Ring and Module Theory

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Preface

This volume is a collection of 13 peer reviewed papers consisting of expository/survey articles and research papers by 24 authors. Many of these papers were presented at the International Conference on Ring and Module Theory held at Hacettepe University in Ankara, Turkey, during August 18–22, 2008.

The selected papers and articles examine wide ranging and cutting edge developments in various areas of Algebra including Ring Theory, Module Theory, Hopf Algebras, and Commutative Algebra. The survey articles are by well-known experts in their respective areas and provide an overview which is useful for researchers in the area, as well as, for researchers looking for new or related fields to investigate. The research papers give a taste of current research. We feel the variety of topics will be of interest to both graduate students and researchers.

We wish to thank the large number of conference participants from over 20 countries, the contributors to this volume, and the referees. Encouragement and support from Hacettepe University, The Scientific and Technological Research Council of Turkey (TÜBİTAK) and Republic of Turkey Ministry of Culture and Tourism are greatly appreciated. We also appreciate Evrim Akalan, Sevil Barın, Canan Celep Yücel, Esra Demiryürek, Özlem Erdoğan, Fatih Karabacak, Didem Kavalcı, Mine Polat, Tuğçe Sivrikaya, Ayşe Sönmez, Figen Takıl, Muharrem Yavuz, Filiz Yıldız and Uğur Yücel for their assistance and efficient arrangement of the facilities which greatly contributed to the success of the conference.

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A Seventy Years Jubilee: The Hopkins-Levitzki Theorem

Toma Albu

Dedicated to the memory of Mark L. Teply (1942–2006)

Abstract. The aim of this expository paper is to discuss various aspects of the Hopkins-Levitzki Theorem (H-LT), including the Relative H-LT, the Absolute or Categorical H-LT, the Latticial H-LT, as well as the Krull dimension-like H-LT.

Mathematics Subject Classification (2000). Primary 16-06, 16P20, 16P40; Secondary 16P70, 16S90, 18E15, 18E40.

Keywords. Hopkins-Levitzki Theorem, Noetherian module, Artinian module, hereditary torsion theory, τ-Noetherian module, τ-Artinian module, quotient category, localization, Grothendieck category, modular lattice, upper continuous lattice, Krull dimension, dual Krull dimension.

1. Introduction

In this expository paper we present a survey of the work done in the last forty years on various extensions of the *Classical Hopkins-Levitzki Theorem*: *Relative, Absolute* or *Categorical, Latticial,* and *Krull dimension-like*.

We shall also illustrate a *general strategy* which consists on putting a *module-theoretical* theorem in a *latticial frame*, in order to translate that theorem to Grothendieck categories and module categories equipped with hereditary torsion theories.

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The (Molien-)Wedderburn-Artin Theorem

One can say that the Modern Ring Theory begun in 1908, when *Joseph Henry Maclagan Wedderburn* (1882–1948) proved his celebrated Classification Theorem for finitely dimensional semi-simple algebras over a field F (see [49]). Before that, in 1893, *Theodor Molien* or *Fedor Eduardovich Molin* (1861–1941) proved the theorem for $F = \mathbb{C}$ (see [36]).

In 1921, *Emmy Noether* (1882–1935) considers in her famous paper [42], for the first time in the literature, the *Ascending Chain Condition* (ACC)

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

for ideals in a commutative ring R.

In 1927, *Emil Artin* (1898–1962) introduces in [17] the *Descending Chain Condition* (DCC)

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

for left/right ideals of a ring and extends the Wedderburn Theorem to rings satisfying both the DCC and ACC for left/right ideals, observing that both ACC and DCC are a good substitute for finite dimensionality of algebras over a field:

THE (MOLIEN-)WEDDERBURN-ARTIN THEOREM. *A ring R is semi-simple if and only if R is isomorphic to a finite direct product of full matrix rings over skew-fields*

$$R \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

Recall that by a *semi-simple ring* one understands a ring R which is left (or right) Artinian and has Jacobson radical or prime radical zero. Since 1927, the (Molien-)Wedderburn-Artin Theorem became a cornerstone of the Noncommutative Ring Theory.

In 1929, Emmy Noether observes (see [43, p. 643]) that the ACC in Artin's extension of the Wedderburn Theorem can be omitted: *Im II. Kapitel werden die Wedderburnschen Resultate neu gewonnen und weitergefürt, ... Und zwar zeigt es sich das der "Vielfachenkettensatz" für Rechtsideale oder die damit identische "Minimalbedingung"* (*in jeder Menge von Rechtsidealen gibt es mindestens ein – in der Menge – minimales*) *als Endlichkeitsbedingung ausreicht* (*Die Wedderburnschen Schlußweissen lassen sich übertragen wenn "Doppelkettensatz" vorausgesezt wird. Vgl. E. Artin* [17]).

It took, however, ten years until it has been proved that always the DCC in a unital ring implies the ACC.

The Classical Hopkins-Levitzki Theorem (H-LT)

One of the most lovely result in Ring Theory is the *Hopkins-Levitzki Theorem*, abbreviated H-LT. This theorem, saying that any right Artinian ring with identity is right Noetherian, has been proved independently in 1939 by *Charles Hopkins*

[27][1] (1902–1939) for left ideals and by *Jacob Levitzki* [31][2] (1904–1956) for right ideals. Almost surely, the fact that the DCC implies the ACC for one-sided ideals in a unital ring was unknown to both E. Noether and E. Artin when they wrote their pioneering papers on chain conditions in the 1920's.

An equivalent form of the H-LT, referred in the sequel also as the *Classical H-LT*, is the following one:

CLASSICAL H-LT. *Let R be a right Artinian ring with identity, and let M_R be a right module. Then M_R is an Artinian module if and only if M_R is a Noetherian module.*

Proof. The standard proof of this theorem, as well as the original one of Hopkins [27, Theorem 6.4] for $M = R$, uses the Jacobson radical J of R. Since R is right Artinian, J is nilpotent and the quotient ring R/J is a semi-simple ring. Let n be a positive integer such that $J^n = 0$, and consider the descending chain of submodules of M_R

$$M \supseteq MJ \supseteq MJ^2 \supseteq \cdots \supseteq MJ^{n-1} \supseteq MJ^n = 0.$$

Since the quotients MJ^k/MJ^{k+1} are killed by J, $k = 0, 1, \ldots, n-1$, each MJ^k/MJ^{k+1} becomes a right module over the semi-simple ring R/J, so each MJ^k/MJ^{k+1} is a semi-simple (R/J)-module.

Now, observe that M_R is Artinian (resp. Noetherian) $\iff$ all MJ^k/MJ^{k+1} are Artinian (resp. Noetherian) R (or R/J)-modules. Since a semi-simple module is Artinian if and only if it is Noetherian, it follows that M_R is Artinian if and only if it is Noetherian, which finishes the proof. □

Extensions of the H-LT

In the last fifty years, especially in the 1970's, 1980's, and 1990's the (Classical) H-LT has been generalized and dualized as follows:

1957 *Fuchs* [21] shows that a left Artinian ring A, not necessarily unital, is Noetherian if and only if the additive group of A contains no subgroup isomorphic to the Prüfer quasi-cyclic p-group $\mathbb{Z}_{p^\infty}$.

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[1] In fact, he proved that any left Artinian ring (called by him *MLI* ring) with left or right identity is left Noetherian (see Hopkins [27, Theorems 6.4 and 6.7]).

[2] The result is however, surprisingly, neither stated nor proved in his paper, though in the literature, including our papers, the Hopkins' Theorem is also wrongly attributed to Levitzki. Actually, what Levitzki proved was that the ACC is superfluous in most of the main results of the original paper of Artin [17] assuming both the ACC and DCC for right ideals of a ring. This is also very clearly stated in the Introduction of his paper: "*In the present note it is shown that the maximum condition can be omitted without affecting the results achieved by Artin.*" Note that Levitzki considers rings which are not necessarily unital, so anyway it seems that he was even not aware about DCC implies ACC in unital rings; this implication does not hold in general in non unital rings, as the example of the ring with zero multiplication associated with any Prüfer quasi-cyclic p-group $\mathbb{Z}_{p^\infty}$ shows. Note also that though all sources in the literature, including Mathematical Reviews, indicate 1939 as the year of appearance of Levitzki's paper in *Compositia Mathematica*, the free reprint of the paper available at http://www.numdam.org indicates 1940 as the year when the paper has been published.

1972 *Shock* [46] provides necessary and sufficient conditions for a non unital Artinian ring and an Artinian module to be Noetherian; his proofs avoid the Jacobson radical of the ring and depend primarily upon the length of a composition series.

1976 *Albu* and *Năstăsescu* [9] prove the *Relative H-LT*, i.e., the H-LT relative to a hereditary torsion theory, but only for commutative unital rings, and conjecture it for arbitrary unital rings.

1978–1979 *Murase* [37] and *Tominaga* and *Murase* [48] show, among others, that a left Artinian ring A, not necessarily unital, is Noetherian if and only J/AJ is finite (where J is the Jacobson radical of R) if and only if the largest divisible torsion subgroup of the additive group of A is 0.

1979 *Miller* and *Teply* [35] prove the *Relative H-LT* for arbitrary unital rings.

1979–1980 *Năstăsescu* [38], [39] proves the *Absolute* or *Categorical H-LT*, i.e., the H-LT for an arbitrary Grothendieck category.

1980 *Albu* [3] proves the *Absolute Dual H-LT* for commutative Grothendieck categories.

1982 *Faith* [20] provides another module-theoretical proof of the Relative H-LT, and gives two interesting versions of it: Δ-Σ and *counter*.

1984 *Albu* [4] establishes the *Latticial H-LT* for upper continuous modular lattices.

1996 *Albu* and *Smith* [12] prove the *Latticial H-LT* for arbitrary modular lattices.

1996 *Albu, Lenagan*, and *Smith* [7] establish a *Krull dimension-like* extension of the Classical H-LT and Absolute H-LT.

1997 *Albu* and *Smith* [13] extend the result of Albu, Lenagan, and Smith [7] from Grothendieck categories to upper continuous modular lattices, using the technique of localization of modular lattices they developed in [12].

In the sequel we shall be discussing in full detail all the extensions of the HL-T for unital rings listed above.

2. The Relative H-LT

The next result is due to Albu and Năstăsescu [9, Théorème 4.7] for commutative rings, conjectured for noncommutative rings by Albu and Năstăsescu [9, Problème 4.8], and proved for arbitrary unital rings by Miller and Teply [35, Theorem 1.4].

Theorem 2.1. (RELATIVE H-LT). *Let R be a ring with identity, and let τ be a hereditary torsion theory on* Mod-R. *If R is a right τ-Artinian ring, then every τ-Artinian right R-module is τ-Noetherian.*

Let us mention that the module-theoretical proofs available in the literature of the Relative H-LT, namely the original one in 1979 due to Miller and Teply [35, Theorem 1.4], and another one in 1982 due to Faith [20, Theorem 7.1 and Corollary 7.2], are very long and complicated.

The importance of the Relative H-LT in investigating the structure of some relevant classes of modules, including injectives as well as projectives, is revealed in Albu and Năstăsescu [10] and Faith [20], where the main body of both these monographs deals with this topic.

We are now going to explain all the terms occurring in the statement above.

Hereditary torsion theories

The concept of *torsion theory* for Abelian categories has been introduced by S.E. Dickson [19] in 1966. For our purposes, we present it only for module categories in one of the many equivalent ways that can be done. Basic torsion-theoretic concepts and results can be found in Golan [23] and Stenström [47].

All rings considered in this paper are associative with unit element $1 \neq 0$, and modules are unital right modules. If R is a ring, then Mod-R denotes the category of all right R-modules. We often write M_R to emphasize that M is a right R-module; $\mathcal{L}(M_R)$, or just $\mathcal{L}(M)$, stands for the lattice of all submodules of M. The notation $N \leqslant M$ means that N is a submodule of M.

A *hereditary torsion theory* on Mod-R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of nonempty subclasses $\mathcal{T}$ and $\mathcal{F}$ of Mod-R such that $\mathcal{T}$ is a *localizing subcategory* of Mod-R in the Gabriel's sense [22] (this means that $\mathcal{T}$ is a Serre class of Mod-R which is closed under direct sums) and $\mathcal{F} = \{ F_R \mid \mathrm{Hom}_R(T, F) = 0, \forall T \in \mathcal{T} \}$. Thus, any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is uniquely determined by its first component $\mathcal{T}$. Recall that a nonempty subclass $\mathcal{T}$ of Mod-R is a *Serre class* if for any short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in Mod-R, one has $X \in \mathcal{T} \Longleftrightarrow X' \in \mathcal{T}$ & $X'' \in \mathcal{T}$, and $\mathcal{T}$ is *closed under direct sums* if for any family $(X_i)_{i \in I}$, I arbitrary set, with $X_i \in \mathcal{T}$, $\forall\, i \in I$, it follows that $\bigoplus_{i \in I} X_i \in \mathcal{T}$.

The prototype of a hereditary torsion theory is the pair $(\mathcal{A}, \mathcal{B})$ in Mod-$\mathbb{Z}$, where $\mathcal{A}$ is the class of all torsion Abelian groups, and $\mathcal{B}$ is the class of all torsion-free Abelian groups.

Throughout this paper $\tau = (\mathcal{T}, \mathcal{F})$ will be a fixed hereditary torsion theory on Mod-R. For any module M_R we denote

$$\tau(M) := \sum_{N \leqslant M,\, N \in \mathcal{T}} N.$$

Since $\mathcal{T}$ is a localizing subcategory of Mod-R, it follows that $\tau(M) \in \mathcal{T}$, and we call it the τ-*torsion submodule* of M. Note that, as for Abelian groups, we have

$$M \in \mathcal{T} \Longleftrightarrow \tau(M) = M \quad \text{and} \quad M \in \mathcal{F} \Longleftrightarrow \tau(M) = 0.$$

The members of $\mathcal{T}$ are called τ-*torsion modules*, while the members of $\mathcal{F}$ are called τ-*torsion-free modules*.

For any $N \leqslant M$ we denote by $\overline{N}$ the submodule of M such that $\overline{N}/N = \tau(M/N)$, called the τ-*closure* or τ-*saturation* of N (in M). One says that N is τ-*closed* or τ-*saturated* if $\overline{N} = N$, or equivalently, if $M/N \in \mathcal{F}$, and the set of all τ-closed submodules of M is denoted by $\mathrm{Sat}_\tau(M)$. It is well known that $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice. Note that though $\mathrm{Sat}_\tau(M)$ is a subset of the lattice $\mathcal{L}(M)$ of all submodules of M, it is not a sublattice, because the sum of two τ-closed submodules of M is not necessarily τ-closed.

Definition 2.2. A module M_R is said to be *τ-Noetherian* (resp. *τ-Artinian*) if $\mathrm{Sat}_\tau(M)$ is a Noetherian (resp. Artinian) poset. The ring R is said to be *τ-Noetherian* (resp. *τ-Artinian*) if the module R_R is *τ-Noetherian* (resp. *τ-Artinian*).

Recall that a partially ordered set, shortly poset, $(P, \leqslant)$ is called *Noetherian* (resp. *Artinian*) if it satisfies the ACC (resp. DCC), i.e., if there is no strictly ascending (resp. descending) chain $x_1 < x_2 < \cdots$ (resp. $x_1 > x_2 > \cdots$) in P.

Relativization

The Relative H-LT nicely illustrates a general direction in Module Theory, namely the so-called *Relativization*. Roughly speaking, this topic deals with the following matter:

Given a property $\mathbb{P}$ in the lattice $\mathcal{L}(M_R)$ investigate the property $\mathbb{P}$ in the lattice $\mathrm{Sat}_\tau(M_R)$.

Since about forty years Module Theorists were dealing with the following problem:

Having a theorem $\mathbb{T}$ on modules, is its relativization τ-$\mathbb{T}$ true?

As we mentioned just after the statement of the Relative H-LT, its known module-theoretical proofs are very long and complicated; so, the relativization of a result on modules is not always a simple job, and as this will become clear with the next statement, sometimes it may be even impossible.

Theorem 2.3. (METATHEOREM). *The relativization $\mathbb{T} \rightsquigarrow \tau$-$\mathbb{T}$ of a theorem $\mathbb{T}$ in Module Theory is not always true/possible.*

Proof. Consider the following lovely theorem (see Lenagan [30, Theorem 3.2]):

$\mathbb{T}$: *If R has right Krull dimension then the prime radical $\mathrm{N}(R)$ is nilpotent.*

The relativization of $\mathbb{T}$ is the following:

τ-$\mathbb{T}$: *If R has right τ-Krull dimension then the τ-prime radical $\mathrm{N}_\tau(R)$ is τ-nilpotent.*

Recall that $\mathrm{N}_\tau(R)$ is the intersection of all τ-closed two-sided prime ideals of R, and a right ideal I of R is said to be τ-nilpotent if $I^n \in \mathcal{T}$ for some integer $n > 0$.

The truth of the relativization τ-$\mathbb{T}$ of $\mathbb{T}$ has been asked by Albu and Smith [11, Problem 4.3]. Surprisingly, the answer is "no" in general, even if R is (left and right) Noetherian, by Albu, Krause, and Teply [6, Example 3.1]. This proves our Metatheorem.

However, τ-$\mathbb{T}$ is true for any ring R and any *ideal invariant* hereditary torsion theory τ, including any commutative ring R and any τ (see Albu, Krause, and Teply [6, Section 6]). □

3. The Absolute (or Categorical) H-LT

The next result is due to Năstăsescu, who actually gave two different short nice proofs: [38, Corollaire 1.3] in 1979, based on the Loewy length, and [39, Corollaire 2] in 1980, based on the length of a composition series.

Theorem 3.1. (ABSOLUTE H-LT). *Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator. Then any Artinian object of $\mathcal{G}$ is Noetherian.*

Recall that a *Grothendieck category* is an Abelian category $\mathcal{G}$, with exact direct limits (or, equivalently, satisfying the axiom AB5 of Grothendieck), and having a generator G (this means that for every object X of $\mathcal{G}$ there exist a set I and an epimorphism $G^{(I)} \twoheadrightarrow X$). A family $(U_j)_{j\in J}$ of objects of $\mathcal{G}$ is said to be a *family of generators* of $\mathcal{G}$ if $\oplus_{j\in J}U_j$ is a generator of $\mathcal{G}$. The Grothendieck category $\mathcal{G}$ is called *locally Noetherian* (resp. *locally Artinian*) if it has a family of Noetherian (resp. Artinian) generators. Also, recall that an object $X \in \mathcal{G}$ is said to be *Noetherian* (resp. *Artinian*) if the lattice $\underline{\mathrm{Sub}}(X)$ of all subobjects of X is Noetherian (resp. Artinian).

Note that J.E. Roos [45] has produced in 1969 an example of a locally Artinian Grothendieck $\mathcal{C}$ category which is not locally Noetherian; thus, the so-called *Locally Absolute H-LT* fails. Even if a locally Artinian Grothendieck category $\mathcal{C}$ has a family of projective Artinian generators, then it is not necessarily locally Noetherian, as an example due to Menini [33] shows. However, the Locally Absolute H-LT is true if the family of Artinian generators of $\mathcal{C}$ is finite (because in this case $\mathcal{C}$ has an Artinian generator), as well as if the Grothendieck category $\mathcal{C}$ is *commutative*, by Albu and Năstăsescu [9, Corollaire 4.38] (see Section 6 for the definition of a commutative Grothendieck category).

Quotient categories and the Gabriel-Popescu Theorem

Clearly, for any ring R with identity element, the category Mod-R is a Grothendieck category. A procedure to construct new Grothendieck categories is by taking the *quotient category* Mod-$R/\mathcal{T}$ of Mod-R modulo any of its localizing subcategories $\mathcal{T}$. The construction of the quotient category of Mod-$R/\mathcal{T}$, or more generally, of the quotient category $\mathcal{A}/\mathcal{C}$ of any locally small Abelian category $\mathcal{A}$ modulo any of its Serre subcategories $\mathcal{C}$ is quite complicated and goes back to Serre's "langage modulo $\mathcal{C}$" (1953), Grothendieck (1957), and Gabriel (1962) [22].

Recall briefly this construction. The objects of the category $\mathcal{A}/\mathcal{C}$ are the same as those of $\mathcal{A}$, while the morphisms in this category are defined not so simple: for every objects X, Y of $\mathcal{A}$, one sets

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X,Y) := \varinjlim_{(X',Y')\in I_{X,Y}} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y'),$$

where $I_{X,Y} := \{(X',Y') \,|\, X' \leqslant X,\ Y' \leqslant Y,\ X/X' \in \mathcal{C},\ Y' \in \mathcal{C}\}$ is considered as an ordered set in an obvious manner, and with this order it is actually a directed set (it is indeed a set because the given Abelian category $\mathcal{A}$ was supposed to be locally small, i.e., the class of all subobjects of every object of $\mathcal{A}$ is a set). Then $\mathcal{A}/\mathcal{C}$ is an Abelian category, and there exists a canonical covariant exact functor

$$T : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

defined as follows: for every objects X, Y of $\mathcal{A}$ and every $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ one sets $T(X) := X$ and $T(f) :=$ the image of f in the inductive limit. It turns out that the exact functor T annihilates $\mathcal{C}$ (i.e., "kills" each $X \in \mathcal{C}$), and, as for quotient modules, the pair $(\mathcal{A}/\mathcal{C}, T)$ is universal for exact functors, which annihilate $\mathcal{C}$, from $\mathcal{A}$ into Abelian categories. Moreover, the given Serre subcategory $\mathcal{C}$ of $\mathcal{A}$ is a localizing subcategory of $\mathcal{A}$ if and only if the functor T has a right adjoint, and in this case the quotient category $\mathcal{A}/\mathcal{C}$ is a Grothendieck category if $\mathcal{A}$ is so. In particular, for any unital ring R, the quotient category Mod-$R/\mathcal{T}$ of Mod-R modulo any of its localizing subcategories $\mathcal{T}$ is a Grothendieck category.

Roughly speaking, the renowned *Gabriel-Popescu Theorem*, discovered exactly forty five years ago, states that in this way we obtain, up to an equivalence of categories, *all* the Grothendieck categories. More precisely,

Theorem 3.2. (The Gabriel-Popescu Theorem). *For any Grothendieck category $\mathcal{G}$ there exist a unital ring R and a localizing subcategory $\mathcal{T}$ of* Mod-R *such that $\mathcal{G} \simeq$* Mod-$R/\mathcal{T}$.

Notice that the ring R and the localizing subcategory $\mathcal{T}$ of Mod-R can be obtained in the following (noncanonical) way: Let U be any generator of the Grothendieck category $\mathcal{G}$, and let R_U be the ring $\mathrm{End}_{\mathcal{G}}(U)$ of endomorphims of U. If $S_U : \mathcal{G} \longrightarrow \text{Mod-}R_U$ is the functor $\mathrm{Hom}_{\mathcal{G}}(U, -)$, then S_U has a left adjoint T_U, $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and $\mathrm{Ker}(T_U) := \{\, M \in \text{Mod-}R_U \mid T_U(M) = 0 \,\}$ is a localizing subcategory of Mod-R_U. Take now as R any such R_U and as $\mathcal{T}$ such a $\mathrm{Ker}(T_U)$.

The reader is referred to Albu and Năstăsescu [10], Gabriel [22], and Stenström [47] for the concepts, constructions, and facts presented in this subsection.

Absolutization

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Then, because $\mathcal{T}$ is a localizing subcategory of Mod-R one can form the quotient category Mod-$R/\mathcal{T}$. Denote by

$$T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

the canonical functor from the category Mod-R to its quotient category Mod-$R/\mathcal{T}$.

Proposition 3.3. (Albu and Năstăsescu [10, Proposition 7.10]). *With the notation above, for every module M_R there exists a lattice isomorphism*

$$\mathrm{Sat}_\tau(M) \simeq \underline{\mathrm{Sub}}(T_\tau(M)).$$

In particular, M is a τ-Noetherian (resp. τ-Artinian) module if and only if $T_\tau(M)$ is a Noetherian (resp. Artinian) object of Mod-$R/\mathcal{T}$.

Absolutization is a technique to pass from τ-relative results in Mod-R to absolute properties in the quotient category Mod-$R/\mathcal{T}$ via the canonical functor T_τ : Mod-$R \longrightarrow$ Mod-$R/\mathcal{T}$. This technique is, in a certain sense, opposite to *relativization*, meaning that absolute results in a Grothendieck category $\mathcal{G}$ can be translated, via the Gabriel-Popescu Theorem, into τ-relative results in Mod-R as follows:

Let U be any generator of the Grothendieck category $\mathcal{G}$, let R_U be the ring $\mathrm{End}_{\mathcal{G}}(U)$ of endomorphims of U. As we have already mentioned above, if $S_U : \mathcal{G} \longrightarrow$ Mod-R_U is the functor $\mathrm{Hom}_{\mathcal{G}}(U, -)$, then S_U has a left adjoint T_U, $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and $\mathrm{Ker}(T_U) := \{ M \in$ Mod-$R_U \mid T_U(M) = 0 \}$ is a localizing subcategory of Mod-R_U. Let now τ_U be the hereditary torsion theory (uniquely) determined by the localizing subcategory $\mathrm{Ker}(T_U)$ of Mod-R_U. Many properties of an object $X \in \mathcal{G}$ can now be translated as τ_U-relative properties of the right R_U-module $S_U(X)$; e.g., $X \in \mathcal{G}$ is an Artinian (resp. Noetherian) object if and only if $S_U(X)$ is a τ_U-Artinian (resp. τ_U-Noetherian) right R_U-module. Observe that this relativization strongly depends on the choice of the generator U of $\mathcal{G}$.

As mentioned before, the two module-theoretical proofs available in the literature of the Relative H-LT due to Miller and Teply [35] and Faith [20], are very long and complicated. On the contrary, the two categorical proofs of the Absolute H-LT due to Năstăsescu [38], [39] are very short and simple.

Using the interaction *relativization* $\longleftrightarrow$ *absolutization*, we shall prove in Section 5 that *Relative H-LT* $\Longleftrightarrow$ *Absolute H-LT*; this means exactly that any of this theorems can be deduced from the other one. In this way we can obtain two short categorical proofs of the Relative H-LT.

However, some module theorists are not so comfortable with categorical proofs of module-theoretical theorems: they cannot touch the elements of an object because categories work only with objects and morphisms and not with elements of an object.

Good news for those people: There exists an alternative, namely the *latticial setting*. Why? If τ is a hereditary torsion theory on Mod-R and M_R is any module then $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice, and if $\mathcal{G}$ is a Grothendieck category then the lattice $\underline{\mathrm{Sub}}(X)$ of all subobjects of any object $X \in \mathcal{G}$ is also an upper continuous modular lattice. Therefore, a strong reason to study such kinds of lattices exists.

A latticial strategy

Let $\mathbb{P}$ be a problem, involving subobjects or submodules, to be investigated in Grothendieck categories or in module categories with respect to hereditary torsion theories. Our *main strategy* in this direction since more than twenty five years consists of the following three steps:

I. *Translate/formulate*, if possible, the problem $\mathbb{P}$ to be investigated in a Grothendieck category or in a module category equipped with a hereditary torsion theory into a *latticial setting*.

II. *Investigate* the obtained problem $\mathbb{P}$ in this *latticial frame*.
III. *Back to basics*, i.e., to Grothedieck categories and module categories equipped with hereditary torsion theories.

The advantage to deal in such a way, is, in our opinion, that this is the most *natural* and the most *simple* as well, because we ignore the specific context of Grothendieck categories and module categories equipped with hereditary torsion theories, focussing only on those latticial properties which are relevant in our given specific categorical or relative module-theoretical problem $\mathbb{P}$. The best illustration of this approach is, as we will see later, that both the *Relative H-LT* and the *Absolute H-LT* are immediate consequences of the so-called *Latticial H-LT*, which will be amply discussed in Sections 4 and 5.

4. The latticial H-LT and latticial dual H-LT

The Classical/Relative/Absolute H-LT deals with the question when a particular Artinian lattice $\mathcal{L}(M_R)/\mathrm{Sat}_\tau(M_R)/\underline{\mathrm{Sub}}(X)$ is Noetherian. Our contention is that the natural setting for the H-LT and its various extensions is *Lattice Theory*, being concerned as it is with descending and ascending chains in certain lattices. Therefore we shall present in this section the Latticial H-LT which gives an exhaustive answer to the following more general question:

When an arbitrary Artinian modular lattice is Noetherian?

The answer, given in an "if and only" form, is due to Albu and Smith [11, Theorem 1.9], and will be discussed in the next subsections.

Lattice background

All lattices considered in this paper are assumed to have a least element denoted by 0 and a last element denoted by 1, and $(L, \leqslant, \wedge, \vee, 0, 1)$, or more simply, just L, will always denote such a lattice. We denote by $\mathcal{M}$ the class of all modular lattices with 0 and 1. The opposite lattice of L will be denoted by L^0. We shall use $\mathbb{N}$ to denote the set $\{0, 1, \ldots\}$ of all natural numbers.

Recall that a lattice L is called *modular* if

$$a \wedge (b \vee c) = b \vee (a \wedge c), \forall\, a, b, c \in L \text{ with } b \leqslant a.$$

A lattice L is said to be *upper continuous* if L is complete and

$$a \wedge (\bigvee_{c\in C} c) = \bigvee_{c\in C} (a \wedge c)$$

for every $a \in L$ and every chain (or, equivalently, directed subset) $C \subseteq L$.

If x, y are elements in L with $x \leqslant y$, then y/x will denote the interval $[x, y]$, i.e.,

$$y/x = \{\, a \in L \mid x \leqslant a \leqslant y \,\}.$$

An element e of L is called *essential* if $e \wedge a \neq 0$ for all $0 \neq a \in L$. Dually, an element s of L is called *superfluous* or *small* if $s \vee b \neq 1$ for all $1 \neq b \in L$, i.e., if s is an essential element of L^0. A *composition series* of a lattice L is a chain $0 = a_0 < a_1 < \cdots < a_n = 1$ in L which has no refinement, except by introducing repetitions of the given elements a_i, and the integer n is called the *length* of the chain. If L is a modular lattice having a composition series, then we say that L is a lattice of *finite length*, and in this case any two composition series of L have the same length, called the *length* of L and denoted by $l(L)$. A modular lattice is of finite length if and only if L is both Noetherian and Artinian.

For all undefined notation and terminology on lattices, the reader is referred to Crawley and Dilworth [18], Grätzer [26], and Stenström [47].

The H-LT and Dual H-LT for arbitrary modular lattices

In this subsection we present a very general form of the H-LT for an arbitrary modular lattice, saying that an Artinian lattice L is Noetherian if and only if it satisfies two conditions, one of which guaranteeing that L has a good supply of essential elements and the second ensuring that there is a bound for the composition lengths of certain intervals of L.

More precisely, consider the following two properties that a lattice L may have ("$\mathcal{E}$" for Essential and "$\mathcal{BL}$" for Bounded Length):

($\mathcal{E}$) *for all* $a \leqslant b$ *in* L *there exists* $c \in L$ *such that* $b \wedge c = a$ *and* $b \vee c$ *is an essential element of* $1/a$ *.*

($\mathcal{BL}$) *there exists a positive integer* n *such that for all* $x < y$ *in* L *with* $y/0$ *having a composition series there exists* $c_{xy} \in L$ *with* $c_{xy} \leqslant y$, $c_{xy} \nleqslant x$, *and* $l(c_{xy}/0) \leqslant n$ *.*

Any pseudo-complemented modular lattice, in particular any upper continuous modular lattice satisfies ($\mathcal{E}$). Also, any Noetherian lattice satisfies ($\mathcal{E}$).

The dual properties of ($\mathcal{E}$) and ($\mathcal{BL}$) are respectively:

($\mathcal{E}^0$) *for all* $a \leqslant b$ *in* L *there exists* $c \in L$ *such that* $a \vee c = b$ *and* $a \wedge c$ *is a superfluous element of* $b/0$ *.*

($\mathcal{BL}^0$) *there exists a positive integer* n *such that for all* $x < y$ *in* L *with* $1/x$ *having a composition series there exists* c_{xy} *in* L *with* $x \leqslant c_{xy}$, $y \nleqslant c_{xy}$, *and* $l(1/c_{xy}) \leqslant n$ *.*

The next result, due to Albu and Smith [12, Theorem 1.9] is the *Latticial H-LT* for an arbitrary modular lattice, which, on one hand, is interesting in its own right, being the most general form of the H-LT we know, and, on the other hand is crucial in proving other versions of the H-LT.

Theorem 4.1. (Latticial H-LT). *Let L be an Artinian modular lattice. Then L is Noetherian if and only if L satisfies both conditions ($\mathcal{E}$) and ($\mathcal{BL}$).*

Since the opposite of a modular lattice is again a modular lattice, it follows that the above result can be dualized as follows (see Albu and Smith [12, Theorem 1.11]):

Theorem 4.2. (Latticial Dual H-LT). *Let L be a Noetherian modular lattice. Then L is Artinian if and only if L satisfies both conditions ($\mathcal{E}^0$) and ($\mathcal{BL}^0$).*

The condition (l^*) and lattice generation

The following condition for a lattice L has been considered in Albu [4]:

(l^*) *there exists a positive integer* n *such that for all* $x < y$ *in* L *there exists* $c_{xy} \in L$ *with* $c_{xy} \leqslant y$, $c_{xy} \nleqslant x$, $c_{xy}/0$ *Artinian, and* $l^*(c_{xy}/0) \leqslant n$.

If A is an Artinian lattice, then $l^*(A)$ denotes the so-called *reduced length* of A, that is $l(1/a^*)$, where a^* is the least element of the set $\{ a \in A \mid 1/a$ is Noetherian$\}$, see Albu [4, Lemma 0.3]. It is clear that for an Artinian lattice L, the condition (l^*) implies the condition $(\mathcal{BL})$.

Recall that if M_R and U_R are two modules, then the module M is said to be *U-generated* if there exists a set I and an epimorphism $U^{(I)} \twoheadrightarrow M$. The fact that M is U-generated can also be expressed as follows: for any proper submodule N of M there exists a submodule P of M which is not contained in N, such that P is isomorphic to a quotient of the module U. Further, M is said to be *completely U-generated* in case every submodule of M is U-generated. These concepts have been naturally extended in Albu [5] to posets as follows:

We say that a poset L is *generated* by a poset G, or is *G-generated*, if for every $a \neq 1$ in L there exist $c \in L$ and $g \in G$ such that $c \nleqslant a$ and $c/0 \simeq 1/g$. The poset L is called *completely generated* by G or *completely G-generated* if for every $b \in L$, the interval $b/0$ is G-generated, that is, for every $a < b$ in L, there exist $c \in L$ and $g \in G$ such that $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$.

Clearly, if the module M is (completely) U-generated, then the lattice $\mathcal{L}(M_R)$ is (completely) $\mathcal{L}(U_R)$-generated, but not conversely.

Note that if L and G are two Artinian lattices, and if L is completely G-generated, then the lattice L satisfies the condition (l^*), and so, also the condition $(\mathcal{BL})$. This immediately implies the following version of the Latticial H-LT (Theorem 4.1) in terms of lattice complete generation:

Theorem 4.3. *If L is a modular Artinian lattice which is completely generated by a modular Artinian lattice G, then L is Noetherian if and only if L satisfies $(\mathcal{E})$.*

The H-LT for upper continuous modular lattices

We present below a version in terms of condition (l^*), due to Albu [4, Corollary 1.8], of the Latticial H-LT for modular lattices which additionally are upper continuous:

Theorem 4.4. (Latticial H-LT for upper continuous lattices). *Let L be an Artinian upper continuous modular lattice. Then L is Noetherian if and only if L satisfies the condition (l^*).*

Observe that Theorem 4.1 is an extension of Theorem 4.4 from upper continuous modular lattices to arbitrary modular lattices. More precisely, the upper continuity from Theorem 4.4 is replaced by the less restrictive condition $(\mathcal{E})$, while the condition (l^*) by the condition $(\mathcal{BL})$.

5. Connections between various forms of the H-LT

In this section we are going to discuss the connections between the *Classical H-LT, Relative H-LT, Absolute H-LT*, and *Latticial H-LT*, and to present the *Faith's* Δ-Σ and *counter* versions of the Relative H-LT.

Latticial H-LT $\Longrightarrow$ Relative H-LT

As mentioned above, the module-theoretical proofs available in the literature of the Relative H-LT (namely, the original one in 1979 due to Miller and Teply [35], and another one in 1982 due to Faith [20]) are very long and complicated. We present below a very short proof based on the Latticial H-LT in terms of complete generation (Theorem 4.3).

So, let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Assume that R is τ-Artinian, and let M_R be a τ-Artinian module. The Relative H-LT states that M_R is a τ-Noetherian module.

Set $G := \mathrm{Sat}_\tau(R_R)$ and $L := \mathrm{Sat}_\tau(M_R)$. Then G and L are Artinian upper continuous modular lattices. We have to prove that M_R is a τ-Noetherian module, i.e., L is a Noetherian lattice. By Theorem 4.3, it is sufficient to check that L is completely G-generated, i.e., for every $a < b$ in L, there exist $c \in L$ and $g \in G$ such that $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$.

Since $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(M/\tau(M))$ we may assume, without loss of generality, that $M \in \mathcal{F}$. Let $a = A < B = b$ in $L = \mathrm{Sat}_\tau(M_R)$. Then, there exists $x \in B \setminus A$. Set $C := \overline{xR}$ and $I = \mathrm{Ann}_R(x)$. We have $R/I \simeq xR \leqslant M \in \mathcal{F}$, so $R/I \in \mathcal{F}$, i.e., $I \in \mathrm{Sat}_\tau(R_R) = G$. Using known properties of lattices of type $\mathrm{Sat}_\tau(N)$, we deduce that

$$[I, R] \simeq \mathrm{Sat}_\tau(R/I) \simeq \mathrm{Sat}_\tau(xR) \simeq \mathrm{Sat}_\tau(\overline{xR}) = \mathrm{Sat}_\tau(C) = [0, C],$$

where the intervals $[I, R]$ and $[0, C]$ are considered in the lattices G and L, respectively. Then, if we denote $c = C$ and $g = I$, we have $c \in L$, $g \in G$, $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$, which shows that L is completely G-generated, as desired.

Absolute H-LT $\Longrightarrow$ Relative H-LT

We are going to show how the Relative H-LT can be deduced from the Absolute H-LT. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Assume that R is τ-Artinian ring, and let M_R be a τ-Artinian module. We pass from Mod-R to the Grothendieck category Mod-$R/\mathcal{T}$ with the use of the canonical functor $T_\tau :$ Mod-$R \longrightarrow$ Mod-$R/\mathcal{T}$. Since R_R is a generator of Mod-R and T_τ is an exact functor we deduce that $T_\tau(R)$ is a generator of Mod-$R/\mathcal{T}$, which is Artinian by Proposition 3.3. Now, again by Proposition 3.3, $T_\tau(M)$ is an Artinian object of Mod-$R/\mathcal{T}$, so, it is also Noetherian by the Absolute H-LT, i.e., M is τ-Noetherian, and we are done.

Relative H-LT $\Longrightarrow$ Absolute H-LT

We prove that the Absolute H-LT is a consequence of the Relative H-LT. Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator U. Set $R_U := \mathrm{End}_{\mathcal{G}}(U)$,