BIRKHÄUSER

Operator Theory
Advances and Applications
Vol. 203

Topics in Operator Theory

Volume 2: Systems and Mathematical Physics

A tribute to Israel Gohberg on the occasion of his 80th birthday

Joseph A. Ball
Vladimir Bolotnikov
J. William Helton
Leiba Rodman
Ilya M. Spitkovsky
Editors

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Operator Theory: Advances and Applications

Vol. 203

Founded in 1979 by Israel Gohberg

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Topics in Operator Theory

Volume 2: Systems and Mathematical Physics

Proceedings of the XIXth International Workshop on Operator Theory and its Applications, College of William and Mary, 2008

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2010 Mathematics Subject Classification: 15, 45, 46, 47, 93

Library of Congress Control Number: 2010920057

Bibliographic information published by Die Deutsche Bibliothek.
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data is available in the Internet at http://dnb.ddb.de

ISBN 978-3-0346-0160-3

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© 2010 Birkhäuser, Springer Basel AG
P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF∞
Printed in Germany

ISBN 978-3-0346-0157-3 Vol. 1 e-ISBN 978-3-0346-0158-0
ISBN 978-3-0346-0160-3 Vol. 2 e-ISBN 978-3-0346-0161-0
ISBN 978-3-0346-0163-4 Set

9 8 7 6 5 4 3 2 1

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Contents

The XIXth International Workshop on Operator Theory and its Applications. II

Joseph A. Ball, Vladimir Bolotnikov, J. William Helton, Leiba Rodman and Ilya M. Spitkovsky

Abstract. Information about the workshop and comments about the second volume of proceedings is provided.

Mathematics Subject Classification (2000). 35-06, 37-06, 45-06, 93-06, 47-06.

Keywords. Operator theory, differential and difference equations, system theory, mathematical physics.

The Nineteenth International Workshop on Operator Theory and its Applications – IWOTA 2008 – took place in Williamsburg, Virginia, on the campus of the College of William and Mary, from July 22 till July 26, 2008. It was held in conjunction with the 18th International Symposium on Mathematical Theory of Networks and Systems (MTNS) in Blacksburg, Virginia (Virginia Tech, July 28–August 1, 2008) and the 9th Workshop on Numerical Ranges and Numerical Radii (July 19–July 21, 2008) at the College of William and Mary. The organizing committee of IWOTA 2008 (Ball, Bolotnikov, Helton, Rodman, Spitkovsky) served also as editors of the proceedings.

IWOTA 2008 celebrated the work and career of Israel Gohberg on the occasion of his 80th birthday, which actually fell on August 23, 2008. We are pleased to present this volume as a tribute to Israel Gohberg.

IWOTA 2008 was a comprehensive, inclusive conference covering many aspects of theoretical and applied operator theory. More information about the workshop can be found on its web site

`http://www.math.wm.edu/~vladi/IWOTA/IWOTA2008.htm`

There were 241 participants at IWOTA 2008, representing 30 countries, including 29 students (almost exclusively graduate students), and 20 young researchers (those who received their doctoral degrees in the year 2003 or later). The scientific program included 17 plenary speakers and 7 invited speakers who gave overview of many topics related to operator theory. The special sessions covered

Israel Gohberg at IWOTA 2008, Williamsburg, Virginia

a broad range of topics: Matrix and operator inequalities; hypercomplex operator theory; the Kadison–Singer extension problem; interpolation problems; matrix completions; moment problems; factorizations; Wiener–Hopf and Fredholm operators; structured matrices; Bezoutians, resultants, inertia theorems and spectrum localization; applications of indefinite inner product spaces; linear operators and linear systems; multivariable operator theory; composition operators; matrix polynomials; indefinite linear algebra; direct and inverse scattering transforms for integrable systems; theory, computations, and applications of spectra of operators.

We gratefully acknowledge support of IWOTA 2008 by the National Science Foundation Grant 0757364, as well as by the individual grants of some organizers, and by various entities within the College of William and Mary: Department of Mathematics, the Office of the Dean of the Faculty of Arts and Sciences, the Office of the Vice Provost for Research, and the Reves Center for International Studies.

One plenary speaker has been sponsored by the International Linear Algebra Society. The organization and running of IWOTA 2008 was helped tremendously by the Conference Services of the College of William and Mary.

The present volume is the second of two volumes of proceedings of IWOTA 2008. Here, papers on systems, differential and difference equations, and mathematical physics are collected. All papers are refereed. The first volume contains papers on operator theory, linear algebra, and analytic functions, as well as a commemorative article dedicated to Israel Gohberg.

August 2009

Added on December 14, 2009:

With deep sadness the editors' final act in preparing this volume is to record that Israel Gohberg passed away on October 12, 2009, aged 81. Gohberg was a great research mathematician, educator, and expositor. His visionary ideas inspired many, including the editors and quite a few contributors to the present volume.

Israel Gohberg was the driving force of iwota. He was the first and the only President of the Steering Committee. In iwota, just as in his other endeavors, Gohberg's charisma, warmth, judgement and stature lead to the lively community we have today.

He will be dearly missed.

The Editors: Joseph A. Ball, Vladimir Bolotnikov, J. William Helton,
Leiba Rodman, Ilya M. Spitkovsky.

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Operator Theory:
Advances and Applications, Vol. 203, 1–12
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Exact Solutions to the Nonlinear Schrödinger Equation

Tuncay Aktosun, Theresa Busse, Francesco Demontis and Cornelis van der Mee

Dedicated to Israel Gohberg on the occasion of his eightieth birthday

Abstract. A review of a recent method is presented to construct certain exact solutions to the focusing nonlinear Schrödinger equation on the line with a cubic nonlinearity. With motivation by the inverse scattering transform and help from the state-space method, an explicit formula is obtained to express such exact solutions in a compact form in terms of a matrix triplet and by using matrix exponentials. Such solutions consist of multisolitons with any multiplicities, are analytic on the entire xt-plane, decay exponentially as $x \to \pm\infty$ at each fixed t, and can alternatively be written explicitly as algebraic combinations of exponential, trigonometric, and polynomial functions of the spatial and temporal coordinates x and t. Various equivalent forms of the matrix triplet are presented yielding the same exact solution.

Mathematics Subject Classification (2000). Primary: 37K15; Secondary: 35Q51, 35Q55.

Keywords. Nonlinear Schrödinger equation, exact solutions, explicit solutions, focusing NLS equation, NLS equation with cubic nonlinearity, inverse scattering transform.

1. Introduction

Our goal in this paper is to review and further elaborate on a recent method [3, 4] to construct certain exact solutions to the focusing nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{1.1}$$

Communicated by J.A. Ball.

with a cubic nonlinearity, where the subscripts denote the corresponding partial derivatives.

The NLS equation has important applications in various areas such as wave propagation in nonlinear media [15], surface waves on deep waters [14], and signal propagation in optical fibers [9–11]. It was the second nonlinear partial differential equation (PDE) whose initial value problem was discovered [15] to be solvable via the inverse scattering transform (IST) method. Recall that the IST method associates (1.1) with the Zakharov-Shabat system

$$\frac{d\varphi(\lambda,x,t)}{dx}=\begin{bmatrix}-i\lambda & u(x,t)\\ -u(x,t)^* & i\lambda\end{bmatrix}\varphi(\lambda,x,t), \tag{1.2}$$

where $u(x,t)$ appears as a potential and an asterisk is used for complex conjugation. By exploiting the one-to-one correspondence between the potential $u(x,t)$ and the corresponding scattering data for (1.2), that method amounts to determining the time evolution $u(x,0)\mapsto u(x,t)$ in (1.1) with the help of solutions to the direct and inverse scattering problems for (1.2). We note that the direct scattering problem for (1.2) consists of determining the scattering coefficients (related to the asymptotics of scattering solutions to (1.2) as $x\to\pm\infty$) when $u(x,t)$ is known for all x. On the other hand, the inverse scattering problem for (1.2) is to construct $u(x,t)$ when the scattering data is known for all λ.

Even though we are motivated by the IST method, our goal is not to solve the initial value problem for (1.1). Our aim is rather to construct certain exact solutions to (1.1) with the help of a matrix triplet and by using matrix exponentials. Such exact solutions turn out to be multisolitons with any multiplicities. Dealing with even a single soliton with multiplicities has not been an easy task in other methods; for example, the exact solution example presented in [15] for a one-soliton solution with a double pole, which is obtained by coalescing two distinct poles into one, contains a typographical error, as pointed out in [13].

In constructing our solutions we make use of the state-space method [6] from control theory. Our solutions are uniquely constructed via the explicit formula (2.6), which uses as input three (complex) constant matrices A, B, C, where A has size $p\times p$, B has size $p\times 1$, and C has size $1\times p$, with p as any positive integer. We will refer to (A,B,C) as a triplet of size p. There is no loss of generality in using a triplet yielding a minimal representation [3, 4, 6], and we will only consider such triplets. As seen from the explicit formula (2.6), our solutions are well defined as long as the matrix $F(x,t)$ defined in (2.5) is invertible. It turns out that $F(x,t)$ is invertible if and only if two conditions are met on the eigenvalues of the constant matrix A; namely, none of the eigenvalues of A are purely imaginary and that no two eigenvalues of A are symmetrically located with respect to the imaginary axis. Our solutions given by (2.6) are globally analytic on the entire xt-plane and decay exponentially as $x\to\pm\infty$ for each fixed $t\in\mathbf{R}$ as long as those two conditions on the eigenvalues of A are satisfied.

In our method [3, 4] we are motivated by using the IST with rational scattering data. For this purpose we exploit the state-space method [6]; namely, we use a matrix triplet (A, B, C) of an appropriate size in order to represent a rational function vanishing at infinity in the complex plane. Recall that any rational function $R(\lambda)$ in the complex plane that vanishes at infinity has a matrix realization in terms of a matrix triplet (A, B, C) as

$$R(\lambda) = -iC(\lambda I - iA)B, \tag{1.3}$$

where I denotes the identity matrix. The smallest integer p in the size of the triplet yields a minimal realization for $R(\lambda)$ in (1.3). A minimal realization is unique up to a similarity transformation. The poles of $R(\lambda)$ coincide with the eigenvalues of (iA).

The use of a matrix realization in the IST method allows us to establish the separability of the kernel of a related Marchenko integral equation [1, 2, 4, 12] by expressing that kernel in terms of a matrix exponential. We then solve that Marchenko integral equation algebraically and observe that our procedure leads to exact solutions to the NLS equation even when the input to the Marchenko equation does not necessarily come from any scattering data. We refer the reader to [3, 4] for details.

The explicit formula (2.6) provides a compact and concise way to express our exact solutions. If such solutions are desired to be expressed in terms of exponential, trigonometric (sine and cosine), and polynomial functions of x and t, this can also be done explicitly and easily by "unpacking" matrix exponentials in (2.6). If the size p in the matrices A, B, C is larger than 3, such expressions become long; however, we can still explicitly evaluate them for any matrix size p either by hand or by using a symbolic software package such as Mathematica. The power of our method is that we can produce exact solutions via (2.6) for any positive integer p. In some other available methods, exact solutions are usually tried to be produced directly in terms of elementary functions without using matrix exponentials, and hence any concrete examples that can be produced by such other methods will be relatively simple and we cannot expect those other methods to produce our exact solutions when p is large.

Our method is generalizable to obtain similar explicit formulas for exact solutions to other integrable nonlinear PDEs where the IST involves the use of a Marchenko integral equation [1, 2, 4, 12]. For example, a similar method has been used [5] for the half-line Korteweg-de Vries equation, and it can be applied to other equations such as the modified Korteweg-de Vries equation and the sine-Gordon equation. Our method is also generalizable to the matrix versions of such integrable nonlinear PDEs. For instance, a similar method has been applied in the third author's Ph.D. thesis [8] to the matrix NLS equation in the focusing case with a cubic nonlinearity.

Our method also easily handles nonsimple bound-state poles and the time evolution of the corresponding bound-state norming constants. In the literature, nonsimple bound-state poles are usually avoided due to mathematical complications. We refer the reader to [13], where nonsimple bound-state poles were investigated and complications were encountered. A systematic treatment of nonsimple bound states has recently been given in the second author's Ph.D. thesis [7].

The organization of our paper is as follows. Our main results are summarized in Section 2 and some explicit examples are provided in Section 3. For the proofs, further results, details, and a summary of other methods to solve the NLS equation exactly, we refer the reader to [3, 4].

2. Main results

In this section we summarize our method to construct certain exact solutions to the NLS equation in terms of a given triplet (A, B, C) of size p. For the details of our method we refer the reader to [3, 4]. Without any loss of generality, we assume that our starting triplet (A, B, C) corresponds to a minimal realization in (1.3). Let us use a dagger to denote the matrix adjoint (complex conjugate and matrix transpose), and let the set $\{a_j\}_{j=1}^m$ consist of the distinct eigenvalues of A, where the algebraic multiplicity of each eigenvalue may be greater than one and we use n_j to denote that multiplicity. We only impose the restrictions that no a_j is purely imaginary and that no two distinct a_j values are located symmetrically with respect to the imaginary axis on the complex plane. Let us set $\lambda_j := ia_j$ so that we can equivalently state our restrictions as that no λ_j will be real and no two distinct λ_j values will be complex conjugates of each other. Our method uses the following steps:

(i) First construct the constant $p \times p$ matrices Q and N that are the unique solutions, respectively, to the Lyapunov equations

$$Q\,A + A^\dagger Q = C^\dagger C, \tag{2.1}$$

$$A\,N + N\,A^\dagger = BB^\dagger. \tag{2.2}$$

In fact, Q and N can be written explicitly in terms of the triplet (A, B, C) as

$$Q = \frac{1}{2\pi}\int_\gamma d\lambda\,(\lambda I + iA^\dagger)^{-1} C^\dagger C (\lambda I - iA)^{-1}, \tag{2.3}$$

$$N = \frac{1}{2\pi}\int_\gamma d\lambda\,(\lambda I - iA)^{-1} BB^\dagger (\lambda I + iA^\dagger)^{-1}, \tag{2.4}$$

where γ is any positively oriented simple closed contour enclosing all λ_j in such a way that all λ_j^* lie outside γ. The existence and uniqueness of the solutions to (2.1) and (2.2) are assured by the fact that $\lambda_j \neq \lambda_j^*$ for all $j = 1, 2, \ldots, m$ and $\lambda_j \neq \lambda_k^*$ for $k \neq j$.

(ii) Construct the $p \times p$ matrix-valued function $F(x, t)$ as

$$F(x, t) := e^{2A^\dagger x - 4i(A^\dagger)^2 t} + Q\, e^{-2Ax - 4iA^2 t} N. \tag{2.5}$$

(iii) Construct the scalar function $u(x,t)$ via

$$u(x,t) := -2B^\dagger F(x,t)^{-1} C^\dagger. \tag{2.6}$$

Note that $u(x,t)$ is uniquely constructed from the triplet (A,B,C). As seen from (2.6), the quantity $u(x,t)$ exists at any point on the xt-plane as long as the matrix $F(x,t)$ is invertible. It turns out that $F(x,t)$ is invertible on the entire xt-plane as long as $\lambda_j \neq \lambda_j^*$ for all $j = 1,2,\dots,m$ and $\lambda_j \neq \lambda_k^*$ for $k \neq j$.

Let us note that the matrices Q and N given in (2.3) and (2.4) are known in control theory as the observability Gramian and the controllability Gramian, respectively, and that it is well known in control theory that (2.3) and (2.4) satisfy (2.1) and (2.2), respectively. In the context of system theory, the invertibility of Q and N is described as the observability and the controllability, respectively. In our case, both Q and N are invertible due to the appropriate restrictions imposed on the triplet (A,B,C), which we will see in Theorem 1 below.

Our main results are summarized in the following theorems. For the proofs we refer the reader to [3, 4]. Although the results presented in Theorem 1 follow from the results in the subsequent theorems, we state Theorem 1 independently to clearly illustrate the validity of our exact solutions to the NLS equation.

Theorem 1. *Consider any triplet (A,B,C) of size p corresponding to a minimal representation in* (1.3)*, and assume that none of the eigenvalues of A are purely imaginary and that no two eigenvalues of A are symmetrically located with respect to the imaginary axis. Then:*

(i) *The Lyapunov equations* (2.1) *and* (2.2) *are uniquely solvable, and their solutions are given by* (2.3) *and* (2.4)*, respectively.*

(ii) *The constant matrices Q and N given in* (2.3) *and* (2.4)*, respectively, are selfadjoint; i.e., $Q^\dagger = Q$ and $N^\dagger = N$. Furthermore, both Q and N are invertible.*

(iii) *The matrix $F(x,t)$ defined in* (2.5) *is invertible on the entire xt-plane, and the function $u(x,t)$ defined in* (2.6) *is a solution to the NLS equation everywhere on the xt-plane. Moreover, $u(x,t)$ is analytic on the entire xt-plane and it decays exponentially as $x \to \pm\infty$ at each fixed $t \in \mathbf{R}$.*

We will say that two triplets (A,B,C) and $(\tilde{A},\tilde{B},\tilde{C})$ are equivalent if they yield the same potential $u(x,t)$ through (2.6). The following result shows that, as far as constructing solutions via (2.6) is concerned, there is no loss of generality is choosing our starting triplet (A,B,C) of size p so that it corresponds to a minimal representation in (1.3) and that all eigenvalues a_j of the matrix A have positive real parts.

Theorem 2. *Consider any triplet $(\tilde{A},\tilde{B},\tilde{C})$ of size p corresponding to a minimal representation in* (1.3)*, and assume that none of the eigenvalues of $\tilde{A}$ are purely imaginary and that no two eigenvalues of $\tilde{A}$ are symmetrically located with respect to the imaginary axis. Then, there exists an equivalent triplet (A,B,C) of* *size p corresponding to a minimal representation in* (1.3) *in such a way that all eigenvalues of A have positive real parts.*

The next two results given in Theorems 3 and 4 show some of the advantages of using a triplet (A, B, C) where all eigenvalues of A have positive real parts. Concerning Theorem 2, we remark that the triplet (A, B, C) can be obtained from $(\tilde{A}, \tilde{B}, \tilde{C})$ and vice versa with the help of Theorem 5 or Theorem 6 given below.

Theorem 3. *Consider any triplet (A, B, C) of size p corresponding to a minimal representation in* (1.3)*. Assume that all eigenvalues of A have positive real parts. Then:*

(i) *The solutions Q and N to* (2.1) *and* (2.2)*, respectively, can be expressed in terms of the triplet (A, B, C) as*

$$Q = \int_0^\infty ds\, [Ce^{-As}]^\dagger [Ce^{-As}], \qquad N = \int_0^\infty ds\, [e^{-As}B][e^{-As}B]^\dagger. \tag{2.7}$$

(ii) *Q and N are invertible, selfadjoint matrices.*

(iii) *Any square submatrix of Q containing the $(1, 1)$-entry or (p, p)-entry of Q is invertible. Similarly, any square submatrix of N containing the $(1, 1)$-entry or (p, p)-entry of N is invertible.*

Theorem 4. *Consider a triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ of size p corresponding to a minimal representation in* (1.3) *and that all eigenvalues a_j of the matrix $\tilde{A}$ have positive real parts and that the multiplicity of a_j is n_j for $j = 1, 2, \ldots, m$. Then, there exists an equivalent triplet (A, B, C) of size p corresponding to a minimal representation in* (1.3) *in such a way that A is in a Jordan canonical form with each Jordan block containing a distinct eigenvalue a_j and having -1 in the superdiagonal entries, and the entries of B consist of zeros and ones. More specifically, we have*

$$A = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_m \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 & \ldots & C_m \end{bmatrix}, \tag{2.8}$$

$$A_j := \begin{bmatrix} a_j & -1 & 0 & \ldots & 0 \\ 0 & a_j & -1 & \ldots & 0 \\ 0 & 0 & a_j & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_j \end{bmatrix}, \quad B_j := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_j := \begin{bmatrix} c_{j(n_j-1)} & \ldots & c_{j1} & c_{j0} \end{bmatrix},$$

where A_j has size $n_j \times n_j$, B_j has size $n_j \times 1$, C_j has size $1 \times n_j$, and the (*complex*) *constant $c_{j(n_j-1)}$ is nonzero.*

We will refer to the specific form of the triplet (A, B, C) given in (2.8) as a standard form.

The transformation between two equivalent triplets can be obtained with the help of the following two theorems. First, in Theorem 5 below we consider the transformation where all eigenvalues of A are reflected with respect to the imaginary axis. Then, in Theorem 6 we consider transformations where only some of the eigenvalues of A are reflected with respect to the imaginary axis.

Theorem 5. *Assume that the triplet* (A, B, C) *of size* p *corresponds to a minimal realization in* (1.3) *and that all eigenvalues of* A *have positive real parts. Consider the transformation*

$$(A, B, C, Q, N, F) \mapsto (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N}, \tilde{F}), \tag{2.9}$$

where (Q, N) *corresponds to the unique solution to the Lyapunov system in* (2.1) *and* (2.2)*, the quantity* F *is as in* (2.5)*,*

$$\tilde{A} = -A^\dagger, \quad \tilde{B} = -N^{-1}B, \quad \tilde{C} = -CQ^{-1}, \quad \tilde{Q} = -Q^{-1}, \quad \tilde{N} = -N^{-1},$$

and $\tilde{F}$ *and* $\tilde{u}$ *are as in* (2.5) *and* (2.6)*, respectively, but by using* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N})$ *instead of* (A, B, C, Q, N) *on the right-hand sides. We then have the following:*

(i) *The matrices* $\tilde{Q}$ *and* $\tilde{N}$ *are selfadjoint and invertible. They satisfy the respective Lyapunov equations*

$$\begin{cases} \tilde{Q}\tilde{A} + \tilde{A}^\dagger\tilde{Q} = \tilde{C}^\dagger\tilde{C}, \\ \tilde{A}\tilde{N} + \tilde{N}\tilde{A}^\dagger = \tilde{B}\tilde{B}^\dagger. \end{cases} \tag{2.10}$$

(ii) *The quantity* F *is transformed as* $\tilde{F} = Q^{-1}FN^{-1}$*. The matrix* $\tilde{F}$ *is invertible at every point on the* xt*-plane.*

To consider the case where only some of eigenvalues of A are reflected with respect to the imaginary axis, let us again start with a triplet (A, B, C) of size p and corresponding to a minimal realization in (1.3), where the eigenvalues of A all have positive real parts. Without loss of any generality, let us assume that we partition the matrices A, B, C as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{2.11}$$

so that the $q \times q$ block diagonal matrix A_1 contains the eigenvalues that will remain unchanged and A_2 contains the eigenvalues that will be reflected with respect to the imaginary axis on the complex plane, the submatrices B_1 and C_1 have sizes $q \times 1$ and $1 \times q$, respectively, and hence A_2, B_2, C_2 have sizes $(p-q) \times (p-q)$, $(p-q) \times 1$, $1 \times (p-q)$, respectively, for some integer q not exceeding p. Let us clarify our notational choice in (2.11) and emphasize that the partitioning in (2.11) is not the same partitioning used in (2.8). Using the partitioning in (2.11), let us write the corresponding respective solutions to (2.1) and (2.2) as

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \tag{2.12}$$

where Q_1 and N_1 have sizes $q\times q$, Q_4 and N_4 have sizes $(p-q)\times(p-q)$, etc. Note that because of the selfadjointness of Q and N stated in Theorem 1, we have

$$Q_1^\dagger = Q_1, \quad Q_2^\dagger = Q_3, \quad Q_4^\dagger = Q_4, \quad N_1^\dagger = N_1, \quad N_2^\dagger = N_3, \quad N_4^\dagger = N_4.$$

Furthermore, from Theorem 3 it follows that Q_1, Q_4, N_1, and N_4 are all invertible.

Theorem 6. *Assume that the triplet* (A, B, C) *partitioned as in* (2.11) *corresponds to a minimal realization in* (1.3) *and that all eigenvalues of* A *have positive real parts. Consider the transformation* (2.9) *with* $(\tilde A, \tilde B, \tilde C)$ *having similar block representations as in* (2.11)*,* (Q, N) *as in* (2.12) *corresponding to the unique solution to the Lyapunov system in* (2.1) *and* (2.2)*,*

$$\tilde A_1 = A_1, \quad \tilde A_2 = -A_2^\dagger, \quad \tilde B_1 = B_1 - N_2 N_4^{-1} B_2, \quad \tilde B_2 = -N_4^{-1} B_2,$$

$$\tilde C_1 = C_1 - C_2 Q_4^{-1} Q_3, \quad \tilde C_2 = -C_2 Q_4^{-1},$$

and $\tilde Q$ *and* $\tilde N$ *partitioned in a similar way as in* (2.12) *and given as*

$$\tilde Q_1 = Q_1 - Q_2 Q_4^{-1} Q_3, \quad \tilde Q_2 = -Q_2 Q_4^{-1}, \quad \tilde Q_3 = -Q_4^{-1} Q_3, \quad \tilde Q_4 = -Q_4^{-1},$$

$$\tilde N_1 = N_1 - N_2 N_4^{-1} N_3, \quad \tilde N_2 = -N_2 N_4^{-1}, \quad \tilde N_3 = -N_4^{-1} N_3, \quad \tilde N_4 = -N_4^{-1},$$

and $\tilde F$ *and* $\tilde u$ *as in* (2.5) *and* (2.6)*, respectively, but by using* $(\tilde A, \tilde B, \tilde C, \tilde Q, \tilde N)$ *instead of* (A, B, C, Q, N) *on the right-hand sides. We then have the following:*

(i) *The matrices* $\tilde Q$ *and* $\tilde N$ *are selfadjoint and invertible. They satisfy the respective Lyapunov equations given in* (2.10)*.*

(ii) *The quantity* F *is transformed according to*

$$\tilde F = \begin{bmatrix} I & -Q_2 Q_4^{-1} \\ 0 & -Q_4^{-1} \end{bmatrix} F \begin{bmatrix} I & 0 \\ -N_4^{-1} N_3 & -N_4^{-1} \end{bmatrix},$$

and the matrix $\tilde F$ *is invertible at every point on the xt-plane.*

(iii) *The triplets* (A, B, C) *and* $(\tilde A, \tilde B, \tilde C)$ *are equivalent; i.e.,* $\tilde u(x,t) = u(x,t)$*.*

3. Examples

In this section we illustrate our method of constructing exact solutions to the NLS equation with some concrete examples.

Example 1. The well-known "n-soliton" solution to the NLS equation is obtained by choosing the triplet (A, B, C) as

$$A = \operatorname{diag}\{a_1, a_2, \dots, a_n\}, \quad B^\dagger = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix},$$

where a_j are distinct (complex) nonzero constants with positive real parts, B contains n entries, and the quantities c_j are complex constants. Note that diag is used to denote the diagonal matrix. In this case, using (2.5) and (2.7) we evaluate the (j,k)-entries of the $n \times n$ matrix-valued functions Q, N, and $F(x,t)$ as

$$N_{jk} = \frac{1}{a_j + a_k^*}, \; Q_{jk} = \frac{c_j^* c_k}{a_j^* + a_k}, \; F_{jk} = \delta_{jk} e^{2a_j^* x - 4i(a_j^*)^2 t} + \sum_{s=1}^{n} \frac{c_j^* c_s \, e^{-2a_s x - 4i a_s^2 t}}{(a_j^* + a_s)(a_s + a_k^*)},$$

where δ_{jk} denotes the Kronecker delta. Having obtained Q, N, and $F(x,t)$, we construct the solution $u(x,t)$ to the NLS equation via (2.6) or equivalently as the ratio of two determinants as

$$u(x,t) = \frac{2}{\det F(x,t)} \begin{vmatrix} 0 & B^\dagger \\ C^\dagger & F(x,t) \end{vmatrix}. \tag{3.1}$$

For example, when $n = 1$, from (3.1) we obtain the single soliton solution

$$u(x,t) = \frac{-8c_1^* (\mathrm{Re}[a_1])^2 \, e^{-2a_1^* x + 4i(a_1^*)^2 t}}{4(\mathrm{Re}[a_1])^2 + |c_1|^2 \, e^{-4x(\mathrm{Re}[a_1]) + 8t(\mathrm{Im}[a_1^2])}},$$

where Re and Im denote the real and imaginary parts, respectively. From (1.1) we see that if $u(x,t)$ is a solution to (1.1), so is $e^{i\theta} u(x,t)$ for any real constant θ. Hence, the constant phase factor $e^{i\theta}$ can always be omitted from the solution to (1.1) without any loss of generality. As a result, we can write the single soliton solution also in the form

$$u(x,t) = 2\,\mathrm{Re}[a_1] \; e^{i\beta(x,t)} \mathrm{sech}\left(2\,\mathrm{Re}[a_1](x - 4t\,\mathrm{Im}[a_1]) - \log\left(\frac{|c_1|}{2\mathrm{Re}[a_1]} \right) \right),$$

where it is seen that $u(x,t)$ has amplitude $2\,\mathrm{Re}[a_1]$ and moves with velocity $4\,\mathrm{Im}[a_1]$ and we have

$$\beta(x,t) := 2x \mathrm{Im}[a_1] + 4t\,\mathrm{Re}[a_1^2].$$

Example 2. For the triplet (A, B, C) given by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \tag{3.2}$$

we evaluate Q and N explicitly by solving (2.1) and (2.2), respectively, as

$$N = \begin{bmatrix} 1/4 & 1 \\ 1 & -1/2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1/4 & -1 \\ -1 & -1/2 \end{bmatrix},$$

and obtain $F(x,t)$ by using (2.5) as

$$F(x,t) = \begin{bmatrix} e^{4x-16it} - e^{2x-4it} + \frac{1}{16} e^{-4x-16it} & \frac{1}{4} e^{-4x-16it} + \frac{1}{2} e^{2x-4it} \\ -\frac{1}{4} e^{-4x-16it} - \frac{1}{2} e^{2x-4it} & e^{-2x-4it} - e^{-4x-16it} + \frac{1}{4} e^{2x-4it} \end{bmatrix}.$$

Finally, using (2.6), we obtain the corresponding solution to the NLS equation as

$$u(x,t) = \frac{8e^{4it}(9e^{-4x} + 16e^{4x}) - 32e^{16it}(4e^{-2x} + 9e^{2x})}{-128\cos(12t) + 4e^{-6x} + 16e^{6x} + 81e^{-2x} + 64e^{2x}}. \tag{3.3}$$

It can independently be verified that $u(x,t)$ given in (3.3) satisfies the NLS equation on the entire xt-plane.

With the help of the results stated in Section 2, we can determine triplets $(\tilde{A},\tilde{B},\tilde{C})$ that are equivalent to the triplet in (3.2).

The following triplets all yield the same $u(x,t)$ given in (3.3):

(i) $\tilde{A}=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix},\ \tilde{B}=\begin{bmatrix}9/\alpha_1\\ -4/\alpha_2\end{bmatrix},\ \tilde{C}=\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix},$

where α_1 and α_2 are arbitrary (complex) nonzero parameters. Note that both eigenvalues of $\tilde{A}$ are positive, whereas only one of the eigenvalues of A in (3.2) is positive.

(ii) $\tilde{A}=\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix},\ \tilde{B}=\begin{bmatrix}16/(9\alpha_3)\\ -4/(9\alpha_4)\end{bmatrix},\ \tilde{C}=\begin{bmatrix}\alpha_3 & \alpha_4\end{bmatrix},$

where α_3 and α_4 are arbitrary (complex) nonzero parameters. Note that the eigenvalues of $\tilde{A}$ in this triplet are negatives of the eigenvalues of A given in (3.2).

(iii) $\tilde{A}=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix},\ \tilde{B}=\begin{bmatrix}1/\alpha_5\\ -1/\alpha_6\end{bmatrix},\ \tilde{C}=\begin{bmatrix}\alpha_5 & \alpha_6\end{bmatrix},$

where α_5 and α_6 are arbitrary (complex) nonzero parameters. Note that $\tilde{A}$ here agrees with A in (3.2).

(iv) $\tilde{A}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix},\ \tilde{B}=\begin{bmatrix}16/\alpha_7\\ -9/\alpha_8\end{bmatrix},\ \tilde{C}=\begin{bmatrix}\alpha_7 & \alpha_8\end{bmatrix},$

where α_7 and α_8 are arbitrary (complex) nonzero parameters. Note that both eigenvalues of $\tilde{A}$ are negative.

(v) Equivalent to (3.2) we also have the triplet $(\tilde{A},\tilde{B},\tilde{C})$ given by

$$\tilde{A}=\begin{bmatrix}\alpha_9 & \alpha_{10}\\ \dfrac{(1-\alpha_9)(\alpha_9-2)}{\alpha_{10}} & 3-\alpha_9\end{bmatrix},$$

$$\tilde{B}=\frac{\begin{bmatrix}5\alpha_{10}^2\alpha_{11}+\alpha_{10}\alpha_{12}-5\alpha_9\alpha_{10}\alpha_{12}\\ 14\alpha_{10}\alpha_{11}-5\alpha_9\alpha_{10}\alpha_{11}+10\alpha_{12}-15\alpha_9\alpha_{12}+5\alpha_9^2\alpha_{12}\end{bmatrix}}{\alpha_{10}^2\alpha_{11}^2+3\alpha_{10}\alpha_{11}\alpha_{12}-2\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}+2\alpha_{12}^2-3\alpha_9\alpha_{12}^2+\alpha_9^2\alpha_{12}^2},$$

$$\tilde{C}=\begin{bmatrix}\alpha_{11} & \alpha_{12}\end{bmatrix},$$

where $\alpha_9,\ \dots,\ \alpha_{12}$ are arbitrary parameters with the restriction that $\alpha_{10}\alpha_{11}\alpha_{12}\neq 0$, which guarantees that the denominator of $\tilde{B}$ is nonzero; when $\alpha_{10}=0$ we must have $\alpha_{11}\alpha_{12}\neq 0$ and choose α_9 as 2 or 1. In fact, the minimality of the triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ guarantees that $\tilde{B}$ is well defined. For example, the triplet is not minimal if $\alpha_{11}\alpha_{12} = 0$. We note that the eigenvalues of $\tilde{A}$ are 2 and 1 and that $\tilde{A}$ here is similar to the matrix $\tilde{A}$ in the equivalent triplet given in (i).

Other triplets equivalent to (3.2) can be found as in (v) above, by exploiting the similarity for the matrix $\tilde{A}$ given in (ii), (iii), and (iv), respectively, and by using (1.3) to determine the corresponding $\tilde{B}$ and $\tilde{C}$ in the triplet.

Acknowledgment

The research leading to this article was supported in part by the U.S. National Science Foundation under grant DMS-0610494, the Italian Ministry of Education and Research (MIUR) under PRIN grant no. 2006017542-003, and INdAM-GNCS.

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Received: March 1, 2009
Accepted: August 20, 2009

Operator Theory:
Advances and Applications, Vol. 203, 13–88
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Robust Control, Multidimensional Systems and Multivariable Nevanlinna-Pick Interpolation

Joseph A. Ball and Sanne ter Horst

Dedicated to Israel Gohberg on the occasion of his 80th birthday

Abstract. The connection between the standard H^∞-problem in control theory and Nevanlinna-Pick interpolation in operator theory was established in the 1980s, and has led to a fruitful cross-pollination between the two fields since. In the meantime, research in H^∞-control theory has moved on to the study of robust control for systems with structured uncertainties and to various types of multidimensional systems, while Nevanlinna-Pick interpolation theory has moved on independently to a variety of multivariable settings. Here we review these developments and indicate the precise connections which survive in the more general multidimensional/multivariable incarnations of the two theories.

Mathematics Subject Classification (2000). Primary: 47A57, 93D09; Secondary: 13F25, 47A56, 47A63, 93B52, 93D15.

Keywords. Model-matching problem, Youla-Kučera parametrization of stabilizing controllers, H^∞-control problem, structured singular value, structured uncertainty, Linear-Fractional-Transformation model, stabilizable, detectable, robust stabilization, robust performance, frequency domain, state space, Givone-Roesser commutative/noncommutative multidimensional linear system, gain-scheduling, Finsler's lemma.

1. Introduction

Starting in the early 1980s with the seminal paper [139] of George Zames, there occurred an active interaction between operator theorists and control engineers in the development of the early stages of the emerging theory of H^∞-control. The cornerstone for this interaction was the early recognition by Francis-Helton-Zames [65] that the simplest case of the central problem of H^∞-control (the sensitivity

Communicated by L. Rodman.

minimization problem) is one and the same as a Nevanlinna-Pick interpolation problem which had already been solved in the early part of the twentieth century (see [110, 105]). For the standard problem of H^∞-control it was known early on that it could be brought to the so-called Model-Matching form (see [53, 64]). In the simplest cases, the Model-Matching problem converts easily to a Nevanlinna-Pick interpolation problem of classical type. Handling the more general problems of H^∞-control required extensions of the theory of Nevanlinna-Pick interpolation to tangential (or directional) interpolation conditions for matrix-valued functions; such extensions of the interpolation theory were pursued by both engineers and mathematicians (see, e.g., [26, 58, 90, 86, 87]). Alternatively, the Model-Matching problem can be viewed as a Sarason problem which is suitable for application of Commutant Lifting theory (see [125, 62]). The approach of [64] used an additional conversion to a Nehari problem where existing results on the solution of the Nehari problem in state-space coordinates were applicable (see [69, 33]). The book of Francis [64] was the first book on H^∞-control and provides a good summary of the state of the subject in 1987.

While there was a lot of work emphasizing the connection of the H^∞-problem with interpolation and the related approach through J-spectral factorization ([26, 90, 91, 86, 87, 33, 24]), we should point out that the final form of the H^∞-theory parted ways with the connection with Nevanlinna-Pick interpolation. When calculations were carried out in state-space coordinates, the reduction to Model-Matching form via the Youla-Kučera parametrization of stabilizing controllers led to inflation of state-space dimension; elimination of non-minimal state-space nodes by finding pole-zero cancellations demanded tedious brute-force calculations (see [90, 91]). A direct solution in state-space coordinates (without reduction to Model-Matching form and any explicit connection with Nevanlinna-Pick interpolation) was finally obtained by Ball-Cohen [24] (via a J-spectral factorization approach) and in the more definitive coupled-Riccati-equation form of Doyle-Glover-Khargonekar-Francis [54]. This latter paper emphasizes the parallels with older control paradigms (e.g., the Linear-Quadratic-Gaussian and Linear-Quadratic-Regulator problems) and obtained parallel formulas for the related H^2-problem. The J-spectral factorization approach was further developed in the work of Kimura, Green, Glover, Limebeer, and Doyle [87, 70, 71]. A good review of the state of the theory to this point can be found in the books of Zhou-Doyle-Glover [141] and Green-Limebeer [72].

The coupled-Riccati-equation solution however has now been superseded by the Linear-Matrix-Inequality (LMI) solution which came shortly thereafter; we mention specifically the papers of Iwasaki-Skelton [78] and Gahinet-Apkarian [66]. This solution does not require any boundary rank conditions entailed in all the earlier approaches and generalizes in a straightforward way to more general settings (to be discussed in more detail below). The LMI form of the solution is particularly appealing from a computational point of view due to the recent advances in semidefinite programming (see [68]). The book of Dullerud-Paganini [57] gives an up-to-date account of these latest developments.

Research in H^∞-control has moved on in a number of different new directions, e.g., extensions of the H^∞-paradigm to sampled-data systems [47], nonlinear systems [126], hybrid systems [23], stochastic systems [76], quantum stochastic systems [79], linear repetitive processes [123], as well as behavioral frameworks [134]. Our focus here will be on the extensions to robust control for systems with structured uncertainties and related H^∞-control problems for multidimensional (N-D) systems – both frequency-domain and state-space settings. In the meantime, Nevanlinna-Pick interpolation theory has moved on to a variety of multivariable settings (polydisk, ball, noncommutative polydisk/ball); we mention in particular the papers [1, 49, 113, 3, 35, 19, 20, 21, 22, 30].

As the transfer function for a multidimensional system is a function of several variables, one would expect that the same connections familiar from the 1-D/single-variable case should also occur in these more general settings; however, while there had been some interaction between control theory and several-variable complex function theory in the older area of systems over rings (see [83, 85, 46]), to this point, with a few exceptions [73, 74, 32], there has not been such an interaction in connection with H^∞-control for N-D systems and related such topics. With this paper we wish to make precise the interconnections which do exist between the H^∞-theory and the interpolation theory in these more general settings. As we shall see, some aspects which are taken for granted in the 1-D/single-variable case become much more subtle in the N-D/multivariable case. Along the way we shall encounter a variety of topics that have gained attention recently, and sometimes less recently, in the engineering literature.

Besides the present Introduction, the paper consists of five sections which we now describe:

(1) In Section 2 we lay out four specific results for the classical 1-D case; these serve as models for the type of results which we wish to generalize to the N-D/multivariable settings.

(2) In Section 3 we survey the recent results of Quadrat [117, 118, 119, 120, 121, 122] on internal stabilization and parametrization of stabilizing controllers in an abstract ring setting. The main point here is that it is possible to parametrize the set of all stabilizing controllers in terms of a given stabilizing controller even in settings where the given plant may not have a double coprime factorization – resolving some issues left open in the book of Vidyasagar [136]. In the case where a double-coprime factorization is available, the parametrization formula is more efficient. Our modest new contribution here is to extend the ideas to the setting of the standard problem of H^∞-control (in the sense of the book of Francis [64]) where the given plant is assumed to have distinct disturbance and control inputs and distinct error and measurement outputs.

(3) In Section 4 we look at the internal-stabilization/H^∞-control problem for multidimensional systems. These problems have been studied in a purely frequency-domain framework (see [92, 93]) as well as in a state-space framework (see [81, 55, 56]). In Subsection 4.1, we give the frequency-domain formulation of the problem.

When one takes the stable plants to consist of the ring of structurally stable rational matrix functions, the general results of Quadrat apply. In particular, for this setting stabilizability of a given plant implies the existence of a double coprime factorization (see [119]). Application of the Youla-Kučera parametrization then leads to a Model-Matching form and, in the presence of some boundary rank conditions, the H^∞-problem converts to a polydisk version of the Nevanlinna-Pick interpolation problem. Unlike the situation in the classical single-variable case, this interpolation problem has no practical necessary-and-sufficient solution criterion and in practice one is satisfied with necessary and sufficient conditions for the existence of a solution in the more restrictive Schur-Agler class (see [1, 3, 35]).

In Subsection 4.2 we formulate the internal-stabilization/H^∞-control problem in Givone-Roesser state-space coordinates. We indicate the various subtleties involved in implementing the state-space version [104, 85] of the double-coprime factorization and associated Youla-Kučera parametrization of the set of stabilizing controllers. With regard to the H^∞-control problem, unlike the situation in the classical 1-D case, there is no useable necessary and sufficient analysis for solution of the problem; instead what is done (see, e.g., [55, 56]) is the use of an LMI/Bounded-Real-Lemma analysis which provides a convenient set of sufficient conditions for solution of the problem. This sufficiency analysis in turn amounts to an N-D extension of the LMI solution [78, 66] of the 1-D H^∞-control problem and can be viewed as a necessary and sufficient analysis of a compromise problem (the "scaled" H^∞-problem).

While stabilization and H^∞-control problems have been studied in the state-space setting [81, 55, 56] and in the frequency-domain setting [92, 93] separately, there does not seem to have been much work on the precise connections between these two settings. The main point of Subsection 4.3 is to study this relationship; while solving the state-space problem implies a solution of the frequency-domain problem, the reverse direction is more subtle and it seems that only partial results are known. Here we introduce a notion of *modal stabilizability* and *modal detectability* (a modification of the notions of *modal controllability* and *modal observability* introduced by Kung-Levy-Morf-Kailath [88]) to obtain a partial result on relating a solution of the frequency-domain problem to a solution of the associated state-space problem. This result suffers from the same weakness as a corresponding result in [88]: just as the authors in [88] were unable to prove that *minimal* (i.e., simultaneously modally controllable and modally observable) realizations for a given transfer matrix exist, so also we are unable to prove that a simultaneously modally stabilizable and modally detectable realization exists. A basic difficulty in translating from frequency-domain to state-space coordinates is the failure of the State-Space-Similarity theorem and related Kalman state-space reduction for N-D systems. Nevertheless, the result is a natural analogue of the corresponding 1-D result.

There is a parallel between the control-theory side and the interpolation-theory side in that in both cases one is forced to be satisfied with a compromise solution: the scaled-H^∞ problem on the control-theory side, and the Schur-Agler class (rather than the Schur class) on the interpolation-theory side. We include some discussion on the extent to which these compromises are equivalent.

(4) In Section 5 we discuss several 1-D variations on the internal-stabilization and H^∞-control problem which lead to versions of the N-D/multivariable problems discussed in Section 4. It was observed early on that an H^∞-controller has good robustness properties, i.e., an H^∞-controller not only provides stability of the closed-loop system associated with the given (or *nominal*) plant for which the control was designed, but also for a whole neighborhood of plants around the nominal plant. This idea was refined in a number of directions, e.g., robustness with respect to additive or multiplicative plant uncertainty, or with respect to uncertainty in a normalized coprime factorization of the plant (see [100]). Another model for an uncertainty structure is the Linear-Fractional-Transformation (LFT) model used by Doyle and coworkers (see [97, 98]). Here a key concept is the notion of structured singular value $\mu(A)$ for a finite square matrix A introduced by Doyle and Safonov [52, 124] which simultaneously generalizes the norm and the spectral radius depending on the choice of uncertainty structure (a C^*-algebra of matrices with a prescribed block-diagonal structure); we refer to [107] for a comprehensive survey. If one assumes that the controller has on-line access to the uncertainty parameters one is led to a gain-scheduling problem which can be identified as the type of multidimensional control problem discussed in Section 4.2 – see [106, 18]; we survey this material in Subsection 5.1. In Subsection 5.2 we review the purely frequency-domain approach of Helton [73, 74] toward gain-scheduling which leads to the frequency-domain internal-stabilization/H^∞-control problem discussed in Section 4.1. Finally, in Section 5.3 we discuss a hybrid frequency-domain/state-space model for structured uncertainty which leads to a generalization of Nevanlinna-Pick interpolation for single-variable functions where the constraint that the norm be uniformly bounded by 1 is replaced by the constraint that the μ-singular value be uniformly bounded by 1; this approach has only been analyzed for very special cases of the control problem but does lead to interesting new results for operator theory and complex geometry in the work of Bercovici-Foias-Tannenbaum [38, 39, 40, 41], Agler-Young [5, 6, 7, 8, 9, 10, 11, 12, 13], Huang-Marcantognini-Young [77], and Popescu [114].

(5) The final Section 6 discusses an enhancement of the LFT-model for structured uncertainty to allow dynamic time-varying uncertainties. If the controller is allowed to have on-line access to these more general uncertainties, then the solution of the internal-stabilization/H^∞-control problem has a form completely analogous to the classical 1-D case. Roughly, this result corresponds to the fact that, with this noncommutative enhanced uncertainty structure, the a priori upper bound $\widehat{\mu}(\mathbf{A})$ for the structured singular value $\mu(\mathbf{A})$ is actually equal to $\mu(\mathbf{A})$, despite the fact that for non-enhanced structures, the gap between μ and $\widehat{\mu}$ can be arbitrarily large (see [133]). In this precise form, the result appears for the first time in the thesis of Paganini [108] but various versions of this type of result have also appeared elsewhere (see [37, 42, 60, 99, 129]). We discuss this enhanced noncommutative LFT-model in Subsection 6.1. In Subsection 6.2 we introduce a noncommutative frequency-domain control problem in the spirit of Chapter 4 of the thesis of Lu [96], where the underlying polydisk occurring in Section 4.1 is now replaced by the noncommutative polydisk consisting of all d-tuples of contraction operators on a fixed separable infinite-dimensional Hilbert space $\mathcal{K}$ and the space of H^∞-functions is replaced by the space of scalar multiples of the noncommutative Schur-Agler class introduced in [28]. Via an adaptation of the Youla-Kučera parametrization of stabilizing controllers, the internal-stabilization/H^∞-control problem can be reduced to a Model-Matching form which has the interpretation as a noncommutative Sarason interpolation problem. In the final Subsection 6.3, we show how the noncommutative state-space problem is exactly equivalent to the noncommutative frequency-domain problem and thereby obtain an analogue of the classical case which is much more complete than for the commutative-variable case given in Section 4.3. In particular, if the problem data are given in terms of state-space coordinates, the noncommutative Sarason problem can be solved as an application of the LMI solution of the H^∞-problem. While there has been quite a bit of recent activity on this kind of noncommutative function theory (see, e.g., [14, 22, 75, 82, 115, 116]), the noncommutative Sarason problem has to this point escaped attention; in particular, it is not clear how the noncommutative Nevanlinna-Pick interpolation problem studied in [22] is connected with the noncommutative Sarason problem.

Finally we mention that each section ends with a "Notes" subsection which discusses more specialized points and makes some additional connections with existing literature.

2. The 1-D systems/single-variable case

Let $\mathbb{C}[z]$ be the space of polynomials with complex coefficients and $\mathbb{C}(z)$ the quotient field consisting of rational functions in the variable z. Let $\mathcal{R}H^\infty$ be the subring of *stable* elements of $\mathbb{C}(z)$ consisting of those rational functions which are analytic and bounded on the unit disk $\mathbb{D}$, i.e., with no poles in the closed unit disk $\overline{\mathbb{D}}$. We assume to be given a plant $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right] : \mathcal{W} \oplus \mathcal{U} \to \mathcal{Z} \oplus \mathcal{Y}$ which is given as a block matrix of appropriate size with entries from $\mathbb{C}(z)$. Here the spaces $\mathcal{U}$, $\mathcal{W}$, $\mathcal{Z}$ and $\mathcal{Y}$ have the interpretation of *control-signal space*, *disturbance-signal space*, *error-signal space* and *measurement-signal space*, respectively, and consist of column vectors of given sizes $n_\mathcal{U}$, $n_\mathcal{W}$, $n_\mathcal{Z}$ and $n_\mathcal{Y}$, respectively, with entries from $\mathbb{C}(z)$. For this plant G we seek to design a controller $K\colon \mathcal{Y} \to \mathcal{U}$, also given as a matrix over $\mathbb{C}(z)$, that stabilizes the feedback system $\Sigma(G, K)$ obtained from the signal-flow diagram in Figure 1 in a sense to be defined precisely below.

Note that the various matrix entries G_{ij} of G are themselves matrices with entries from $\mathbb{C}(z)$ of compatible sizes (e.g., G_{11} has size $n_\mathcal{Z} \times n_\mathcal{W}$) and K is a matrix over $\mathbb{C}(z)$ of size $n_\mathcal{U} \times n_\mathcal{Y}$.

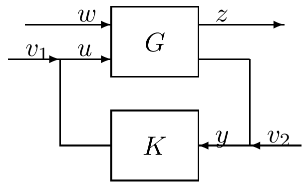

FIGURE 1. Feedback with tap signals

The system equations associated with the signal-flow diagram of Figure 1 can be written as

$$\begin{bmatrix} I & -G_{12} & 0 \\ 0 & I & -K \\ 0 & -G_{22} & I \end{bmatrix} \begin{bmatrix} z \\ u \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ G_{21} & 0 & I \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}. \tag{2.1}$$

Here v_1 and v_2 are tap signals used to detect stability properties of the internal signals u and y. We say that the system $\Sigma(G,K)$ is *well posed* if there is a well-defined map from $\left[\begin{smallmatrix} w \\ v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ u \\ y \end{smallmatrix}\right]$. It follows from a standard Schur complement computation that the system is well posed if and only if $\det(I - G_{22}K) \neq 0$, and that in that case the map from $\left[\begin{smallmatrix} w \\ v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ u \\ y \end{smallmatrix}\right]$ is given by

$$\begin{bmatrix} z \\ u \\ y \end{bmatrix} = \Theta(G,K) \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

where

$$\Theta(G,K) := \begin{bmatrix} I & -G_{12} & 0 \\ 0 & I & -K \\ 0 & -G_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ G_{21} & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} & G_{12}[I + K(I - G_{22}K)^{-1}G_{22}] & G_{12}K(I - G_{22}K)^{-1} \\ K(I - G_{22}K)^{-1}G_{21} & I + K(I - G_{22}K)^{-1}G_{22} & K(I - G_{22}K)^{-1} \\ (I - G_{22}K)^{-1}G_{21} & (I - G_{22}K)^{-1}G_{22} & (I - G_{22}K)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21} & G_{12}(I - KG_{22})^{-1} & G_{12}(I - KG_{22})^{-1}K \\ (I - KG_{22})^{-1}KG_{21} & (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ [I + G_{22}(I - KG_{22})^{-1}K]G_{21} & G_{22}(I - KG_{22})^{-1} & I + G_{22}(I - KG_{22})^{-1}K \end{bmatrix}. \tag{2.2}$$

We say that the system $\Sigma(G,K)$ is *internally stable* if $\Sigma(G,K)$ is well posed and, in addition, if the map $\Theta(G,K)$ maps $\mathcal{R}H^\infty_{\mathcal{W}} \oplus \mathcal{R}H^\infty_{\mathcal{U}} \oplus \mathcal{R}H^\infty_{\mathcal{Y}}$ into $\mathcal{R}H^\infty_{\mathcal{Z}} \oplus \mathcal{R}H^\infty_{\mathcal{U}} \oplus \mathcal{R}H^\infty_{\mathcal{Y}}$, i.e., stable inputs w, v_1, v_2 are mapped to stable outputs z, u, y. Note that this is the same as the condition that the entries of $\Sigma(G,K)$ be in $\mathcal{R}H^\infty$.

We say that the system $\Sigma(G,K)$ *has performance* if $\Sigma(G,K)$ is internally stable and in addition the transfer function T_{zw} from w to z has supremum-norm over the unit disk bounded by some tolerance which we normalize to be equal to 1:

$$\|T_{zw}\|_\infty := \sup\{\|T_{zw}(\lambda)\| \colon \lambda \in \mathbb{D}\} \le 1.$$

Here $\|T_{zw}(\lambda)\|$ refers to the induced operator norm, i.e., the largest singular value for the matrix $T_{zw}(\lambda)$. We say that the system $\Sigma(G,K)$ has *strict performance* if in addition $\|T_{zw}\|_\infty < 1$. The *stabilization problem* then is to describe all (if any exist) internally stabilizing controllers K for the given plant G, i.e., all $K \in \mathbb{C}(z)^{n_\mathcal{U} \times n_\mathcal{Y}}$ so that the associated closed-loop system $\Sigma(G,K)$ is internally stable. The *standard H^∞-problem* is to find all internally stabilizing controllers which in addition achieve performance $\|T_{zw}\|_\infty \le 1$. The strictly suboptimal H^∞-problem is to describe all internally stabilizing controllers which also achieve strict performance $\|T_{zw}\|_\infty < 1$.

2.1. The model-matching problem

Let us now consider the special case where $G_{22} = 0$, so that G has the form $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & 0 \end{smallmatrix}\right]$. In this case well-posedness is automatic and $\Theta(G,K)$ simplifies to

$$\Theta(G,K) = \begin{bmatrix} G_{11} + G_{12}KG_{21} & G_{12} & G_{12}K \\ KG_{21} & I & K \\ G_{21} & 0 & I \end{bmatrix}.$$

Thus internal stability for the closed-loop system $\Sigma(G,K)$ is equivalent to stability of the four transfer matrices G_{11}, G_{12}, G_{21} and K. Hence internal stabilizability of G is equivalent to stability of G_{11}, G_{12} and G_{21}; when the latter holds a given K internally stabilizes G if and only if K itself is stable.

Now assume that G_{11}, G_{12} and G_{21} are stable. Then the H^∞-performance problem for G consists of finding stable K so that $\|G_{11} + G_{12}KG_{21}\|_\infty \le 1$. Following the terminology of [64], the problem is called the Model-Matching Problem. Due to the influence of the paper [125], this problem is usually referred to as the Sarason problem in the operator theory community; in [125] it is shown explicitly how the problem can be reduced to an interpolation problem.

In general control problems the assumption that $G_{22} = 0$ is an unnatural assumption. However, after making a change of coordinates using the Youla-Kučera parametrization or the Quadrat parametrization, discussed below, it turns out that the general H^∞-problem can be reduced to a model-matching problem.

2.2. The frequency-domain stabilization and H^∞ problem

The following result on characterization of stabilizing controllers is well known (see, e.g., [64] or [136, 137] for a more general setting).

Theorem 2.1. *Suppose that we are given a rational matrix function $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$ of size $(n_\mathcal{Z} + n_\mathcal{Y}) \times (n_\mathcal{W} + n_\mathcal{U})$ with entries in $\mathbb{C}(z)$ as above. Assume that G is stabilizable, i.e., there exists a rational matrix function K of size $n_\mathcal{U} \times n_\mathcal{Y}$ so that the nine transfer functions in* (2.2) *are all stable. Then a given rational matrix function K stabilizes G if and only if K stabilizes G_{22}, i.e., $\Theta(G,K)$ in* (2.2) *is*