\frac{t}{\hat{r}_n}\right)^{\beta} T(\hat{r}_n),$$

this implies that $\{\hat{r}_n\}$ satisfies (3) in Definition 1.1.1. It is easy to show $\{\hat{r}_n\}$ satisfies other conditions of the Pólya peak.

This completes the proof of Theorem 1.1.3. □

Chuang considered in [4] the type function and in [3] the Pólya peak sequence of a continuous real function and revealed some relations between the type function and the Pólya peak sequence by demonstrating their existence simultaneously starting from a basic theorem, that is, Theorem 1 of [3] or Lemma 4.4 of [4]. In fact, we easily obtain a sequence of the Pólya peak of order $\lambda(T)$ from the type function, for an example, a careful calculation implies that a sequence of positive real numbers $\{r_n\}$ with $U(r_n) = (1+o(1))T(r_n)$ must be a Pólya peak sequence of $T(r)$ of order $\lambda(T)$. Drasin and Shea [5] obtained a necessary and sufficient condition for existence of a sequence of Pólya peaks of order β which satisfies only (1) and (3) listed in Definition 1.1.1. Set

$$\lambda^*(T) = \sup\left\{\tau : \limsup_{x,A\to\infty} \frac{T(Ax)}{A^\tau T(x)} = \infty\right\}$$

and

$$\mu_*(T) = \inf\left\{\tau : \liminf_{x,A\to\infty} \frac{T(Ax)}{A^\tau T(x)} = 0\right\}.$$

It is proved in [5] that $\mu_*(T) \leqslant \mu(T) \leqslant \lambda(T) \leqslant \lambda^*(T)$ and if $\mu_* < \infty$, then a sequence of Pólya peaks of order β satisfying only (1) and (3) listed in Definition 1.1.1 exists if and only if $\mu_* \leqslant \beta \leqslant \lambda^*$ and $\beta < \infty$. However, we do not know if this condition is sufficient to the existence of our Pólya peak sequence. Usually, we call λ^* and μ_* respectively the Pólya order and Pólya lower order of $T(r)$.

Generally, there exists no Pólya peak sequence of $T(r)$ whose lower order is of infinite order. However, we have the following, which will be often used in the sequel.

Lemma 1.1.3. *Let $T(r)$ be an increasing and non-negative continuous function with the infinite order and F a set of positive real numbers having finite logarithmic measure. Then given a sequence $\{s_n\}$ of positive real numbers, there exists an unbounded sequence $\{r_n\}$ of positive real numbers outside F such that*

$$\frac{T(t)}{t^{s_n}} \leqslant \mathrm{e}\frac{T(r_n)}{r_n^{s_n}}, \qquad 1 \leqslant t \leqslant r_n.$$

Proof. Since $T(r)$ is of infinite order, for a fixed s_n we have

$$\limsup_{t\to\infty}\frac{T(t)}{t^{s_n}}=\infty$$

and it is easy to see that we can find a sequence $\{\hat{r}_m\}$ such that $\hat{r}_m > 2^{nm}$ and $\hat{r}_{m+1} > e^{1/s_n}\hat{r}_m$ and

$$\frac{T(t)}{t^{s_n}} \leqslant \frac{T(\hat{r}_m)}{\hat{r}_m^{s_n}}, \qquad 1\leqslant t\leqslant \hat{r}_m.$$

Set

$$F_n=\bigcup_{m=1}^{\infty}[\hat{r}_m, e^{1/s_n}\hat{r}_m].$$

Then

$$\int_{F_n}\frac{dt}{t}=\sum_{m=1}^{\infty}\int_{\hat{r}_m}^{e^{1/s_n}\hat{r}_m}\frac{dt}{t}=\sum_{m=1}^{\infty}\frac{1}{s_n}=\infty$$

so that $F_n\setminus F$ has the infinite logarithmic measure. We can find a $r_n\in F_n\setminus F$ such that for some m, $\hat{r}_m\leqslant r_n\leqslant e^{1/s_n}\hat{r}_m$ and choose a r'_n in $[\hat{r}_m, r_n]$ such that

$$\frac{T(r'_n)}{r'^{s_n}_n}=\max\left\{\frac{T(t)}{t^{s_n}}:\hat{r}_m\leqslant t\leqslant r_n\right\}.$$

Thus for $1\leqslant t\leqslant r_n$, we have

$$\frac{T(t)}{t^{s_n}}\leqslant\frac{T(r'_n)}{r'^{s_n}_n}\leqslant\left(\frac{r_n}{r'_n}\right)^{s_n}\frac{T(r_n)}{r_n^{s_n}}\leqslant e\frac{T(r_n)}{r_n^{s_n}}.$$

The desired sequence $\{r_n\}$ has been attained. □

1.1.3 The Regularity of a Real Function

We first of all consider the density and the logarithmic density of a Lebesgue measurable set on the positive real axis. However, we begin with a general case, which will bring us some benefits.

An absolutely continuous function $\psi(r)$ on an interval $[a,b]$ has finite derivative almost everywhere in the sense of Lebesgue and $\psi'(r)\in L^1([a,b])$ and for each $r\in[a,b]$

$$\psi(r)=\psi(a)+\int_a^r\psi'(t)dt$$

and an indefinite integral of a function in $L^1([a,b])$ is absolutely continuous. A convex function is absolutely continuous and its right (left) derivative is non-decreasing. We say that an increasing function $\psi(r)$ is a convex function of another increasing $\varphi(r)$ if the right (left) derivative $d\psi(t)/d\varphi(t)$ exists and is non-decreasing.

We denote by m the Lebesgue measure on the positive real axis. Let E be a Lebesgue measurable subset of the positive real axis and $\psi(r)$ a positive and absolutely continuous function of r for $r \geqslant r_0$. Following Barry [1], we define the ψ-measure of $E(r) = E \cap [r_0, r]$ by

$$\psi\_m(E(r)) = \int_{E(r)} \psi'(t)\mathrm{d}t$$

and the upper and lower ψ-densities, respectively, of E by

$$\psi\_\overline{\underline{\mathrm{dens}}}E = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\psi\_m(E(r))}{\psi(r)}.$$

When $\psi(r)$ is taken to be r, we obtain the definition of the upper and lower densities of E, denoted by $\overline{\mathrm{dens}}E$ and $\underline{\mathrm{dens}}E$ and when $\psi(r)$ is $\log r$, we have the upper and lower logarithmic densities of E, denoted by $\overline{\log\mathrm{dens}}E$ and $\underline{\log\mathrm{dens}}E$. When $\psi\_\overline{\mathrm{dens}}E = \psi\_\underline{\mathrm{dens}}E$, it is said that E has a ψ-density and we use notation $\psi\_\mathrm{dens}E$ to denote the common value and in this case, specially we have the definition of the density and logarithmic density of a set.

It is easy to see that for a set E on the positive real axis with the finite logarithmic measure, i.e., $\int_E t^{-1}\mathrm{d}t < \infty$, we have $\mathrm{dens}E = 0$. Actually, it follows from the following equation

$$m(E(r)) = m(E(\sqrt{r})) + m(E \cap [\sqrt{r}, r]) \leqslant \sqrt{r} + r\int_{E\cap[\sqrt{r},r]} t^{-1}\mathrm{d}t = o(r).$$

The following is Lemma 1 of Barry [1].

Lemma 1.1.4. *Let $\psi(r)$ and $\varphi(r)$ be positive, increasing, unbounded and absolutely continuous functions of r, and $\psi(r)$ a convex function of $\varphi(r)$ for $r \geqslant r_0$. Then*

$$\psi\_\underline{\mathrm{dens}}E \leqslant \varphi\_\underline{\mathrm{dens}}E \leqslant \varphi\_\overline{\mathrm{dens}}E \leqslant \psi\_\overline{\mathrm{dens}}E.$$

Proof. According to the definition of the upper ψ-density of a set, given arbitrarily $\varepsilon > 0$, for $t \geqslant r_1(\varepsilon) > r_0$, we have

$$\psi\_m(E(t)) < (\psi\_\overline{\mathrm{dens}}E + \varepsilon)\psi(t).$$

Noticing that $\mathrm{d}\psi(t)/\mathrm{d}\varphi(t)$ is non-decreasing in t, in view of the formula for integration by parts, we have for $r > r_1$

$$\begin{aligned}
\varphi_m(E(r)) &= \int_{E(r)} \mathrm{d}\varphi(t) = \int_{E(r)} \left(\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}\right)^{-1} \mathrm{d}\psi(t)\\
&= \int_{r_0}^{r} \left(\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}\right)^{-1} \mathrm{d}\psi_m(E(t))\\
&= \psi_m(E(t))\left(\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}\right)^{-1}\Bigg|_{r_0}^{r} + \int_{r_0}^{r} \psi_m(E(t))\mathrm{d}\left[-\left(\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}\right)^{-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&< (\psi\_\overline{\mathrm{dens}}E+\varepsilon)\psi(r)\left(\frac{\mathrm{d}\psi(r)}{\mathrm{d}\varphi(r)}\right)^{-1}\\
&\quad+\int_{r_0}^{r_1}\psi\_m(E(t))\mathrm{d}\left[-\left(\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}\right)^{-1}\right]\\
&\quad+\int_{r_1}^{r}(\psi\_\overline{\mathrm{dens}}E+\varepsilon)\psi(t)\mathrm{d}\left[-\left(\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}\right)^{-1}\right]\\
&= O(1)+(\psi\_\overline{\mathrm{dens}}E+\varepsilon)\left[\psi(r)\left(\frac{\mathrm{d}\psi(r)}{\mathrm{d}\varphi(r)}\right)^{-1}-\psi(t)\left(\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}\right)^{-1}\Bigg|_{r_1}^{r}\right.\\
&\quad\left.+\int_{r_1}^{r}\left(\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}\right)^{-1}\mathrm{d}\psi(t)\right]\\
&= O(1)+(\psi\_\overline{\mathrm{dens}}E+\varepsilon)(O(1)+\varphi(r)).
\end{aligned}
$$

Thus

$$\limsup_{r\to\infty}\frac{\varphi\_m(E(r))}{\varphi(r)}\leqslant \psi\_\overline{\mathrm{dens}}E.$$

The remainder inequality follows from this by taking complements. □

Specially, from Lemma 1.1.4 we get

$$\underline{\mathrm{dens}}E\leqslant \log\_\,\underline{\mathrm{dens}}E\leqslant \log\_\,\overline{\mathrm{dens}}E\leqslant \overline{\mathrm{dens}}E,$$

for r is a convex function of $\log r$.

Generally, a monotone continuous function may be complicated in the sense of its regular behavior and such an irregular behavior may cause difficulties to our discussion. However, fortunately, after a small set is ignored, such a function possess some regularities which are sufficient in certain discussions. The following is a fundamental lemma of E. Borel.

Lemma 1.1.5. *Let $T(r)$ be a non-decreasing continuous function in $[r_0,+\infty)$ such that $T(r_0)\geqslant 1$. Then with possible exception of values of r in a set with measure at most 2, we have*

$$T\left(r+\frac{1}{T(r)}\right)<2T(r).$$

The following is Lemma 10.1 of Edrei and Fuchs [7], a modified version of the Borel Lemma 1.1.5.

Lemma 1.1.6. *Let $\psi(r)$ and $\varphi(r)$ be two positive functions on the positive real axis. Assume that for $r\geqslant r_0\geqslant 0$, $\psi(r)$ is non-decreasing while $\varphi(r)$ is non-increasing and that for some $r_1(>r_0)$ and a given positive number c, $\psi(r_1)>r_0+c$. Set*

$$E=\{r\geqslant r_1:\psi(r+\varphi(\psi(r)))\geqslant\psi(r)+c\}.$$

Then we have

$$m(E(a,A)) \leqslant \frac{1}{c}\int_{\psi(a)-c}^{\psi(A)} \varphi(t)\mathrm{d}t,$$

provided that $r_1 \leqslant a < A < +\infty$*, where* $E(a,A)$ *stands for the intersection of* E *with the interval* (a,A).

Proof. Under the assumption that $\psi(r_1) > r_0 + c$, it is easy to see that $\psi(r) - c > r_0$ and $\varphi(t)$ is non-increasing for $t \geqslant \psi(r) - c$ and $r \geqslant r_1$.

Assume conversely that Lemma 1.1.6 is false, that is,

$$m(E(a_0,A)) \geqslant \varepsilon + \frac{1}{c}\int_{\psi(a_0)-c}^{\psi(A)} \varphi(t)\mathrm{d}t, \tag{1.1.6}$$

for three fixed numbers $\varepsilon > 0$, a_0 and A with $r_1 \leqslant a_0 < A < \infty$.

Put

$$\lambda(x) = \inf_{r\in E(x,A)} \{r\}$$

and in view of the definition of the infimum we can find $b_1 \in E(a_0,A)$ with $\lambda(a_0) \leqslant b_1 < \lambda(a_0) + \frac{\varepsilon}{2}$. Set $a_1 = b_1 + \varphi(\psi(b_1))$ and since $b_1 \in E$,

$$\psi(a_1) \geqslant \psi(b_1) + c.$$

Next we want to get the similar estimate from below of $m(E(a_1,A))$ to (1.1.6). Notice that if $a_1 \leqslant A$, $m(E(a_1,A)) = m(E(a_0,A)) - m(E(a_0,a_1))$, and to the end we respectively estimate $m(E(a_0,A))$ and $m(E(a_0,a_1))$ as follows: as $\varphi(r)$ is non-increasing, we have

$$\begin{aligned} m(E(a_0,a_1)) &\leqslant a_1 - \lambda(a_0) = (a_1 - b_1) + (b_1 - \lambda(a_0)) \\ &\leqslant \varphi(\psi(b_1)) + \frac{\varepsilon}{2} \\ &\leqslant \frac{1}{c}\int_{\psi(b_1)-c}^{\psi(b_1)} \varphi(t)\mathrm{d}t + \frac{\varepsilon}{2} \end{aligned}$$

and in view of (1.1.6) and $A \geqslant b_1 \geqslant a_0$, we have

$$\begin{aligned} m(E(a_0,A)) &\geqslant \varepsilon + \frac{1}{c}\int_{\psi(a_0)-c}^{\psi(A)} \varphi(t)\mathrm{d}t \\ &= \varepsilon + \frac{1}{c}\int_{\psi(b_1)}^{\psi(A)} \varphi(t)\mathrm{d}t + \frac{1}{c}\int_{\psi(a_0)-c}^{\psi(b_1)} \varphi(t)\mathrm{d}t \\ &\geqslant \varepsilon + \frac{1}{c}\int_{\psi(b_1)}^{\psi(A)} \varphi(t)\mathrm{d}t + \frac{1}{c}\int_{\psi(b_1)-c}^{\psi(b_1)} \varphi(t)\mathrm{d}t \\ &\geqslant \frac{\varepsilon}{2} + \frac{1}{c}\int_{\psi(b_1)}^{\psi(A)} \varphi(t)\mathrm{d}t + m(E(a_0,a_1)). \end{aligned}$$

This implies that $E(a_1,A)$ is not empty and $a_1 < A$ so that,

$$m(E(a_1,A)) \geqslant \frac{\varepsilon}{2} + \frac{1}{c}\int_{\psi(a_1)-c}^{\psi(A)} \varphi(t)\mathrm{d}t > 0.$$

Starting from this inequality we may repeat our previous construction with a_0 replaced by a_1 and ε by $\varepsilon/2$ and thus such construction can be repeated infinitely to obtain a sequence of intervals $[b_k, a_k]$ such that

$$a_0 \leqslant b_1 < a_1 \leqslant b_2 < a_2 \leqslant \cdots < A$$

and $b_k \in E$. Since $\psi(r)$ is non-decreasing, we have

$$\psi(b_{k+1}) \geqslant \psi(a_k) > \psi(b_k) + c,$$

so that $\psi(A) \geqslant \psi(b_{k+1}) \geqslant \psi(b_1) + kc$. This is impossible and therefore Lemma 1.1.6 is proved. □

Corollary 1.1.1. *Under the same assumption as in Lemma 1.1.6, assume, in addition, that*

$$\int^{\infty} \varphi(t)\mathrm{d}t < \infty.$$

Then E has only finite measure. In particular, let $T(r)$ be a continuous non-decreasing function of r with $T(r) > 1$. Then for $\varepsilon > 0$

$$T(r\mathrm{e}^{\alpha(r)}) \leqslant \mathrm{e}^c T(r),\ \alpha(r) = \frac{1}{(\log T(r))^{1+\varepsilon}}$$

holds for all r possibly outside a set of finite logarithmic measure.

Proof. The first part is obvious and we provide proof for the latter part only.

Set

$$\psi(r) = \log T(\mathrm{e}^r),\ \ \varphi(r) = \frac{1}{r^{1+\varepsilon}}.$$

It is obvious that $\psi(r)$ and $\varphi(r)$ satisfy the assumption of the first part. Since

$$\psi(\log r + \varphi(\psi(\log r))) = \log T(r\mathrm{e}^{\alpha(r)}) \text{ and } \psi(\log r) + c = \log \mathrm{e}^c T(r),$$

the first part implies that

$$E = \{x = \log r : T(r\mathrm{e}^{\alpha(r)}) \geqslant \mathrm{e}^c T(r)\}$$

has finite measure and therefore $F = \{r : \log r \in E\}$ has finite logarithmic measure by the formula for integration by substitution. Thus, the latter part has been proved. □

In the theory of value distribution, we have to often avoid some exceptional sets from the situation we consider, whence the following result is useful in treating this case.

Lemma 1.1.7. *Let $\psi(r)$ and $\varphi(r)$ be non-decreasing positive functions. Assume that*

$$\psi(r) \leqslant \varphi(r)$$

for all r possibly outside a set E with $\overline{\text{dens}}E < 1$. Then for each k with $(1-\overline{\text{dens}}E)^{-1} < k < +\infty$, for all sufficiently large r we have

$$\psi(r) \leqslant \varphi(kr).$$

If E is of finite measure or of finite logarithmic measure, then for each $k > 1$ and all sufficiently large r the above inequality is true.

Proof. Suppose for some $(1-\overline{\text{dens}}E)^{-1} < k < +\infty$ there exists an unbounded sequence $\{r_n\}$ such that $\psi(r_n) > \varphi(kr_n)$. Set $F = \bigcup_{n=1}^{\infty}[r_n, kr_n]$. Then

$$\overline{\text{dens}}F \geqslant \limsup_{n\to\infty} \frac{1}{kr_n} m(F(kr_n)) \geqslant \limsup_{n\to\infty} \frac{1}{kr_n}(kr_n - r_n) = \frac{k-1}{k} > \overline{\text{dens}}E.$$

This asserts an existence of a $r \in F \setminus E$ and so for some n, $r_n \leqslant r \leqslant kr_n$. Therefore in view of the monotonicity of ψ and φ, we have

$$\psi(r_n) \leqslant \psi(r) \leqslant \varphi(r) \leqslant \varphi(kr_n).$$

This contradicts the hypothesis about r_n and Lemma 1.1.7 follows. $\square$

We remark that from Lemma 1.1.7 it follows that if $\log\_\,\overline{\text{dens}}E < 1$, then for $k > (1-\log\_\,\overline{\text{dens}}E)^{-1}$ and all sufficiently large r we have

$$\psi(r) \leqslant \varphi(r^k).$$

The following is due to Hayman [9].

Lemma 1.1.8. *Let $T(r)$ be a non-negative, non-constant and non-decreasing continuous function for $r \geqslant a$ with the order λ and lower order μ. Given two real numbers C_1 and C_2 greater than 1, set*

$$G = G(C_1, C_2) = \{r:\ T(C_1 r) \geqslant C_2 T(r)\}.$$

Then

$$\overline{\log\text{dens}}G \leqslant \lambda \frac{\log C_1}{\log C_2} \text{ and } \underline{\log\text{dens}}G \leqslant \mu \frac{\log C_1}{\log C_2}.$$

Proof. Set $r_1 = \inf\{r \geqslant 1 : r \in G\}$. Suppose that r_n has been chosen. Take $r_{n+1} = \inf\{r \geqslant C_1 r_n :\ r \in G\}$, and thus we inductively obtain a sequence of positive numbers $\{r_n\}$ such that $G \subset \bigcup_{n=1}^{\infty}[r_n, C_1 r_n]$. For $r \geqslant r_1$ with $r \in G$, we have $r_q \leqslant r < C_1 r_q$ for some $q \geqslant 1$. This implies that

$$\log\_\, m(G(r)) = \int_{G(r)} \frac{\mathrm{d}t}{t} \leqslant \sum_{k=1}^{q} \int_{r_k}^{C_1 r_k} \frac{\mathrm{d}t}{t} = q \log C_1.$$

where $G(r) = G \cap [1, r]$.

Now we want to estimate q. Generally it is easy to see that

$$T(r_{n+1}) \geqslant T(C_1 r_n) \geqslant C_2 T(r_n)$$

so that

$$T(r_n) \geqslant C_2^{n-1} T(r_1)$$

and therefore,

$$q \leqslant 1 + \frac{1}{\log C_2} \log \frac{T(r_q)}{T(r_1)} \leqslant 1 + \frac{1}{\log C_2} \log \frac{T(r)}{T(1)}.$$

This deduces that

$$\frac{\log_{-} m(G(r))}{\log r} \leqslant \frac{\log C_1}{\log r} + \frac{\log C_1}{\log C_2} \frac{\log T(r) - \log T(1)}{\log r},$$

from which the desired inequalities follows directly by letting $r \to \infty$. □

1.1.4 Quasi-invariance of Inequalities

We begin the subsection with quasi-invariance of inequality under differentiation, that is to say, establish the following, the first part of which was proved in Barry [1].

Lemma 1.1.9. *Let $\psi(r)$ be non-decreasing and $\varphi(r)$ non-constant, non-decreasing and convex for $r \geqslant a$. Assume that*

$$0 \leqslant \psi(r) \leqslant \varphi(r), \ r \notin W$$

for a subset W of $[a, \infty)$ with $\tau = \varphi_{-}\overline{\mathrm{dens}} W < 1$. Then for arbitrary $K > 1/(1-\tau)$, we have

$$\underline{\mathrm{dens}} E \geqslant \frac{K-1}{K} - \tau, \ E = \{r : \psi'(r) \leqslant K\varphi'(r)\}.$$

Further, if $\psi(r)$ is convex, for all sufficiently large r we have

$$\psi'(r) \leqslant K\varphi'(dr), \ d > \frac{K}{(1-\tau)K - 1} > 0. \tag{1.1.7}$$

Proof. From the convexity of $\varphi(r)$, it is easy to see that $\varphi(r) \to \infty$ as $r \to \infty$ and $\varphi'(r)$ is non-negative and monotone non-decreasing and $\varphi(r)$ is absolutely continuous. Set

$$F = \{r : \psi'(r) \geqslant K\varphi'(r)\}$$

and $r' = \sup\{x \in F \setminus W : x \leqslant r\}$ for $r \geqslant a$. Then, for $r > a$, we have

$$
\begin{aligned}
\int_{F(r)} \varphi'(t)\mathrm{d}t &= \int_{(F\setminus W)(r)} \varphi'(t)\mathrm{d}t + \int_{W(r)} \varphi'(t)\mathrm{d}t \\
&= \int_{(F\setminus W)(r')} \varphi'(t)\mathrm{d}t + \int_{W(r)} \varphi'(t)\mathrm{d}t \\
&\leqslant K^{-1} \int_{F(r')} \psi'(t)\mathrm{d}t + \int_{W(r)} \varphi'(t)\mathrm{d}t \\
&\leqslant K^{-1} \int_{a}^{r'} \psi'(t)\mathrm{d}t + \int_{W(r)} \varphi'(t)\mathrm{d}t \\
&\leqslant K^{-1}(\psi(r') - \psi(a)) + \int_{W(r)} \varphi'(t)\mathrm{d}t \\
&\leqslant K^{-1}\varphi(r') - K^{-1}\psi(a) + \int_{W(r)} \varphi'(t)\mathrm{d}t \\
&\leqslant K^{-1}\varphi(r) - K^{-1}\psi(a) + \int_{W(r)} \varphi'(t)\mathrm{d}t
\end{aligned}
$$

and, thus, $\varphi\_\overline{\mathrm{dens}}F \leqslant K^{-1} + \tau$ and in view of Lemma 1.1.4 we get $\overline{\mathrm{dens}}F \leqslant K^{-1} + \tau$ and equivalently $\underline{\mathrm{dens}}E \geqslant 1 - K^{-1} - \tau$.

(1.1.7) follows from application of Lemma 1.1.7, for $\psi'(r)$ is non-decreasing under the assumption of convexity of $\psi(r)$ and $(1-\overline{\mathrm{dens}}F)^{-1} < (1-K^{-1}-\tau)^{-1} = \frac{K}{(1-\tau)K-1}$. □

Hayman and Stewart [10], and Hayman and Rossi [11] investigated the case of any order derivatives. The following result was obtained in [10]: if $\psi(r)$ and $\varphi(r)$ and their derivatives up to $n-1$ order are non-negative, non-decreasing and convex for $r \geqslant a$, then from $0 \leqslant \psi(r) \leqslant \varphi(r)$ for all $r \geqslant a$, we have

$$
\psi^{(n)}(r) \leqslant Kn!\left(\frac{\mathrm{e}}{n}\right)^n \varphi^{(n)}(r) \tag{1.1.8}
$$

on a set E of r with positive lower density depending only on K, n and φ but not on ψ and furthermore, Hayman and Rossi [11] proved that $\underline{\mathrm{dens}}E \geqslant (\sqrt[n]{K}-1)/(\sqrt[n]{K}-1+n)$. What we should emphasize is that in Hayman and Stewart's result, the inequality (1.1.8) holds on the above fixed set E for any function $\psi(r)$ satisfying those assumptions determined by a given function $\varphi(r)$. Naturally we ask whether the set E in Lemma 1.1.9 is independent of $\psi(r)$, which concerns a question posed in page 256 of [10].

Finally, we consider quasi-invariance of inequality under integration. Here are two non-negative, non-decreasing real functions $A(r)$ and $B(r)$. If for all r in $[r_0, +\infty)$ but outside a subset E, we have

$$
A(r) \leqslant B(r), \tag{1.1.9}
$$

then could we compare $\int_{r_0}^{r} A(t)\mathrm{d}t$ to $\int_{r_0}^{r} B(t)\mathrm{d}t$? This is an important question in the value distribution of meromorphic functions. In terms of (1.1.9), we have

$$\begin{aligned}\int_{r_0}^{r} A(t)\mathrm{d}t &= \int_{[r_0,r]\setminus E} A(t)\mathrm{d}t + \int_{E\cap[r_0,r]} A(t)\mathrm{d}t\\ &\leqslant \int_{[r_0,r]\setminus E} B(t)\mathrm{d}t + \int_{E\cap[r_0,r]} A(t)\mathrm{d}t\\ &\leqslant \int_{r_0}^{r} B(t)\mathrm{d}t + \int_{E\cap[r_0,r]} A(t)\mathrm{d}t.\end{aligned}$$

Obviously, we cannot directly control $\int_{E\cap[r_0,r]} A(t)\mathrm{d}t$ in terms of $\int_{r_0}^{r} B(t)\mathrm{d}t$, but we can hope to use $\int_{r_0}^{r} A(t)\mathrm{d}t$ to control it. The following result realizes this purpose, which is a generalization of Lemma 9 of Eremenko and Sodin [8] but the basic idea is due to them.

Lemma 1.1.10. *Let E be a measurable subset of $[r_0,+\infty)$ and $\varepsilon > 0$ and let $\varphi(x)$ be a positive non-increasing function in $[r_0,+\infty)$ such that $\int^{\infty} \varphi(t)\mathrm{d}t = +\infty$. Then there exists a subset E^* of $[r_0,+\infty)$ with*

$$\int_{E^*(r)} \varphi(t)\mathrm{d}t \leqslant \frac{2}{\varepsilon}\int_{E(r)} \varphi(t)\mathrm{d}t \tag{1.1.10}$$

such that for any non-negative, non-decreasing function $\psi(x)$ and $r \notin E^*$ and any $\tau < r$, we have

$$\int_{E(\tau,r)} \psi(t)\mathrm{d}t < 2\varepsilon \int_{\tau}^{r} \psi(t)\mathrm{d}t. \tag{1.1.11}$$

Proof. Define

$$E^* = \left\{ r \geqslant r_0 : \exists x = x(r) < r \text{ such that } \int_{E(x,r)} \varphi(t)\mathrm{d}t \geqslant \varepsilon \int_{x(r)}^{2r-x(r)} \varphi(t)\mathrm{d}t \right\}.$$

It is obvious that s is the center point of the interval $(x(s), 2s - x(s))$ and so for a fixed $r \geqslant r_0$, $\{(x(s), 2s - x(s)) : s \in E^*(r)\}$ is a covering of $E^*(r)$. As $E^*(r)$ is a bounded, closed set, there exist finitely many intervals $\{(x(s_j), 2s_j - x(s_j)) : 1 \leqslant j \leqslant q\}$ to cover $E^*(r)$ and each point in $E^*(r)$ is covered at most two times. Thus, as $s_j \in E^*$, we have

$$\begin{aligned}\int_{E^*(r)} \varphi(t)\mathrm{d}t &\leqslant \sum_{j=1}^{q} \int_{x(s_j)}^{2s_j - x(s_j)} \varphi(t)\mathrm{d}t\\ &\leqslant \frac{1}{\varepsilon}\sum_{j=1}^{q} \int_{E(x(s_j),s_j)} \varphi(t)\mathrm{d}t\\ &\leqslant \frac{2}{\varepsilon}\int_{E\cap(\cup_{j=1}^{q}(x(s_j),s_j))} \varphi(t)\mathrm{d}t\\ &\leqslant \frac{2}{\varepsilon}\int_{E(r)} \varphi(t)\mathrm{d}t.\end{aligned}$$

Now let us prove (1.1.11). For $r \notin E^*$ and for all $r_0 \leqslant t \leqslant r$, we set $\eta(t) = \int_{E(t,r)} \varphi(t)\mathrm{d}t$ and, then, have

$$\eta(t) < \varepsilon \int_t^{2r-t} \varphi(x)\mathrm{d}x.$$

Noting that for $t < r$, $\varphi(2r-t) \leqslant \varphi(t)$ and $\eta(t)$ is non-increasing, but $\frac{\psi(t)}{\varphi(t)}$ is non-decreasing, we have

$$\begin{aligned}
&2\varepsilon \int_\tau^r \psi(t)\mathrm{d}t - \int_{E[\tau,r]} \psi(t)\mathrm{d}t \\
&\geqslant \varepsilon \int_\tau^r \frac{\psi(t)}{\varphi(t)}(\varphi(2r-t)+\varphi(t))\mathrm{d}t - \int_{E[\tau,r]} \psi(t)\mathrm{d}t \\
&= \int_\tau^r \frac{\psi(t)}{\varphi(t)}\mathrm{d}\left(-\varepsilon \int_t^{2r-t} \varphi(x)dx\right) + \int_\tau^r \frac{\psi(t)}{\varphi(t)}\mathrm{d}\eta(t) \\
&= \int_\tau^r \frac{\psi(t)}{\varphi(t)}\mathrm{d}\left(\eta(t) - \varepsilon \int_t^{2r-t} \varphi(x)\mathrm{d}x\right) \\
&\geqslant \frac{\psi(t)}{\varphi(t)}\left(\eta(t) - \varepsilon \int_t^{2r-t} \varphi(x)\mathrm{d}x\right)_\tau^r - \int_\tau^r \left(\eta(t) - \varepsilon \int_t^{2r-t} \varphi(x)\mathrm{d}x\right)\mathrm{d}\frac{\psi(t)}{\varphi(t)} \\
&= \frac{\psi(\tau)}{\varphi(\tau)}\left(\varepsilon \int_\tau^{2r-\tau} \varphi(x)\mathrm{d}x - \eta(\tau)\right) + \int_\tau^r \left(\varepsilon \int_t^{2r-t} \varphi(x)\mathrm{d}x - \eta(t)\right)\mathrm{d}\frac{\psi(t)}{\varphi(t)} \\
&\geqslant 0.
\end{aligned}$$

This yields (1.1.11). □

We make a remark on Lemma 1.1.10. If $\int_E \varphi(t)\mathrm{d}t < +\infty$, then $\int_{E^*} \varphi(t)\mathrm{d}t < +\infty$ and hence if $\varphi(t) \equiv 1$ or $\varphi(t) = 1/t$, that is to say, E is of finite measure or of finite logarithmic measure, then so is E^* in turn. Further, we can take into account the density of E and E^* in view of (1.1.10). Set $\phi(t) = \int^t \varphi(x)\mathrm{d}x$. Then we have

$$\phi\text{-}\overline{\underline{\mathrm{dens}}}E^* \leqslant \frac{2}{\varepsilon}\phi\text{-}\overline{\underline{\mathrm{dens}}}E$$

so that when $\phi\text{-}\overline{\underline{\mathrm{dens}}}E = 0$, we have $\phi\text{-}\overline{\underline{\mathrm{dens}}}E^* = 0$. What we should stress is that E^* in Lemma 1.1.10 does not rely on $\psi(r)$.

Now let us turn to answer the question mentioned before Lemma 1.1.10. Assume (1.1.9) holds for all r outside a set E with the properties $\int_E \varphi(t)\mathrm{d}t < +\infty$ for a $\varphi(x)$ stated in Lemma 1.1.10. Take a sequence of positive numbers $\{\varepsilon_j\}$ such that $0 < \varepsilon_j \leqslant 1$ and $\varepsilon_j \to 0$ as $j \to \infty$. In view of Lemma 1.1.10, we have E_j^* for each ε_j such that $\int_{E_j^*} \varphi(t)\mathrm{d}t < +\infty$ and for $r \notin E_j^*$

$$\int_{E(\tau,r)} \psi(t)\mathrm{d}t \leqslant \varepsilon_j \int_\tau^r \psi(t)\mathrm{d}t \tag{1.1.12}$$

for any non-negative, non-decreasing function $\psi(x)$. There exist a sequence of positive numbers $\{r_j\}$ such that $r_{j-1} < r_j \to \infty$ and

$$\int_{E_j^*\cap[r_j,\infty)} \varphi(t)\mathrm{d}t < \frac{1}{2^j}.$$

Define

$$E^* = (E_1^*\cap[r_0,r_1])\cup\bigcup_{j=1}^{\infty} E_j^*\cap[r_j,r_{j+1}]. \tag{1.1.13}$$

Then we have

$$\begin{aligned}\int_{E^*}\varphi(t)\mathrm{d}t &\leqslant \int_{E_1^*\cap[r_0,r_1]}\varphi(t)\mathrm{d}t+\sum_{j=1}^{\infty}\int_{E_j^*\cap[r_j,r_{j+1}]}\varphi(t)\mathrm{d}t\\ &< \int_{E_1^*\cap[r_0,r_1]}\varphi(t)\mathrm{d}t+1\\ &< +\infty.\end{aligned}$$

Now define a function $\varepsilon(r)$ by $\varepsilon(r)=\varepsilon_j$ for $r_j\leqslant r<r_{j+1}$ $(j=1,2,\cdots)$ and $\varepsilon(r)=\varepsilon_1$ for $r_0\leqslant r<r_1$. Obviously, $\varepsilon(r)\to 0$ as $r\to\infty$. For $r\notin E^*$, we have $r_j\leqslant r<r_{j+1}$ for some $j\in\mathbb{N}$ but $r\notin E_j^*$ and thus (1.1.12) holds. Further, in terms of (1.1.9), we can get

$$\begin{aligned}\int_{\tau}^{r}A(t)\mathrm{d}t &= \int_{[\tau,r]\setminus E}A(t)\mathrm{d}t+\int_{E\cap[\tau,r]}A(t)\mathrm{d}t\\ &\leqslant \int_{[\tau,r]\setminus E}B(t)\mathrm{d}t+\varepsilon_j\int_{\tau}^{r}A(t)\mathrm{d}t\end{aligned}$$

so that

$$(1-\varepsilon(r))\int_{\tau}^{r}A(t)\mathrm{d}t\leqslant\int_{\tau}^{r}B(t)\mathrm{d}t. \tag{1.1.14}$$

Now we consider the case when $\phi\_\overline{\mathrm{dens}}E=0$ and $\phi(r)\to\infty$ as $r\to\infty$. Then there exists a set E_j^* for each ε_j such that $\phi\_\overline{\mathrm{dens}}E_j^*=0$ and for $r\notin E_j^*$ we have (1.1.12). Take a r_j by induction on j such that $r_j>r_{j-1}$ and for $r\geqslant r_j$, we have

$$\frac{1}{\phi(r)}\int_{E_j^*(r)}\varphi(t)\mathrm{d}t<\frac{\varepsilon_j}{2^j}$$

and $\frac{\phi(r_{j-1})}{\phi(r)}<\varepsilon_j$. Define E^* by (1.1.13). Then for $r\notin E^*$, (1.1.14) holds.

Below we prove $\phi\_\overline{\mathrm{dens}}E^*=0$ for this case. For arbitrary $\varepsilon>0$, there exists a $N\in\mathbb{N}$ such that for all $j>N$, $\varepsilon_j<\varepsilon$. For $r\geqslant r_N$, we have $r_M\leqslant r<r_{M+1}$ for some $M\in\mathbb{N}$ with $M\geqslant N$ and therefore

$$
\begin{aligned}
\frac{1}{\phi(r)}\int_{E^*(r)}\varphi(t)\mathrm{d}t &= \frac{1}{\phi(r)}\sum_{j=1}^{M-1}\int_{E_j^*[r_j,r_{j+1}])}\varphi(t)\mathrm{d}t \\
&\quad + \frac{1}{\phi(r)}\left(\int_{E_1^*(r_1)}\varphi(t)\mathrm{d}t + \int_{E_M^*[r_M,r]}\varphi(t)\mathrm{d}t\right) \\
&< \sum_{j=1}^{M-1}\frac{\phi(r_{j+1})}{\phi(r)}\frac{\varepsilon_j}{2^j} + \frac{\phi(r_1)}{\phi(r)}\frac{\varepsilon_1}{2} + \frac{\varepsilon_M}{2^M} \\
&< \sum_{j=1}^{M-2}\varepsilon_M\frac{\varepsilon_j}{2^j} + \frac{\varepsilon_{M-1}}{2^{M-1}} + \frac{1}{2}\varepsilon_M + \frac{\varepsilon_M}{2^M} \\
&< \varepsilon_M + \frac{\varepsilon_{M-1}}{2^{M-1}} + \frac{1}{2}\varepsilon_M + \frac{\varepsilon_M}{2^M} < 3\varepsilon
\end{aligned}
$$

taking note that $\frac{\phi(r_{j+1})}{\phi(r)} \leqslant \frac{\phi(r_{M-1})}{\phi(r)} < \varepsilon_M$ for $1 \leqslant j \leqslant M-2$. This implies $\phi_{-}\overline{\mathrm{dens}}E^* = 0$.

For the case when $\phi_{-}\underline{\mathrm{dens}}E = 0$, we can attain the corresponding result whose proof is left to the reader. Let us formulate the above result as a lemma stated as follows.

Lemma 1.1.11. *Let E and $\varphi(x)$ be given as in Lemma 1.1.10. Then there exists a set E^* such that if (1.1.9) holds for $r \notin E$, we have (1.1.14) for $r \notin E^*$ with properties that:*

(1) *if* $\int_E \varphi(t)\mathrm{d}t < +\infty$, *then* $\int_{E^*} \varphi(t)\mathrm{d}t < +\infty$;

(2) *if* $\phi_{-}\overline{\mathrm{dens}}E = 0$ *(*$\phi_{-}\underline{\mathrm{dens}}E = 0$, *respectively), then* $\phi_{-}\overline{\mathrm{dens}}E^* = 0$ *(*$\phi_{-}\underline{\mathrm{dens}}E^* = 0$*), where* $\phi(t) = \int^t \varphi(x)\mathrm{d}x$.

1.2 Integral Formula and Integral Inequalities

For completeness and in order to bring the reader convenience in their readings, this section recall the Green formula and collect several integral inequalities. They are useful in the sequel and certain proofs will be provided taking into account that they are not easy to find or not well-known in the general literatures.

1.2.1 The Green Formula for Functions with Two Real Variables

Various characteristics, except the Ahlfors-Shimizu's, of a meromorphic function, we introduce in the next chapter, stem from the Green formula for functions with two real variables.

Let U be a domain in $\mathbb{C}$ surrounded by finitely many piecewise differentiable simple curves and let $X(x,y)$ and $Y(x,y)$ be two continuous differentiable functions in the closure of U. Then we have the Green formula

$$\int\!\!\int_U \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) \mathrm{d}\sigma = \int_{\partial U} X\mathrm{d}x + Y\mathrm{d}y$$

where $\mathrm{d}\sigma$ is the area element. We mean by $\mathrm{d}s$ the arc element, and by $\boldsymbol{n}$ the inner normal of ∂U with respect to U, and by Δ the Laplacian.

Assume further that $X(x,y)$ and $Y(x,y)$ are the second order continuous differentiable functions in the closure of U. In view of the Green formula, we have the following

$$\begin{aligned}
\int_{\partial U} Y \frac{\partial X}{\partial \boldsymbol{n}} \mathrm{d}s &= \int_{\partial U} Y \left(\frac{\partial X}{\partial x}\cos\alpha + \frac{\partial X}{\partial y}\cos\beta\right) \mathrm{d}s \\
&= -\int_{\partial U} \left(Y\frac{\partial X}{\partial x}\mathrm{d}y - Y\frac{\partial X}{\partial y}\mathrm{d}x\right) \\
&= -\int\!\!\int_U Y\Delta X \mathrm{d}\sigma - \int\!\!\int_U \left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial Y}{\partial y}\right) \mathrm{d}\sigma,
\end{aligned}$$

where $\boldsymbol{n} = (\cos\alpha, \cos\beta)$. Thus

$$\int\!\!\int_U (X\Delta Y - Y\Delta X)\mathrm{d}\sigma = \int_{\partial U} \left(Y\frac{\partial X}{\partial \boldsymbol{n}} - X\frac{\partial Y}{\partial \boldsymbol{n}}\right) \mathrm{d}s. \tag{1.2.1}$$

This formula is known as the second Green formula. We have two special formulae:

If $X(x,y)$ and $Y(x,y)$ are harmonic in U, that is, $\Delta X = 0 = \Delta Y$, then

$$\int_{\partial U} \left(X\frac{\partial Y}{\partial \boldsymbol{n}} - Y\frac{\partial X}{\partial \boldsymbol{n}}\right) \mathrm{d}s = 0 \tag{1.2.2}$$

and

$$\int_{\partial U} \frac{\partial X}{\partial \boldsymbol{n}} \mathrm{d}s = 0. \tag{1.2.3}$$

Furthermore, if U is doubly connected and Γ is the outer boundary and γ the inner boundary, then

$$\oint_\Gamma \left(X\frac{\partial Y}{\partial \boldsymbol{n}} - Y\frac{\partial X}{\partial \boldsymbol{n}}\right) \mathrm{d}s = \oint_\gamma \left(X\frac{\partial Y}{\partial \boldsymbol{n}} - Y\frac{\partial X}{\partial \boldsymbol{n}}\right) \mathrm{d}s. \tag{1.2.4}$$

These formulae will be used often in the next chapter.

1.2.2 Several Integral Inequalities

Let $(X, \mathscr{A}, \mu)$ be an arbitrary measure space. For a positive real number p, $L^p(X, \mathscr{A}, \mu)$ is the set of all real-valued $\mathscr{A}$-measurable function f defined μ-a.e. on X such that $\int_X |f(x)|^p \mathrm{d}\mu(x)$ exists and is finite. We write L^p for $L^p(X, \mathscr{A}, \mu)$ where confusion seems impossible. Define for $f \in L^p$