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Visions in Mathematics

GAFA 2000 Special Volume, Part I

N. Alon
J. Bourgain
A. Connes
M. Gromov
V. Milman
Editors

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Reprint of the 2000 Edition

Birkhäuser

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Foreword

The meeting "Visions in Mathematics – Towards 2000" took place mainly at Tel Aviv University in August 25-September 3, 1999, with a few days at the Sheraton-Moriah Hotel at the Dead Sea Health Resort. The meeting included about 45 lectures by some of the leading researchers in the world, in most areas of mathematics and a number of discussions in different directions, organized in various forms.

The goals of the conference, as defined by the scientific committee, consisting of N. Alon, J. Bourgain, A. Connes, M. Gromov and V. Milman, were to discuss the importance, methods, future and unity/diversity of mathematics as we enter the 21st Century, to consider the relation between mathematics and related areas and to discuss the past and future of mathematics as well as its interaction with Science.

A new format of mathematical discussions developed by the end of the Conference into an interesting addition to the more standard form of lectures and questions.

We believe that the meeting succeeded in giving a wide panorama of mathematics and mathematical physics, but we did not touch upon the interaction of mathematics with the experimental sciences.

This is the first part of the proceedings of the meeting. The second part will appear later this year.

It is a pleasure to thank Mrs. Miriam Hercberg for her great technical help in the preparation of this manuscript.

N. Alon　　J. Bourgain

A. Connes　　M. Gromov

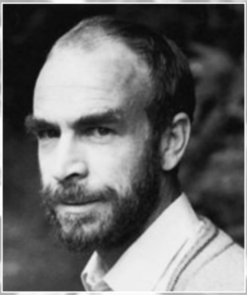
V. Milman

GAFA2000

Tel Aviv, Israel

W. T. Gowers

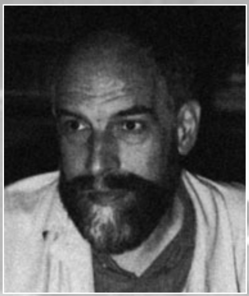
S. Bloch



M. Gromov

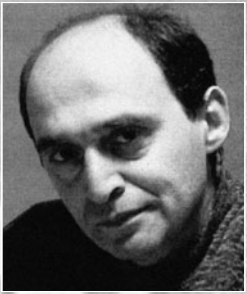
J. Bourgain

V. Kac



J. Fröhlich

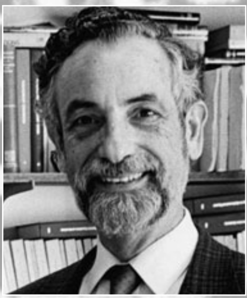
D. Kazhdan


S. Klainerman


Y. Ne'eman


A. Kupiainen


S. Novikov


E. H. Lieb


Ya. G. Sinai


L. Lovász


E. M. Stein

Visions in Mathematics – Towards 2000
Scientific Program

Wednesday, August 25

Lev Auditorium, Tel Aviv University

09:15-09:30 Opening remarks

Geometry *Chair: L. Polterovich*

09:30-10:15 *M. Gromov*: Geometry as the art of asking questions

10:30-11:15 *H. Hofer*: Holomorphic curves and real three-dimensional dynamics

11:45-12:30 *Y. Eliashberg*: Symplectic field theory

Ergodic Theory, Dynamic Systems *Chair: S. Mozes*

14:00-14:45 *H. Furstenberg*: Dynamical methods in diophantine problems

15:00-15:45 *G. Margulis*: Diophantine approximation, lattices and flows on homogeneous spaces

16:15-17:00 *Y. Sinai*: On some problems in the theory of dynamical systems and mathematical physics

17:15- Discussion with introduction by *Y. Sinai*

Thursday, August 26

Lev Auditorium, Tel Aviv University

Mathematical Physics *Chair: A. Jaffe*

09:00-09:45 *J. Fröhlich*: Large quantum systems

10:00-10:45 *Y. Ne'eman*: Physics as geometry - Plato vindicated

11:15-12:00 *A. Connes*: Non-commutative geometry

Computer Science *Chair: N. Alon*

14:00-14:45 *P. Shor*: Mathematical problems in quantum information theory

15:00-15:45 *A. Razborov*: Complexity of proofs and computation

16:15-17:00 *A. Wigderson*: Some fundamental insights of computational complexity

17:15-18:00 *M. Rabin*: The mathematics of trust and adversity

Friday, August 27

Moriah Hotel, Dead Sea

Chair: A. Connes

14:00-14:45 *A. Jaffe*: Mathematics of quantum fields

15:00-15:45 *S. Novikov*: Topological phenomena in real physics

16:00-16:45 Discussion on Mathematical Physics with introduction by *A. Connes*

17:00-18:00 Discussion on Geometry with introduction by *M. Gromov*

Sunday, August 29

Moriah Hotel, Dead Sea

10:00- Discussion on Mathematics in the Real World (image, applications etc.), with introduction by *R. Coifman*

14:00- Discussion on Computer Science and Discrete Mathematics with introduction by *M. Rabin*

Monday, August 30

Lev Auditorium, Tel Aviv University

Analysis *Chair: A. Olevskiĭ*

09:00-09:45 *R. Coifman*: Challenges in analysis 1

10:00-10:45 *P. Jones*: Challenges in analysis 2

11:15-12:00 *E. Stein*: Some geometrical concepts arising in harmonic analysis

12:10-12:40 Discussion

Number Theory *Chair: Z. Rudnick*

14:30-15:15 *H. Iwaniec*: Automorphic forms in recent developments of analytic number theory

15:30-16:15 *P. Sarnak*: Some problems in number theory and analysis

16:45-17:30 *D. Zagier*: On "q" (or "Connections between modular forms, combinatorics and topology")

17:45- Discussion: "The unreasonable effectiveness of modular forms" introduced by *P. Sarnak*

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Lev Auditorium, Tel Aviv University

Discrete Mathematics *Chair: A. Wigderson*

09:00-09:45 *L. Lovász*: Discrete and continuous: two sides of phenomena

10:00-10:45 *N. Alon*: Probabilistic and algebraic methods in discrete mathematics

11:15-12:00 *G. Kalai*: An invitation to Tverberg's theorem

12:10-12:40 Discussion

Analysis *Chair: E. Stein*

14:30-15:15 *T. Gowers*: Rough structure and crude classification

15:30-16:15 *J. Bourgain*: Some problems in Hamiltonian PDE's

16:45-17:30 *V. Milman*: Topics in geometric analysis

17:45- Discussion

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Lev Auditorium, Tel Aviv University

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10:00-10:45 *V. Voevodsky*: Motivic homotopy types

11:15-12:00 *D. Kazhdan*: The lifting problems and crystal base

12:10-12:40 Discussion

Algebra *Chair: D. Kazhdan*

14:30-15:15 *A. Beilinson*: Around geometric Langlands

15:30-16:15 *J. Bernstein*: Equivariant derived categories

16:45-17:30 *V. Kac*: Classification of infinite-dimensional simple groups of supersymmetries and quantum field theory

17:45- Discussion with introduction by *D. Kazhdan*

Thursday, September 2

Lev Auditorium, Tel Aviv University

Chair: M. Gromov

09:00-09:45 *R. MacPherson*: On the applications of topology

10:00-10:45 *M. Kontsevich*: Smooth and compact

11:15-12:00 *D. Sullivan*: String interactions in topology

12:10-12:40 Discussion

Chair: G. Kalai

14:30-15:15 *I. Aumann*: Mathematical game theory: Looking backward and forward

15:30-16:15 *E. Hrushovski*: Logic and geometry

16:45- Discussion: The role of homotopical algebra in physics, with introductions by *D. Sullivan* and *M. Kontsevich*

Friday, September 3

Lev Auditorium, Tel Aviv University

Mathematical Physics and PDE *Chair: V. Zakharov*

09:00-09:45 *T. Spencer*: Universality and statistical mechanics

09:50-10:35 *E. Lieb*: The mathematics of the second law of thermodynamics

11:00-11:45 *A. Kupiainen*: Lessons for turbulence

11:50-12:35 *S. Klainerman*: Some general remarks concerning nonlinear PDE's

12:40-13:10 Discussion

13:10-13:30 Closing remarks

13:30- Continuation of Discussion

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GAFA Geometric And Functional Analysis

GAUß–MANIN DETERMINANT CONNECTIONS AND PERIODS FOR IRREGULAR CONNECTIONS

Spencer Bloch and Hélène Esnault

Abstract

Gauß–Manin determinant connections associated to irregular connections on a curve are studied. The determinant of the Fourier transform of an irregular connection is calculated. The determinant of cohomology of the standard rank 2 Kloosterman sheaf is computed modulo 2 torsion. Periods associated to irregular connections are studied in the very basic $\exp(f)$ case, and analogies with the Gauß–Manin determinant are discussed.

Everything's so awful reg'lar a body can't stand it.

The Adventures of Tom Sawyer
Mark Twain

1 Introduction

A very classical area of mathematics, at the borderline between applied mathematics, algebraic geometry, analysis, mathematical physics and number theory is the theory of systems of linear differential equations (connections). There is a vast literature focusing on regular singular points, Picard-Fuchs differential equations, Deligne's Riemann-Hilbert correspondence and its extension to $\mathcal{D}$-modules, and more recently various index theorems in geometry.

The classification of irregular singular points in terms of Stokes structures is explained in [M, chap. IV]. It involves the Riemann-Hilbert correspondence for holonomic $\mathcal{D}$-modules with regular singular points, together with a filtration related to the rate of growth of local solutions.

Nonetheless, there are some very modern themes in the subject which remain virtually untouched. On the arithmetic side, one may ask, for example, how irregular connections can be incorporated in the modern theory of motives? How deep are the apparent analogies between wild ramification in characteristic p and irregularity? Can one define periods for irregular connections? If so, do the resulting period matrices have anything to do with ϵ-factors for ℓ-adic representations? On the geometric side, one can ask for a theory of characteristic classes $c_i(E,\nabla)$ for irregular connections such that c_0 is the rank of the connection and c_1 is the isomorphism class of the determinant. With these, one can try to attack the Riemann-Roch problem.

In this note, we describe some conjectures and examples concerning what might be called a families index theorem for irregular connections. Let

$$f : X \to S \tag{1.1}$$

be a smooth, projective family of curves over a smooth base S. Let $\mathcal{D}$ be an effective relative divisor on X. Let E be a vector bundle on X, $\nabla : E \to E \otimes \Omega^1_X(\mathcal{D})$ an integrable, absolute connection with poles along $\mathcal{D}$. The relative de Rham cohomology $\mathbb{R}f_*(\Omega^*_{X-\mathcal{D}} \otimes E)$ inherits a connection (Gauß–Manin connection), and the families index or Riemann-Roch problem is to describe the isomorphism class of the line bundle with connection

$$\det \mathbb{R}f_*(\Omega^*_{X-\mathcal{D}} \otimes E)\,. \tag{1.2}$$

The Gauß–Manin construction is fairly standard and we do not recall it in detail. By way of example, we cite two classical formulas (Gauß hypergeometric and Bessel functions, respectively):

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z) = \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a} du$$

$$J_n(z) = \frac{1}{2\pi i} \int_{S_0} u^{-n} \exp \frac{z}{2}\left(u - \frac{1}{u}\right) \frac{du}{u} \qquad (S_0 = \text{circle about } 0)\,.$$

In both cases, the integrand is a product of a solution of a rather simple degree 1 differential equation in u, the solution being either

$$u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a} \quad \text{or} \quad u^{-n} \exp \tfrac{z}{2}\left(u + \tfrac{1}{u}\right)\,,$$

with an algebraic 1-form (du or du/u). The integral is taken over a chain in the u-plane. The resulting functions $F(a,b;c;z)$ and $J_n(z)$ satisfy Gauß–Manin equations, which are much more interesting degree 2 equations in z.

It is not our purpose to go further into the classical theory, but, to understand the role of the determinant, we remark that in each of the above cases, there is a second path and a second algebraic 1-form such that the two integrals, say $f_1(z)$ and $f_2(z)$, satisfy the same second order equation. The Wronskian determinant

$$\begin{vmatrix} f_1 & f_2 \\ \frac{df_1}{dz} & \frac{df_2}{dz} \end{vmatrix}$$

satisfies the degree 1 equation given by the determinant of Gauß–Manin.

We are working here with algebraic de Rham cohomology. Analytically (i.e. permitting coordinate and basis transformations with essential singularities on $\mathcal{D}$), the bundle $E^{\mathrm{an}}|_{X-\mathcal{D}}$ can be transformed (locally on S) to have regular singular points along $\mathcal{D}$, but the algebraic problem we pose is more subtle.

If $\mathcal{D}=\emptyset$, or, more generally, if ∇ has regular singular points, the answer is known:

$$\det \mathbb{R}f_*(\Omega^*_{X-\mathcal{D}}\otimes E) = f_*\big((\det E^\vee, -\det\nabla)\cdot c_1(\omega_{X/S})\big)\,. \quad (1.3)$$

(In the case of regular singular points, the c_1 has to be taken as a relative class [ST], [BE1].)

In a recent article [BE2], we proved an analogous formula in the case when E was irregular and rank 1. For a suitable $\mathcal{D}$, the relative connection induces an isomorphism

$$\nabla_{X/S,\mathcal{D}} : E|_{\mathcal{D}} \cong E|_{\mathcal{D}} \otimes \big(\omega(\mathcal{D})/\omega\big)\,,$$

i.e. a trivialization of $\omega(\mathcal{D})|_{\mathcal{D}}$. The connection pulls back from a rank 1 connection $(\mathcal{E},\nabla_{\mathcal{E}})$ on the relative Picard scheme $\mathrm{Pic}(X,\mathcal{D})$, and the Gauß–Manin determinant connection is obtained by evaluating $(\mathcal{E},\nabla_{\mathcal{E}})$ at the priviledged point $(\omega(\mathcal{D}),\nabla_{X/S,\mathcal{D}})\in\mathrm{Pic}(X,\mathcal{D})$.

We want now to consider two sorts of generalizations. First, we formulate an analogous conjecture for higher rank connections which are admissible in a suitable sense. We prove two special cases of this conjecture, computing the determinant of the Fourier transform of an arbitrary connection and, up to 2-torsion, the determinant of cohomology of the basic rank 2 Kloosterman sheaf.

Second, we initiate in the very simplest of cases $E=\mathcal{O}$, $\nabla(1)=df$, the study of periods for irregular connections. Let $m=\deg f+1$. We are led to a stationary phase integral calculation over the subvariety of $\mathrm{Pic}(\mathbb{P}^1, m\cdot\infty)$ corresponding to trivializations of $\omega(m\cdot\infty)$ at ∞. The subtle part of the integral is concentrated at the same point

$$\big(\omega(m\cdot\infty),\nabla_{X/S,m\cdot\infty}\big)\in\mathrm{Pic}(\mathbb{P}^1,m\cdot\infty)$$

mentioned above.

Our hope is that the geometric methods discussed here will carry over to the sort of arithmetic questions mentioned above. We expect that the distinguished point given by the trivialization of $\omega(\mathcal{D})$ plays some role in the calculation of ϵ-factors for rank 1 ℓ-adic sheaves and that the higher rank conjectures have some ℓ-adic interpretation as well.

We would like to thank Pierre Deligne for sharing his unpublished letters with us. The basic idea of using the relative Jacobian to study ϵ-factors for rank 1 sheaves we learned from him. We also have gotten considerable inspiration from the works of T. Saito and T. Terasoma cited in the bibliography. The monograph [K] is an excellent reference for Kloosterman sheaves, and the subject of periods for exponential integrals is discussed briefly at the end of [Ko].

2 The Conjecture

Let S be a smooth scheme over a field k of characteristic 0. We consider a smooth family of curves $f : X \to S$ and a vector bundle E with an absolute, integrable connection $\nabla : E \to E \otimes \Omega^1_X(*D)$. Here $D \subset X$ is a divisor which is smooth over S. We are interested in the determinant of the Gauß–Manin connection

$$\det \mathbb{R}f_*(\Omega^*_{(X-D)/S} \otimes E) \in \mathrm{Pic}^\nabla(S) := \mathbb{H}^1(S, \mathcal{O}_S^\times \xrightarrow{d\log} \Omega^1_S) \cong \Gamma(S, \Omega^1_S/d\log \mathcal{O}_S^\times)\,. \quad (2.1)$$

PROPOSITION 2.1. *Let $K = k(S)$ be the function field of S. Then the restriction map $\mathrm{Pic}^\nabla(S) \to \mathrm{Pic}^\nabla(\mathrm{Spec}(K))$ is an injection.*

Proof. In view of the interpretation of Pic^∇ as $\Gamma(\Omega^1/d\log \mathcal{O}^\times)$, the proposition follows from the fact that for a meromorphic function g on S, we have g regular at point $s \in S$ if and only if dg/g is regular at s. □

Thus, we do not lose information by taking the base S to be the spectrum of a function field, $S = \mathrm{Spec}(K)$. We shall restrict ourselves to that case.

Let

$$\mathcal{D} = \sum_{x \in D(\overline{K})} m_x x \quad (2.2)$$

be an effective divisor supported on D. Suppose that the relative connection has poles of order bounded by $\mathcal{D}$, i.e.

$$\nabla_{X/S} : E \to E \otimes \omega(\mathcal{D})\,, \quad (2.3)$$

and that $\mathcal{D}$ is minimal with this property. (We frequently write ω in place of $\Omega_{X/S}$.) We view $\mathcal{D}$ as an artinian scheme with sheaf of functions $\mathcal{O}_\mathcal{D}$ and relative dualizing sheaf $\omega_\mathcal{D} := \omega(\mathcal{D})/\omega$. (To simplify, we do not use the notation $\omega_{\mathcal{D}/K}$).

PROPOSITION 2.2. *Define $E_{\mathcal{D}} := E \otimes \mathcal{O}_{\mathcal{D}}$. Then $\nabla_{X/S}$ induces a function-linear map*

$$\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \omega_{\mathcal{D}} \,.$$

Proof. Straightforward. □

Let $j : X - D \hookrightarrow X$. There are now two relative de Rham complexes we might wish to study

$$j_* j^* E \to j_* j^* (E \otimes \omega)$$
$$E \to E \otimes \omega(\mathcal{D}) \,.$$

The first is clearly the correct one. For example, its relative cohomology carries the Gauß–Manin connection. On the other hand, the second complex is sometimes easier to study, as the relative cohomology of the sheaves involved has finite dimension over K. Since we are primarily interested in the irregular case, that is when points of $\mathcal{D}$ have multiplicity ≥ 2, the following proposition clarifies the situation.

PROPOSITION 2.3. *Let notation be as above, and assume every point of $\mathcal{D}$ has multiplicity ≥ 2. Then the natural inclusion of complexes*

$$\iota : \big\{E \to E \otimes \omega(\mathcal{D})\big\} \hookrightarrow \big\{j_* j^* E \to j_* j^* (E \otimes \omega)\big\}$$

is a quasiisomorphism if and only if $\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}$ in Proposition 2.2 is an isomorphism.

Proof. Let $\mathcal{D} : h = 0$ be a local defining equation for $\mathcal{D}$. Write $\mathcal{D} = \mathcal{D}' + D$ where D is the reduced divisor with support $= \mathrm{supp}(\mathcal{D})$. We claim first that the map ι is a quasiisomorphism if and only if for all $n \geq 1$ the map G defined by the commutative diagram

$$\begin{array}{ccc} E/E(-\mathcal{D}) & \xrightarrow{\nabla_{X/S,\mathcal{D}} - n\cdot id \otimes \frac{dh}{h}} & \big(E(\mathcal{D}')/E(-D)\big) \otimes \omega(D) \\ \cong \big\downarrow \text{“}\cdot h^{-n}\text{”} & & \cong \big\downarrow \text{“}\cdot h^{-n}\text{”} \\ E(n\mathcal{D})/E\big((n-1)\mathcal{D}\big) & \xrightarrow{G} & \big(E(n\mathcal{D}+\mathcal{D}')/E((n-1)\mathcal{D}+\mathcal{D}')\big) \otimes \omega(D) \end{array}$$

given by $h^{-n}e \mapsto h^{-n}\nabla_{X/S,\mathcal{D}}(e) - nh^{-n-1}e \otimes dh$ is a quasi-isomorphism. This follows by considering the cokernel of ι

$$j_* j^* E/E \to j_* j^* (E \otimes \omega)/E \otimes \omega(\mathcal{D})$$

and filtering by order of pole. The assertion of the proposition follows because $nh^{-n-1}e \otimes dh$ has a pole of order strictly smaller than the multiplicity of $(n+1)\mathcal{D}$ at every point of $\mathcal{D}$. □

We will consider only the case

$$\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \cong E_{\mathcal{D}} \otimes \omega_{\mathcal{D}} \,. \tag{2.4}$$

We now consider the sheaf $j_*j^*\Omega^1_X$ of absolute (i.e. relative to k) 1-forms on X with poles on D. Let $\mathcal{D} = \sum m_x x$ be an effective divisor supported on D as above, and write $\mathcal{D}' = \mathcal{D} - D$.

Definition 2.4. *The sheaf $\Omega^p_X\{\mathcal{D}\} \subset j_*j^*\Omega^p_X$ is defined locally around a point $x \in D$ with local coordinate z by*

$$\Omega^p_X\{\mathcal{D}\}_x = \Omega^p_X(\mathcal{D}')_x + \Omega^{p-1}_{X,x} \wedge \frac{dz}{z^{m_x}} \,.$$

The graded sheaf $\bigoplus_p \Omega^p_X\{\mathcal{D}\}$ is stable under the exterior derivative and independent of the choice of local coordinates at the points of D. One has exact sequences

$$0 \to f^*\Omega^p_S(\mathcal{D}') \to \Omega^p_X\{\mathcal{D}\} \to \Omega^{p-1}_S(\mathcal{D}') \otimes \omega(D) \to 0. \tag{2.5}$$

Definition 2.5. *An integrable absolute connection on E will be called admissible if there exists a divisor $\mathcal{D}$ such that $\nabla : E \to E \otimes \Omega^1_X\{\mathcal{D}\}$ and such that $\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \cong E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}$.*

Remark 2.6. *When E has rank 1, there always exists a $\mathcal{D}$ such that ∇ is admissible for $\mathcal{D}$ ([BE2, Lemma 3.1]). In higher rank, this need not be true, even if $\nabla_{X/S,\mathcal{D}}$ is an isomorphism for some $\mathcal{D}$. For example, let $\eta \in \Omega^1_K$ be a closed 1-form. Let $n \geq 1$ be an integer and let $c \in k$, $c \neq 0$. The connection matrix*

$$A = \begin{pmatrix} \frac{cdz}{z^m} & \frac{\eta}{z^n} \\ 0 & \frac{cdz}{z^m} - \frac{ndz}{z} \end{pmatrix}$$

satisfies $dA + A \wedge A = 0$ for all $m, n \in \mathbb{N}$, but the resulting integrable connection is not admissible for $n > m$, although $\nabla_{X/S,(0)}$ is an isomorphism for $c \neq n$ if $m = 1$. Note in this case it is possible to change basis to get an admissible connection. We don't know what to expect in general. There do exist connections for which $\nabla_{X/S,\mathcal{D}}$ is not an isomorphism for any $\mathcal{D}$, for example, if one takes a sum of rank 1 connections with different m_x as above (see notations (2.2)) and a local basis adapted to this direct sum decomposition.

Henceforth, $S = \mathrm{Spec}(K)$ is the spectrum of a function field, and we consider only integrable, admissible connections $\nabla : E \to E \otimes \Omega^1_X\{\mathcal{D}\}$ with $\mathcal{D} \neq \emptyset$. In sections 3 and 4 we will see many importnat examples (Fourier transforms, Kloosterman sheaves) of admissible connections. By abuse of notation, we write

$$H^*_{DR/S}(E) := \mathbb{H}^*\big(X, E \to E \otimes \omega(\mathcal{D})\big) \,. \tag{2.6}$$

We assume (Prop. 2.3) this group coincides with $H^*_{DR}(E|_{X-D})$. The isomorphism class of the Gauß–Manin connection on the K-line $\det H^*_{DR/S}(E)$ is determined by an element

$$\det H^*_{DR/S}(E) \in \Omega^1_{K/k}/d\log(K^\times)$$

which we would like to calculate.

Suppose first that E is a line bundle. Twisting E by $\mathcal{O}(\delta)$ for some divisor δ supported on the irregular part of the divisor D, we may assume $\deg E = 0$. In this case, the result (the main theorem in [BE2]) is the following. Since E has rank 1, $\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \cong E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}$ can be interpreted as a section of $\omega_{\mathcal{D}}$ which generates this sheaf as an $\mathcal{O}_{\mathcal{D}}$-module. The exact sequence

$$0 \to \omega \to \omega(\mathcal{D}) \to \omega_{\mathcal{D}} \to 0 \tag{2.7}$$

yields an element $\partial\nabla_{X/S,\mathcal{D}} \in H^1(X,\omega) \cong K$ which is known to equal $\deg E = 0$. Thus, we can find some $s \in H^0(X,\omega(\mathcal{D}))$ lifting $\nabla_{X/S,\mathcal{D}}$. We write (s) for the divisor of s as a section of $\omega(\mathcal{D})$ (so (s) is disjoint from D). Then the result is

$$\det H^*_{DR/S}(E) \cong -f_*\big((s)\cdot E\big)\,. \tag{2.8}$$

(When (s) is a disjoint union of K-points, the notation on the right simply means to restrict E with its absolute connection to each of the points and then tensor the resulting K-lines with connection together.) Notice that unlike the classical Riemann-Roch situation (e.g. (1.3)) the divisor (s) depends on $(E,\nabla_{X/S})$.

Another way of thinking about (2.8) will be important when we consider periods. It turns out that the connection (E,∇) pulls back from a rank 1 connection $(\mathcal{E},\nabla_{\mathcal{E}})$ on the relative Picard scheme $\mathrm{Pic}(X,\mathcal{D})$ whose points are isomorphism classes of line bundles on X with trivializations along $\mathcal{D}$. The pair $(\omega(\mathcal{D}),\nabla_{X/D,\mathcal{D}})$ determine a point $t \in \mathrm{Pic}(X,\mathcal{D})(K)$, and (2.8) is equivalent to

$$\det H^*_{DR/S}(E) \cong -(\mathcal{E},\nabla_{\mathcal{E}})|_t\,. \tag{2.9}$$

Let $\omega^\times_{\mathcal{D}} \subset \omega_{\mathcal{D}}$ be the subset of elements generating $\omega_{\mathcal{D}}$ as an $\mathcal{O}_{\mathcal{D}}$-module. Let $\partial : \omega_{\mathcal{D}} \to H^1(X,\omega) = K$. Define $\tilde{B} = \omega^\times_{\mathcal{D}} \cap \partial^{-1}(0)$. One has a natural action of $K^\times$, and the quotient $\omega^\times_{\mathcal{D}}/K^\times$ is identified with isomorphism classes of trivializations of $\omega(\mathcal{D})|_{\mathcal{D}}$, and hence with a subvariety of $\mathrm{Pic}(X,\mathcal{D})$. One has

$$t \in B := \tilde{B}/K^\times \subset \omega^\times_{\mathcal{D}}/K^\times \subset \mathrm{Pic}(X,\mathcal{D})\,. \tag{2.10}$$

The relation between $t, B, \mathcal{E}$ is the following. $\mathcal{E}|_B \cong \mathcal{O}_B$, so the connection $\nabla_{\mathcal{D}}|_B$ is determined by a global 1-form Ξ. Then

$$\Xi(t) = 0 \in \Omega^1_{B/K} \otimes K(t)\,. \tag{2.11}$$

Indeed, t is the unique point on B where the relative 1-form Ξ/K vanishes (cf. [BE2, Lemma 3.10]).

Now suppose the rank of E is > 1. We will see when we consider examples in the next section that $\det H^*_{DR/S}(E)$ depends on more than just the connection on $\det E$ (Remark 3.3). Thus, it is hard to imagine a simple formula like (2.8). Indeed, there is no obvious way other than by taking the determinant to get rank 1 connections on X from E. The truly surprising thing is that if we rewrite (2.8) algebraically we find a formula which does admit a plausible generalization. We summarize the results, omitting proofs (which are given in detail in [BE2]). For each $x_i \in D$, choose a local section s_i of $\Omega^1_X\{\mathcal{D}\}$ whose image in $\omega(\mathcal{D})$ generates at x_i. Write the local connection matrix in the form

$$A_i = g_i s_i + \frac{\eta_i}{z_i^{m_i-1}} \tag{2.12}$$

with z_i a local coordinate and $\eta_i \in f^*\Omega^1_S$. Let s be a meromorphic section of $\omega(\mathcal{D})$ which is congruent to the $\{s_i\}$ modulo $\mathcal{D}$. Define

$$c_1\big(\omega(\mathcal{D}), \{s_i\}\big) := (s) \in \mathrm{Pic}(X, \mathcal{D})\,. \tag{2.13}$$

As in (2.8), we can define

$$f_*\big(c_1(\omega(\mathcal{D}), \{s_i\}) \cdot \det(E, \nabla)\big) \in \Omega^1_K / d\log K^\times\,. \tag{2.14}$$

We further define

$$\big\{c_1(\omega(\mathcal{D})), \nabla\big\} := f_*\big(c_1(\omega(\mathcal{D}), \{s_i\}) \cdot \det(E, \nabla)\big) - \sum_i \mathrm{res}\,\mathrm{Tr}(dg_i g_i^{-1} A_i)\,. \tag{2.15}$$

Here res refers to the map

$$\Omega^2_X\{\mathcal{D}\} \to \Omega^1_K \otimes \omega(\mathcal{D}) \to \Omega^1_K \otimes \omega_{\mathcal{D}} \xrightarrow{\text{transfer}} \Omega^1_K\,. \tag{2.16}$$

Conjecture 2.7. *Let $\nabla : E \to E \otimes \Omega^1_X\{\mathcal{D}\}$ be an admissible connection as in Definition 2.5. Then*

$$\det H^*_{DR/S}(X - D, E) = -\big\{c_1(\omega(\mathcal{D})), \nabla\big\} \in \Omega^1_K / d\log(K^\times) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

Our main objective here is to provide evidence for this conjecture. Of course, one surprising fact is that the right-hand side is independent of choice of gauge, etc. Again, the proof is given in detail in [BE2] but we reproduce two basic lemmas. There are function linear maps

$$\nabla_{X,\mathcal{D}} : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \big(\Omega^1_X\{\mathcal{D}\}/\Omega^1_X\big)\,; \quad \nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}\,, \tag{2.17}$$

and it makes sense to consider the commutator

$$[\nabla_{X,\mathcal{D}}, \nabla_{X/S,\mathcal{D}}] : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \left(\Omega^1_X\{\mathcal{D}\}/\Omega^1_X\right) \otimes \omega_{\mathcal{D}}\,.$$

LEMMA 2.8. $\qquad [\nabla_{X,\mathcal{D}}, \nabla_{X/S,\mathcal{D}}] = 0.$

Proof. With notation as in (2.12), we take $s_i = dz_i/z_i^{m_i}$, where z_i is a local coordinate. Integrability implies

$$dA_i = dg_i \wedge \tfrac{dz_i}{z_i^{m_i}} + d\Big(\tfrac{\eta_i}{z_i^{m_i-1}}\Big) = A_i^2 = [\eta_i, g_i]\tfrac{dz_i}{z_i^{2m_i-1}} + \epsilon \quad (2.18)$$

with $\epsilon \in \Omega^2_K \otimes K(X)$. Multiplying through by $z_i^{m_i}$, we conclude that $[\eta_i/z_i^{m_i-1}, g_i]$ is regular on D, which is equivalent to the assertion of the lemma. □

The other lemma which will be useful in evaluating the right-hand term in (2.15) is

LEMMA 2.9. *We consider the situation from* (2.12) *and* (2.15) *at a fixed* $x_i \in D$. *For simplicity, we drop the* i *from the notation. Assume* $ds = 0$. *Then*

$$\operatorname{res}\operatorname{Tr}(dgg^{-1}A) = \operatorname{res}\operatorname{Tr}\left(dgg^{-1}\tfrac{\eta}{z^{m-1}}\right).$$

Proof. We must show $\operatorname{res}\operatorname{Tr}(dgs) = 0$. Using (2.18) and $\operatorname{Tr}[g,\eta] = 0$ we reduce to showing $0 = \operatorname{res}\operatorname{Tr}(d(\eta z^{1-m})) \in \Omega^1_K$. Since $\eta \in f^*\Omega^1_K$, we may do the computation formally locally and replace d by d_z. The desired vanishing follows because an exact form has no residues. □

With Lemma 2.8, we can formulate the conjecture in a more invariant way in terms of an AD-cocycle on X. Recall [E]

$$AD^2(X) := \mathbb{H}^2(X, \mathcal{K}_2 \xrightarrow{d\log} \Omega^2_X) \cong H^1(X, \Omega^2_X/d\log\mathcal{K}_2)\,. \quad (2.19)$$

The AD-groups are the cones of cycle maps from Chow groups to Hodge cohomology, and as such they carry classes for bundles with connections. There is a general trace formalism for the AD-groups, but in this simple case the reader can easily deduce from the right-hand isomorphism in (2.19) a trace map

$$f_* : AD^2(X) \to AD^1(S) = \Omega^1_K/d\log K^\times. \quad (2.20)$$

When the connection ∇ has no poles (or more generally, when it has regular singular points) it is possible to define a class

$$\epsilon = c_1(\omega)\cdot c_1(E,\nabla) \in AD^2(X) \quad (2.21)$$

with $f_*(\epsilon) = [\det H^1_{DR/S}(X,E)]$. Remarkably, though it no longer has the product description (2.21), one can associate such a class to any admissible connection.

Fix the divisor $\mathcal{D}$ and consider tuples $\{E,\nabla,\mathcal{L},\mu\}$ where (E,∇) is an admissible, absolute connection, $\mathcal{L}$ is a line bundle on X, and $\mu : E_{\mathcal{D}} \cong E_{\mathcal{D}}\otimes\mathcal{L}_{\mathcal{D}}$. We require

$$0 = [\mu, \nabla_{X,\mathcal{D}}] : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \mathcal{L}_{\mathcal{D}} \otimes \left(\Omega^1_X\{\mathcal{D}\}/\Omega^1_X\right). \tag{2.22}$$

Of course, the example we have in mind, using Lemma 2.8, is

$$\{E,\nabla\} := \left\{E,\nabla,\omega(\mathcal{D}),\nabla_{X/S,\mathcal{D}}\right\}. \tag{2.23}$$

To such a tuple satisfying (2.22), we associate a class $\epsilon(E,\nabla,\mathcal{L},\mu) \in AD^2(X)$ as follows. Choose cochains $c_{ij} \in GL(r,\mathcal{O}_X)$ for E, $\lambda_{ij} \in \mathcal{O}_X^\times$ for $\mathcal{L}$, $\mu_i \in GL(r,\mathcal{O}_{\mathcal{D}})$ for μ, and $\omega_i \in M(r\times r,\Omega^1_X\{\mathcal{D}\})$ for ∇. Choose local liftings $\tilde{\mu_i} \in GL(r,\mathcal{O}_X)$ for the μ_i.

Proposition 2.10. *The Cech hypercochain*

$$\left(\{\lambda_{ij},\det(c_{jk})\}, d\log\lambda_{ij}\wedge \mathrm{Tr}(\omega_j), \mathrm{Tr}(-d\tilde{\mu_i}\tilde{\mu_i}^{-1}\wedge\omega_i)\right)$$

represents a class

$$\epsilon(E,\nabla,\mathcal{L},\mu) \in \mathbb{H}^2\left(X,\mathcal{K}_2 \to \Omega^2_X\{\mathcal{D}\} \to \Omega^2_X\{\mathcal{D}\}/\Omega^2_X\right) \cong AD^2(X).$$

This class is well defined independent of the various choices. Writing $\epsilon(E,\nabla) = \epsilon(E,\nabla,\omega(\mathcal{D}),\nabla_{X/S,\mathcal{D}})$, *we have*

$$f_*\epsilon(E,\nabla) = \left\{c_1(\omega(\mathcal{D})),\nabla\right\}$$

where the right-hand side is defined in (2.15).

Proof. Again the proof is given in detail in [BE2] and we omit it. □

As a consequence, we can restate the main conjecture:

Conjecture 2.11. *Let ∇ be an integrable, admissible, absolute connection as above. Then*

$$\det H^*_{DR/S}(E,\nabla) = -f_*\epsilon(E,\nabla).$$

To finish this section, we would like to show that behind the quite technical cocyle written in Proposition 2.10, there is an algebraic group playing a role similar to $\mathrm{Pic}(X,\mathcal{D})$ in the rank 1 case. Let G be the algebraic group whose K-points are isomorphism classes $(\mathcal{L},\mu)$, where $\mathcal{L}$ is an invertible sheaf, and $\mu : E_{\mathcal{D}} \to E_{\mathcal{D}}\otimes\mathcal{L}_{\mathcal{D}}$ is an isomorphism commuting with $\nabla_{X,\mathcal{D}}$. It is endowed with a surjective map $q : G \to \mathrm{Pic}(X)$. As noted, G contains the special point $(\omega(\mathcal{D}),\nabla_{X/S,\mathcal{D}})$. The cocyle of Proposition 2.10 defines a class in $\mathbb{H}^2(X\times_K G,\mathcal{K}_2 \to \Omega^2_{X\times G}\{\mathcal{D}\times G\} \to \Omega^2_{X\times G}\{\mathcal{D}\times G\}/\Omega^2_{X\times G})$. Taking its trace (2.20), one obtains a class in $(\mathcal{L}(E),\nabla(E)) \in AD^1(G)$, that is a rank one connection on G. Then $f_*\epsilon(E,\nabla)$ is simply the restriction of $(\mathcal{L}(E),\nabla(E))$ to the special point $(\omega(\mathcal{D}),\nabla_{X/S,\mathcal{D}})$.

Now we want to show that this special point, as in the rank 1 case, has a very special meaning. By analogy with (2.10), we define

$$\tilde{B} = \big(\mathrm{Ker}(\mathrm{Hom}(E_{\mathcal{D}}, E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}) \xrightarrow{\mathrm{res\,Tr}} K)\big) \cap \mathrm{Isom}(E_{\mathcal{D}}, E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}) \tag{2.24}$$

$$B = \tilde{B}/K^{\times} \subset G\,. \tag{2.25}$$

We observe that Lemma 2.9 shows that $g \in B$. Choosing a local trivialization $\omega_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}\frac{dz}{z^m}$ and a local trivialization of $E_{\mathcal{D}}$, we write $\nabla_{X/S}|_{\mathcal{D}}$ as a matrix $g\frac{dz}{z^m}$, with $g \in GL_r(\mathcal{O}_{\mathcal{D}})$. Θ is then identified with a translation invariant form on the restriction of scalars $\mathrm{Res}_{\mathcal{D}/K}GL_r$

$$\Theta := \big(\mathcal{L}(E), \nabla_{G/K}(E)\big) = \mathrm{res\,Tr}\left(d\mu\mu^{-1}g\tfrac{dz}{z^m}\right)\,. \tag{2.26}$$

The assumption that $g \in B$ implies that Θ descends to an invariant form on $\mathrm{Res}_{\mathcal{D}/K}GL_r/\mathbb{G}_m \supset G_0$, where $G_0 := \{(\mathcal{O}, \mu) \in G\}$. By invariance, it gives rise to a form on the G_0-torsor $G_{\omega(\mathcal{D})} := \{(\omega(\mathcal{D}), \mu) \in G\}$. We have

$$B \subset G_{\omega(\mathcal{D})} \subset G\,.$$

Let $S \subset G_0$ be the subgroup of points stabilizing B.

Proposition 2.12. *$\Theta|_B$ vanishes at a point $t \in B$ if and only if t lies in the orbit $g \cdot S$.*

Proof. Write $th = g$. Write the universal element in $\mathrm{Res}_{\mathcal{D}/K}GL_r$ as a matrix $X = \sum_{k=0}^{m-1}(X_{ij}^{(k)})_{ij}z^k$. The assertion that $\Theta|_B$ vanishes in the fibre at t means

$$\mathrm{res\,Tr}\Big(\sum_k \big(d(X_{ij}^{(k)})_{ij}z^k\big)h\tfrac{dz}{z^m}\Big)(t) = a\Big(\sum_i dX_{ii}^{(m-1)}\Big)(t)$$

for some $a \in K$. Note this is an identity of the form

$$0 = \sum c_{ij}^{(k)}dX_{ij}^{(k)} \in \Omega^1_{G/K} \otimes K(t)\,; \quad c_{ij}^{(k)} \in K\,. \tag{2.27}$$

We first claim that in fact this identity holds already in $\Omega^1_{G/K}$. To see this, write $\mathcal{G} = \mathrm{Res}_{\mathcal{D}/K}GL_r$. Note $\Omega^1_{\mathcal{G}}$ is a free module on generators $dX_{ij}^{(k)}$. Also, $G \subset \mathcal{G}$ is defined by the equations

$$\Big[\sum_{k=0}^{m-1}(X_{ij}^{(k)})_{ij}z^k, \sum_{k=0}^{m-1}(g_{ij}^{(k)})_{ij}z^k\Big] = 0\,,$$

$$\Big[\sum_{k=0}^{m-1}(X_{ij}^{(k)})_{ij}z^k, \sum_{k=0}^{m-2}(\eta_{ij}^{(k)})_{ij}z^k\Big] = 0\,,$$

which are of the form

$$\sum_{i,j,k} b_{ijp}^{(k)} X_{ij}^{(k)} = 0\,, \qquad p = 1,2,\dots,M\,;\ b_{ijp}^{(k)} \in K\,,$$

that is are linear equations in the $X_{ij}^{(k)}$ with K-coefficients. Thus, we have an exact sequence

$$0 \to N^\vee \to \Omega^1_{\mathcal{G}/K} \otimes \mathcal{O}_G \to \Omega^1_{G/K} \to 0\,, \tag{2.28}$$

where $N^\vee$ is generated by K-linear combinations of the $dX_{ij}^{(k)}$. We have, therefore, a reduction of structure of the sequence (2.28) from $\mathcal{O}_G$ to K, and therefore $\Omega^1_{G/K} \cong \Omega^1_0 \otimes_K \mathcal{O}_G$, where $\Omega^1_0 \subset \Omega^1_{G/K}$ is the K-span of the $dX_{ij}^{(k)}$. Hence, any K-linear identity among the $dX_{ij}^{(k)}$ which holds at a point on G holds everywhere on G. As a consequence, we can integrate to an identity

$$\operatorname{res}\operatorname{Tr}\left(Xh\tfrac{dz}{z^m}\right) = a\Big(\sum_i X_{ii}^{(m-1)}\Big) + \kappa \tag{2.29}$$

with $\kappa \in K$. If we specialize $X \to t$ we find

$$0 = \operatorname{res}\operatorname{Tr}\left(g\tfrac{dz}{z^m}\right) = a \cdot \operatorname{res}\operatorname{Tr}\left(t\tfrac{dz}{z^m}\right) + \kappa = \kappa\,. \tag{2.30}$$

We conclude from (2.29) and (2.30) that $h \in S$. □

3 The Fourier Transform

In this section we calculate the Gauß–Manin determinant line for the Fourier transform of a connection on $\mathbb{P}^1 - D$ and show that it satisfies the Conjecture 2.7. Let $\mathcal{D} = \sum m_\alpha \alpha$ be an effective k-divisor on $\mathbb{P}^1_k$. Let $\mathcal{E} = \oplus_r \mathcal{O}$ be a rank r free bundle on $\mathbb{P}^1_k$, and let $\Psi : \mathcal{E} \to \mathcal{E} \otimes \omega(\mathcal{D})$ be a k-connection on $\mathcal{E}$. Let $(\mathcal{L}, \Xi)$ denote the rank 1 connection on $\mathbb{P}^1 \times \mathbb{P}^1$ with poles on $\{0,\infty\} \times \{0,\infty\}$ given by $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$ and $\Xi(1) = d(z/t)$. Here z, t are the coordinates on the two copies of the projective line. Let $K = k(t)$. We have a diagram

$$\begin{array}{ccc} (\mathbb{P}^1_z - \mathcal{D}) \times \mathbb{P}^1_t & \hookleftarrow & (\mathbb{P}^1_z - \mathcal{D}) \times \operatorname{Spec}(K) \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p_2} \\ \mathbb{P}^1_z - \mathcal{D} & & \operatorname{Spec}(K) \end{array} \tag{3.1}$$

The Gauß–Manin determinant of the Fourier transform is given at the generic point by

$$\det H^*_{DR/K}\big((\mathbb{P}^1_z - \mathcal{D})_K,\ p_1^*(\mathcal{E},\Psi)\otimes(\mathcal{L},\Xi)\big) \\ = \det H^*_{DR/K}\big((\mathbb{P}^1_z - \mathcal{D})_K,(E,\nabla)\big) \quad (3.2)$$

with $E := p_1^*\mathcal{E}\otimes\mathcal{L}|_{(\mathbb{P}^1_z-\mathcal{D})_K}$ and $\nabla = \Psi\otimes 1 + 1\otimes\Xi$. We have the following easy

Remark 3.1. *Write*

$$\Psi = \sum_\alpha \sum_{i=1}^{m_\alpha} \frac{g_i^\alpha dz}{(z-\alpha)^i} + d(g_1^\infty z + \ldots + g^\infty_{m_\infty -1} z^{m_\infty -1}) \qquad (3.3)$$

where $g_i^\alpha \in M(r\times r, k)$. *Then*

$$\nabla = \Psi + \tfrac{dz}{t} - \tfrac{zdt}{t^2} \qquad (3.4)$$

is admissible if and only if either

(i) $m_\infty \le 2$ *and* $g^\alpha_{m_\alpha}$ *is invertible for all* $\alpha \neq \infty$, *or*
(ii) $m_\infty \ge 3$, $g^\alpha_{m_\alpha}$ *is invertible for all* $\alpha\neq\infty$, *and* $g^\infty_{m_\infty-1}$ *is invertible.*

Theorem 3.2. *The connection* (E,∇) *satisfies Conjecture 2.7.*

Proof. We first consider the case when Ψ has a pole of order ≤ 1 at infinity, so the $g_i^\infty = 0$ in (3.3). A basis for

$$H^0\Big(\mathbb{P}^1_K, E\otimes\omega\Big(\sum m_\alpha\alpha + 2\infty\Big)\Big)$$

is given by

$$e_j\otimes dz\,;\quad e_j\otimes\frac{dz}{(z-\alpha)^i}\,,\ 1\le i\le m_\alpha\,,\ 1\le j\le r\,. \qquad (3.5)$$

$H^0_{DR/K} = (0)$ and $H^1_{DR/K} = \operatorname{coker}(H^0(E)\to H^0(E\otimes\omega(\sum m_\alpha\alpha + 2\infty)))$ has basis

$$e_j\otimes\frac{dz}{(z-\alpha)^i}\,;\quad 1\le i\le m_\alpha\,,\ 1\le j\le r\,. \qquad (3.6)$$

To compute the Gauß–Manin connection, we consider the diagram (here $\mathcal{D} = \sum m_\alpha\alpha + 2\infty$ and $\mathcal{D}' = \mathcal{D} - D = \sum(m_\alpha - 1)\alpha + \infty$)

$$\begin{array}{ccccccccc}
 & & & & H^0(E) & = & H^0(E) & & \\
 & & & & \downarrow{\scriptstyle \nabla_X} & & \downarrow{\scriptstyle \nabla_{X/S}} & & \\
0 & \longrightarrow & H^0(E(\mathcal{D}'))\otimes\Omega^1_K & \longrightarrow & H^0(E\otimes\Omega^1_{\mathbb{P}^1}\{\mathcal{D}\}) & \xrightarrow{a} & H^0(E\otimes\omega(\mathcal{D})) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nabla_{X/S}\otimes 1} & & \downarrow{\scriptstyle \nabla_X} & & & & \\
 & & H^0(E(\mathcal{D}')\otimes\omega(\mathcal{D}))\otimes\Omega^1_K & \xrightarrow{\cong} & H^0(E\otimes\Omega^2_{\mathbb{P}^1}\{\mathcal{D}+\mathcal{D}'\}) & & & &
\end{array} \qquad (3.7)$$

One deduces from this diagram the Gauß–Manin connection

$$H^1_{DR/K}(E) \cong \mathrm{coker}(\nabla_{X/S}) \xrightarrow{\nabla_{GM}} H^1_{DR/K}(E) \otimes \Omega^1_K; \quad w \mapsto \nabla_X(a^{-1}(w)). \tag{3.8}$$

We may choose $a^{-1}\big(e_j \otimes \frac{dz}{(z-\alpha)^i}\big) = e_j \otimes \frac{dz}{(z-\alpha)^i}$, so by (3.4)

$$\nabla_{GM}\left(e_j \otimes \frac{dz}{(z-\alpha)^i}\right) = \nabla_X\left(e_j \otimes \frac{dz}{(z-\alpha)^i}\right) = e_j \otimes \frac{zdz \wedge dt}{(z-\alpha)^i t^2}. \tag{3.9}$$

In $H^1_{DR/K} \cong \mathrm{coker}(\nabla_{X/S})$ we have the identity

$$e_j \otimes dz = -t\Psi e_j. \tag{3.10}$$

We conclude

$$\nabla_{GM}\left(e_j \otimes \frac{dz}{(z-\alpha)^i}\right) = \begin{cases} \left(e_j \otimes \frac{dz}{(z-\alpha)^{i-1}} + \alpha e_j \otimes \frac{dz}{(z-\alpha)^i}\right) \wedge \frac{dt}{t^2} & 2 \le i \le m_\alpha \\ \left(-t\Psi e_j + \alpha e_j \otimes \frac{dz}{z-\alpha}\right) \wedge \frac{dt}{t^2} & i = 1. \end{cases} \tag{3.11}$$

In particular, the determinant connection, which is given by $\mathrm{Tr}\nabla_{GM}$, can now be calculated:

$$\mathrm{Tr}\nabla_{GM} = \sum_\alpha \tfrac{rm_\alpha \alpha dt}{t^2} - \mathrm{Tr}\sum_\alpha \tfrac{g_1^\alpha dt}{t}. \tag{3.12}$$

We compare this with the conjectured value which is the negative of (2.15). Define

$$F(z) := \sum_\alpha \frac{1}{(z-\alpha)^{m_\alpha}} - 1 = \frac{G(z)}{(z-\alpha)^{m_\alpha}}; \quad s := F(z)dz. \tag{3.13}$$

One has

$$c_1\left(\omega(\mathcal{D}), \left\{\frac{dz}{(z-\alpha)^{m_\alpha}}, dz\right\}\right) = (G), \tag{3.14}$$

the divisor of zeroes of G. We need to compute $(\det E, \det \nabla)|_{(G)}$. We have

$$\begin{aligned} G(z) &= \sum_\alpha \prod_{\beta \neq \alpha} (z-\beta)^{m_\beta} - \prod_\alpha (z-\alpha)^{m_\alpha} \\ &= -z^{\sum m_\alpha} + \Big(\sum m_\alpha \alpha + \#\{\alpha \,|\, m_\alpha = 1\}\Big) z^{(\sum m_\alpha)-1} + \ldots \end{aligned} \tag{3.15}$$

Note that the coefficients of G do not involve t, so the dz part of the connection dies on (G) and we get

$$\mathrm{Tr}\nabla|_{(G)} = -\tfrac{rz\,dt}{t^2}\Big|_{(G)} = -\tfrac{r\,dt}{t^2} \sum_{\substack{\beta \\ G(\beta)=0}} \beta$$

$$= -\tfrac{r\,dt}{t^2}\Big(\sum m_\alpha \alpha + \#\{\alpha \mid m_\alpha = 1\}\Big)\,. \tag{3.16}$$

It remains to evaluate the correction terms $\mathrm{res}\,\mathrm{Tr}(dgg^{-1}A)$ occurring in (2.15). In the notation of (2.12), $\eta/z^{m-1} = -zdt/t^2$, and by Lemma 2.9 we have $\mathrm{res}\,\mathrm{Tr}(dgg^{-1}A) = -\,\mathrm{res}\,\mathrm{Tr}\left(dgg^{-1}\tfrac{zdt}{t^2}\right)$. Clearly, the only contribution comes at $z = \infty$. Take $u = z^{-1}$. At ∞ the connection is

$$A = -\left(\sum_\alpha \sum_i \frac{g_i^\alpha u^i}{(1-u\alpha)^i} + \frac{1}{t}\right)\frac{du}{u^2} - \frac{dt}{ut^2}\,. \tag{3.17}$$

We rewrite this in the form $A = gs + \frac{\eta}{u}$ as in (2.12) with s as in (3.13) and $\eta = -dt/t^2$. We find

$$g = \frac{\sum_{i,\alpha} \frac{g_i^\alpha u^i}{(1-u\alpha)^i} + t^{-1}}{\sum_\alpha \frac{u^{m_\alpha}}{(1-u\alpha)^{m_\alpha}} - 1} = \frac{\kappa}{v}\,, \tag{3.18}$$

(defining κ and v to be the numerator and denominator, respectively). Then

$$\begin{aligned}\mathrm{res}\,\mathrm{Tr}(dgg^{-1}A) &= -\,\mathrm{res}\,\mathrm{Tr}\left(dgg^{-1}\tfrac{dt}{ut^2}\right)\\ &= \big(-\,\mathrm{res}\,\mathrm{Tr}(d\kappa\kappa^{-1}u^{-1}) + r\cdot \mathrm{res}\,\mathrm{Tr}(dvv^{-1}u^{-1})\big)\tfrac{dt}{t^2}\\ &= \Big(-t\sum \mathrm{Tr}(g_1^\alpha) - r\#\{\alpha \mid m_\alpha = 1\}\Big)\tfrac{dt}{t^2}\,. \end{aligned}\tag{3.19}$$

Combining (3.19), (3.16), and (3.12) we conclude

$$\mathrm{Tr}\nabla_{GM} = -\big(\mathrm{Tr}\nabla|_{(G)} - \mathrm{res}\,\mathrm{Tr}(dgg^{-1}A)\big)\,, \tag{3.20}$$

which is the desired formula.

We turn now to the case where Ψ has a pole of order ≥ 2 at infinity. We write

$$\Psi = \sum_\alpha \sum_{i=1}^{m_\alpha} \frac{g_i^\alpha dz}{(z-\alpha)^i} + g^\infty dz\,; \quad g^\infty = g_2^\infty + \ldots + g_{m_\infty}^\infty z^{m_\infty - 2} \tag{3.21}$$

$$\nabla = \Psi + \tfrac{dz}{t} - \tfrac{zdt}{t^2}\,. \tag{3.22}$$

A basis for $\Gamma(\mathbb{P}^1, \omega(\sum m_\alpha \alpha + m_\infty \infty))$ is given by

$$e_j \otimes \frac{dz}{(z-\alpha)^i}\,; \quad 1 \leq i \leq m_\alpha\,; \quad e_j \otimes z^i dz\,;\ 0 \leq i \leq m_\infty - 2\,. \tag{3.23}$$