Selected Works in Probability and Statistics

Selected Works of Donald L. Burkholder

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The subjects of the volumes have been selected by an editorial board consisting of Anirban DasGupta, Peter Hall, Jim Pitman, Michael Sörensen, and Jon Wellner.

Donald Burkholder

Preface

This book is a celebration of the writing of Donald Burkholder and of the explosions of martingale theory and its reach in the last forty-five years, both spearheaded by Burkholder. It contains reprints of most of Burkholder's publications and lists of his publications and students, two commentaries on his work, one by Gilles Pisier and the other by Rodrigo Bañuelos and Burgess Davis, and a brief biographical introduction. We are grateful to Jim Pitman and to Springer-Verlag. Together they made this book as well as many other selected or collected works possible. We thank Rodrigo Bañuelos and Gilles Pisier for their contributions. We also thank our very capable editor at Springer, John Kimmel.

April 2010 Burgess Davis and Renming Song

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Introduction

When martingales are studied by Don Burkholder good things happen. Maybe the way he proves a theorem turns out to be so important in a subject seemingly far from probability that a person involved in it asks, "Who is this guy Burkholder?"—overheard by someone who decidedly did know who Burkholder was outside an AMS special session on control theory. Or a technique, the Burkholder-Gundy good-λ method, is developed which is so fundamental that a good part of the mathematical world now uses it. The simple definition of martingale, at least when molded by Burkholder, seems to capture an essence which is at least part of what makes many mathematical objects tick.

Don Burkholder was born in 1927 and grew up on his family's farm, close to Octavia, Nebraska. He played on the Octavia High School basketball team, where, according to Don, if you were male and wanted to play you probably could since the school was that small. Don's team once made it to the county finals.

Don graduated from high school with the other three members of his class in 1945. He attended Earlham College, an excellent small liberal arts college in Indiana, and while there, seriously considered becoming a poet. Fortunately, at least for readers of this volume, he did not choose this career.

He met his fellow student and future wife Jean at Earlham, and they married in 1950, the year they both graduated. They had a daughter Kathleen, now deceased, and have two sons, William of Palo Alto, California and Peter of Bloomington, Indiana, and one grandchild.

Jean was very active in the community and served on the Urbana school board for twenty-two years. When she left that job, hundreds of people turned out at the reception in her honor. Don and Jean now live in Urbana, Illinois, within walking distance of the University of Illinois campus.

Don went to graduate school at the University of North Carolina to study under the great mathematical statistician, Wassily Hoeffding. He joined the Mathematics Department of the University of Illinois at Urbana-Champaign (UIUC) in 1955, where he has remained ever since.

Early in his career, Don did some very nice work in statistics, but as time passed his focus turned toward probability. A young professor working in Joseph Doob's Department would naturally become very familiar with martingales. In 1966, Don wrote the paper, *Martingale Transforms*, and the rest, as they say, is history, although in this case it is history in progress because the influence of his work continues to expand.

Don was promoted to Professor of Mathematics in 1964 and was named a CAS Professor at the Center for Advanced Study at UIUC in 1978. He retired in 1998 and is now Professor Emeritus of Mathematics at the Center for Advanced Study.

He has lectured widely, and given an invited lecture to the International Congress of Mathematicians, as well as the Institute of Mathematical Statistics Wald Lecture, a Mordell Lecture at Cambridge University, a Zygmund Lecture at the University of Chicago, and lecture series at Saint-Flour and CIGMA. He is a fellow of the Institute of Mathematical Statistics, of the Society of Industrial and Applied Mathematics, and of the American Academy of Arts and Sciences.

Don was elected a member of the United States National Academy of Sciences in 1992. He has served as editor of the *Annals of Mathematical Statistics* and as president of the Institute of Mathematical Statistics.

Don likes to read and he loves to walk. If you were at a meeting with Don and he suggested going for a walk, he was not talking about a stroll. He walked fast and far.

His mathematical career was unusually long lived. He did some of his deepest and most original work in his late fifties. Until just a few years ago he continued to do quite a lot of refereeing. He has said that once, when feeling overworked, he urged an editor who had sent him a paper to review to please send the paper on to a referee who was under eighty.

Don is unfailingly courteous and thoughtful. He has helped many mathematicians, and even as he rose to the top of his profession, he remained open and encouraging to those around him.

Complete Author Bibliography

Mathematical Articles

[M1] D. L. Burkholder. On a class of stochastic approximation processes. *Ann. Math. Statist.*, 27:1044–1059, 1956.

[M2] D. L. Burkholder. On the existence of a best approximation of one distribution function by another of a given type. *Ann. Math. Statist.*, 30:738–742, 1959.

[M3] D. L. Burkholder. Effect on the minimal complete class of tests of changes in the testing problem. *Ann. Math. Statist.*, 31:325–331, 1960.

[M4] D. L. Burkholder. Sufficiency in the undominated case. *Ann. Math. Statist.*, 32:1191–1200, 1961.

[M5] D. L. Burkholder and Y. S. Chow. Iterates of conditional expectation operators. *Proc. Amer. Math. Soc.*, 12:490–495, 1961.

[M6] D. L. Burkholder. On the order structure of the set of sufficient subfields. *Ann. Math. Statist.*, 33:596–599, 1962.

[M7] D. L. Burkholder. Semi-Gaussian subspaces. *Trans. Amer. Math. Soc.*, 104:123–131, 1962.

[M8] D. L. Burkholder. Successive conditional expectations of an integrable function. *Ann. Math. Statist.*, 33:887–893, 1962.

[M9] D. L. Burkholder and R. A. Wijsman. Optimum properties and admissibility of sequential tests. *Ann. Math. Statist.*, 34:1–17, 1963.

[M10] D. L. Burkholder. Maximal inequalities as necessary conditions for almost everywhere convergence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 3:75–88 (1964), 1964.

[M11] D. L. Burkholder. Martingale transforms. *Ann. Math. Statist.*, 37:1494–1504, 1966.

[M12] D. L. Burkholder. Independent sequences with the Stein property. *Ann. Math. Statist.*, 39:1282–1288, 1968.

[M13] D. Burkholder. Point estimation. In *International Encyclopedia of the Social Sciences*, volume 5, pages 142–150. 1968.

[M14] D. L. Burkholder and R. F. Gundy. Extrapolation and interpolation of quasi-linear operators on martingales. *Acta Math.*, 124:249–304, 1970.

[M15] D. L. Burkholder. Inequalities for operators on martingales. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 551–557. Gauthier-Villars, Paris, 1971.

[M16] D. L. Burkholder. Martingale inequalities. In *Martingales (Report Meeting, Oberwolfach, 1970)*, pages 1–8. Lecture Notes in Math., Vol. 190. Springer, Berlin, 1971.

[M17] D. L. Burkholder, R. F. Gundy, and M. L. Silverstein. A maximal function characterization of the class H^p. *Trans. Amer. Math. Soc.*, 157:137–153, 1971.

[M18] D. L. Burkholder, B. J. Davis, and R. F. Gundy. Integral inequalities for convex functions of operators on martingales. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pages 223–240, Berkeley, Calif., 1972. Univ. California Press.

[M19] D. L. Burkholder and R. F. Gundy. Distribution function inequalities for the area integral. *Studia Math.*, 44:527–544, 1972. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, VI.

[M20] D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42, 1973.

[M21] D. L. Burkholder and R. F. Gundy. Boundary behaviour of harmonic functions in a half-space and Brownian motion. *Ann. Inst. Fourier (Grenoble)*, 23(4):195–212, 1973.

[M22] D. L. Burkholder. H^p spaces and exit times of Brownian motion. *Bull. Amer. Math. Soc.*, 81:556–558, 1975.

[M23] D. L. Burkholder. One-sided maximal functions and H^p. *J. Functional Analysis*, 18:429–454, 1975.

[M24] D. L. Burkholder. Harmonic analysis and probability. In *Studies in harmonic analysis (Proc. Conf., DePaul Univ., Chicago, Ill., 1974)*, pages 136–149. MAA Stud. Math., Vol. 13. Math. Assoc. Amer., Washington, D.C., 1976.

[M25] D. L. Burkholder. Brownian motion and classical analysis. In *Probability (Proc. Sympos. Pure Math., Vol. XXXI, Univ. Illinois, Urbana, Ill., 1976)*, pages 5–14. Amer. Math. Soc., Providence, R.I., 1977.

[M26] D. L. Burkholder. Exit times of Brownian motion, harmonic majorization, and Hardy spaces. *Advances in Math.*, 26(2):182–205, 1977.

[M27] D. L. Burkholder. Boundary value estimation of the range of an analytic function. *Michigan Math. J.*, 25(2):197–211, 1978.

[M28] D. L. Burkholder and T. Shintani. Approximation of L^1-bounded martingales by martingales of bounded variation. *Proc. Amer. Math. Soc.*, 72(1):166–169, 1978.

Indicates article found in this volume.

[M29] D. Burkholder. Point estimation. In W. H. Kruskal and J. M. Tanur, editors, *International Encyclopedia of Statistics*, volume 1, pages 251–259. Free Press, 1978. Reprint, with new material added, of the 1968 contribution to the International Encyclopedia of the Social Sciences.

[M30] D. L. Burkholder. Martingale theory and harmonic analysis in Euclidean spaces. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 2*, Proc. Sympos. Pure Math., XXXV, Part, pages 283–301. Amer. Math. Soc., Providence, R.I., 1979.

[M31] D. L. Burkholder. A sharp inequality for martingale transforms. *Ann. Probab.*, 7(5):858–863, 1979.

[M32] D. L. Burkholder. Weak inequalities for exit times and analytic functions. In *Probability theory (Papers, VIIth Semester, Stefan Banach Internat. Math. Center, Warsaw, 1976)*, volume 5 of *Banach Center Publ.*, pages 27–34. PWN, Warsaw, 1979.

[M33] D. L. Burkholder. Brownian motion and the Hardy spaces H^p. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, pages 97–118. Academic Press, London, 1980.

[M34] D. L. Burkholder. A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. *Ann. Probab.*, 9(6):997–1011, 1981.

[M35] D. L. Burkholder. Martingale transforms and the geometry of Banach spaces. In *Probability in Banach spaces, III (Medford, Mass., 1980)*, volume 860 of *Lecture Notes in Math.*, pages 35–50. Springer, Berlin, 1981.

[M36] D. L. Burkholder. A nonlinear partial differential equation and the unconditional constant of the Haar system in L^p. *Bull. Amer. Math. Soc. (N.S.)*, 7(3):591–595, 1982.

[M37] D. L. Burkholder. A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 270–286. Wadsworth, Belmont, CA, 1983.

[M38] D. L. Burkholder. Boundary value problems and sharp inequalities for martingale transforms. *Ann. Probab.*, 12(3):647–702, 1984.

[M39] D. L. Burkholder. An elementary proof of an inequality of R. E. A. C. Paley. *Bull. London Math. Soc.*, 17(5):474–478, 1985.

[M40] D. L. Burkholder. An extension of a classical martingale inequality.In *Probability theory and harmonic analysis (Cleveland, Ohio, 1983)*, volume 98 of *Monogr. Textbooks Pure Appl. Math.*, pages 21–30. Dekker, New York, 1986.

[M41] D. L. Burkholder. Martingales and Fourier analysis in Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 61–108. Springer, Berlin, 1986.

[M42] D. L. Burkholder. A sharp and strict L^p-inequality for stochastic integrals. *Ann. Probab.*, 15(1):268–273, 1987.

[M43] D. L. Burkholder. A proof of Pełczyński's conjecture for the Haar system. *Studia Math.*, 91(1):79–83, 1988.

Indicates article found in this volume.

[M44] D. L. Burkholder. Sharp inequalities for martingales and stochastic integrals. *Astérisque*, (157-158):75–94, 1988. Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987).

[M45] D. L. Burkholder. Differential subordination of harmonic functions and martingales. In *Harmonic analysis and partial differential equations (El Escorial, 1987)*, volume 1384 of *Lecture Notes in Math.*, pages 1–23. Springer, Berlin, 1989.

[M46] D. L. Burkholder. On the number of escapes of a martingale and its geometrical significance. In *Almost everywhere convergence (Columbus, OH, 1988)*, pages 159–178. Academic Press, Boston, MA, 1989.

[M47] D. L. Burkholder. Explorations in martingale theory and its applications. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 1–66. Springer, Berlin, 1991.

[M48] D. Burkholder. Sharp probabilty bounds for Ito processes. In J. K. Ghosh, S. K. Mitra, K. R. Parthasarathy, and B. L. S. P. Rao, editors, *Statistics and Probability: A Raghu Raj Bahadur Festschrift*, pages 135–145. Wiley Eastern, 1993.

[M49] D. L. Burkholder. Strong differential subordination and stochastic integration. *Ann. Probab.*, 22(2):995–1025, 1994.

[M50] D. L. Burkholder. Sharp norm comparison of martingale maximal functions and stochastic integrals. In *Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994)*, volume 52 of *Proc. Sympos. Appl. Math.*, pages 343–358, Providence, RI, 1997. Amer. Math. Soc.

[M51] D. L. Burkholder. Some extremal problems in martingale theory and harmonic analysis. In *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Math., pages 99–115. Univ. Chicago Press, Chicago, IL, 1999.

[M52] D. L. Burkholder. Martingales and singular integrals in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 233–269. North-Holland, Amsterdam, 2001.

[M53] D. L. Burkholder. The best constant in the Davis inequality for the expectation of the martingale square function. *Trans. Amer. Math. Soc.*, 354(1):91–105 (electronic), 2002.

Discussion Contributions

[D1] J. F. C. Kingman. Discussion of *Subadditive ergodic theory* by J. F. C. Kingman. *Ann. Probability*, 1:883–909, 1973. Also includes discussion by Daryl Daley, H. Kesten, P. Ney, Frank Spitzer and J. M. Hammersley, and a reply by the author.

[D2] D. L. Burkholder. Comments on (5), (14), (22), (30), and (31). In A. Bellow, C. E. Kenig, and P. Malliavin, editors, *Selected papers of Alberto P. Calderón*, pages 553–558. Amer. Math. Soc., Providence, RI, 2008. Numbers refer to papers in the book, not to references in this bibliography.

Indicates article found in this volume.

Other Contributions

[C1] D. Burkholder. Joseph L. Doob, 1910–2004. *IMS Bulletin*, 33(6):8, 2004.

[C2] D. Burkholder and P. Protter. Joseph Leo Doob, 1910–2004. *Stochastic Process. Appl.*, 115(7):1061–1072, 2005.

[C3] D. Burkholder. Foreword. In *Joseph Doob: A Collection of Mathematical Articles in His Memory*, volume 50 of *Illinois J. Math. Volume 50, Number 1-4*, page v (electronic). 2006.

[C4] D. Burkholder. Joseph Leo Doob (1910–2004). *Illinois J. Math.*, 50(1-4):vii–viii (electronic), 2006.

[C5] D. Burkholder. Frank B. Knight, 1933–2007. *IMS Bulletin*, 36(5):6, 2007.

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List of Don Burkholder's Students

Roger Farrell, 1959
Morris Katz, 1959
Rajinder Singh, 1960
Gus Haggstrom, 1963
Ata Al-Hussaini, 1964
P. Warwick Millar, 1967
Burgess Davis, 1968
William Barlow, 1971
Ross Kindermann, 1978
Terry McConnell, 1981
Kwok-Pui Choi, 1987
Gang Wang, 1989
Jinsik Lee, 1992
William Hammack, 1994
Changsun Choi, 1995
Robert Bauer, 1997
Jennifer Steichen, 2000 (Co-director: Bruce Hajek)
Magdalena Musat, 2002 (Co-director: Marius Junge)
Jiyeon Suh, 2003

DONALD BURKHOLDER'S WORK IN MARTINGALES AND ANALYSIS

RODRIGO BAÑUELOS AND BURGESS DAVIS

1. INTRODUCTION

The two mathematicians who have most advanced martingale theory in the last seventy years are Joseph Doob and Donald Burkholder. Martingales as a remarkably flexible tool are used throughout probability and its applications to other areas of mathematics. They are central to modern stochastic analysis. And martingales, which can be defined in terms of fair games, lie at the core of mathematical finance. Burkholder's research has profoundly advanced not only martingale theory but also, via martingale connections, harmonic and functional analysis.

The work of Burkholder and Gundy on martingales in the late sixties and early seventies, which followed Burkholder's seminal 1966 paper *Martingale Transforms* [29], led to applications in analysis which revolutionized parts of this subject. Burkholder's outstanding work in the geometry of Banach spaces, described by Gilles Pisier in this volume, arose from his extension of martingale inequalities to settings beyond Hilbert spaces where the square function approach used in [29] fails. His work in the eighties and nineties on martingale inequalities with emphasis on identifying best constants has become of great importance recently in the investigations of two well known open problems. One of these concerns optimal L^p bounds for a singular integral operator (the two dimensional Hilbert transform) and their ramifications in quasiconformal mappings. The other relates to a longstanding conjecture in the calculus of variations dealing with rank-one convex and quasiconvex functions. These conjectures, which have received much attention in recent years largely due to the beautiful and original techniques developed by Burkholder in his work on sharp martingale inequalities, come from fields which on the surface are far removed from martingales.

We will describe in some detail a remarkable technique discovered by Burkholder and Gundy, which shows how certain integral inequalities between two nonnegative functions on a measure space follow from inequalities involving only parts of their distribution. This seemingly simple but incredibly elegant technique, often, and here, referred to as "the good–λ method," revolutionized the way probabilists and analysts think of norm comparison problems. It is now widely used in areas of mathematics which involve integrals and operators.

It is interesting to note that since 1973, Burkholder has written only two papers with a co-author and that he has written more than one paper only with Richard Gundy. The papers [56] of Burkholder and Gundy and [59] of Burkholder, Gundy, and Silverstein are exceptionally important. The results of [56] include the good–λ inequalities and fundamental integral inequalities comparing the maximal function and the square function, or quadratic variation, of martingales having controlled jumps or continuous paths. A very large share of the extensive applications of these kinds of martingale inequalities, both in probability and other areas of mathematics, involve continuous path martingales. The paper [59] strikingly improved and completed work of Hardy and Littlewood on the characterization of the Hardy H^p spaces via the integrability of certain maximal functions. While probabilistic techniques had already gained the respect of many analysts studying harmonic functions and potential theory, due in part

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to earlier work of Doob, Kakutani, Wiener and others, this landmark paper had a profound influence in harmonic analysis and propelled many analysts to learn probability.

The next section begins with a brief introduction to the good–λ method, in the context of its original application to martingales. We then trace the rest of the development of the theory of martingale square functions and transforms in the late sixties and early seventies, pioneered by Burkholder. We follow this with a discussion of [59] and the subsequent study of H^p theory by a number of researchers, and much more on the surprisingly rich good-λ inequalities. In the final sections we discuss Burkholder's later work on sharp martingale inequalities and some of the remarkable spread of his ideas over other areas of mathematics.

2. MARTINGALE INEQUALITIES

Brownian motion stopped at a stopping time is a continuous martingale, and the continuous martingale inequalities of [56] follow from their validity just for stopped Brownian motions. We will elaborate on this later. We will use $B = \{B_t;\ t \geq 0\}$ to denote standard Brownian motion. This means that B is a stochastic process with continuous paths, that its increment $B_t - B_s$ over the interval $[s, t]$ has a normal distribution with mean 0 and variance $t - s$, that its increments over each of a collection of disjoint intervals are independent, and that $B_0 = 0$. We recall that if the random variable τ is a stopping time for B then $\tau \geq 0$ and if $P(\tau > s) > 0$ and $t > s$, the conditional distribution, given $\tau > s$, of $B_t - B_s$ is normal with mean zero and variance $t - s$. The maximal function of B up to the stopping time τ will be denoted by $B^*_\tau = \sup\{|B_s| \colon 0 \leq s < \tau\}$. The following theorem is from [56].

Theorem 2.1. *Let Φ be a continuous nondecreasing function on $[0, \infty)$ satisfying $\Phi(0) = 0$ and $\Phi(2\lambda) \leq K\Phi(\lambda)$, $\lambda \geq 0$, for some constant K. Then there are positive constants c and C, which depend only on K, such that for any stopping time τ for B,*

$$cE\Phi(\sqrt{\tau}) \leq E\Phi(B^*_\tau) \leq CE\Phi(\sqrt{\tau}). \tag{2.1}$$

Remark 2.1. *Two important examples of functions Φ satisfying this "moderate" growth property are $\Phi(x) = x^p$, $0 < p < \infty$, and $\Phi(x) = x + x\ln^+(x)$.*

To illustrate the good-λ method used by Burkholder and Gundy in [56], we give a direct proof of the left hand side of (2.1) in the case $\Phi(x) = x$ which gives $c = \frac{1}{1200}$. This proof, which requires virtually no specialized knowledge, is a slight alteration of the proof in [56], as it uses summation rather than integration. Later, in Theorems 4.1and 4.2, we present a general form of the good–λ method, together with inequalities for stopped Brownian motion, which imply Theorem 2.1.

Denote the integers by $\mathbf{Z}$. Let $a_k \geq 0$, $k \in \mathbf{Z}$, satisfy $\lim_{k \to -\infty} a_k = 0$ and $a_{k+1} \leq 2a_k$. For $0 < r < 1$, let $J(r) = \{k \colon a_{k+1} > ra_k\}$. If k is in $J(r)$, but none of $k + i$, for $1 \leq i \leq m$, are in $J(r)$, then

$$\sum_{i=1}^{m} a_{k+i} \leq a_{k+1}(1 + r + r^2 + \ldots r^{m-1}) < \frac{2a_k}{1 - r},$$

which implies

$$\sum_{k \in J(r)} a_k \geq \frac{1 - r}{3 - r} \sum_{k \in \mathbf{Z}} a_k. \tag{2.2}$$

The k in $J(r)$ are the "good" k. Now nonnegative random variables X satisfy

$$EX \leq \sum_{k \in \mathbf{Z}} 2^k P(X \geq 2^k) \leq 2EX. \tag{2.3}$$

2

If N is a standard normal random variable then, using tables or that the density of N is bounded by $\frac{1}{\sqrt{2\pi}}$, we get $P\left(|N| < \frac{1}{10}\right) < \frac{1}{12}$, so for an event A,

$$P\left(|N| \geq \frac{1}{10}, A\right) \geq \frac{1}{6}, \text{ if } P(A) \geq \frac{1}{4}. \tag{2.4}$$

Let $A_k = \{\sqrt{\tau} \geq 2^k\}, k \in \mathbf{Z}$, and let $J = \{k\colon P(A_{k+1}) \geq \frac{P(A_k)}{4}\}$. The left hand side of (2.3), and (2.2) with $r = 1/2$ and $a_k = 2^k P(A_k)$, give

$$E\sqrt{\tau} \leq 5\Sigma_{k\in J} 2^k P(A_k). \tag{2.5}$$

Since $2B_t^* \geq |B_a - B_b|$, if $0 \leq a \leq b \leq t$, $2B_\tau^* \geq |B_{2^{2(k+1)}} - B_{2^{2k}}|$ on A_{k+1}. With (2.4) this gives

$$P(20B_\tau^* \geq 2^k) \geq P(2B_\tau^* \geq \frac{1}{10} 2^k \sqrt{3}, A_{k+1}) \geq \frac{1}{6} P(A_k),\ k \in J,$$

which with the right side of (2.3) and (2.5) yields

$$2E20B_\tau^* \geq \sum_{k\in\mathbf{Z}} P(20B_\tau^* \geq 2^k) 2^k \geq \frac{1}{6} \sum_{k\in J} P(A_k) 2^k \geq \frac{1}{30} E\sqrt{\tau}.$$

As noted in [31], Skorohod and others had before [56] proved the inequalities (2.1) for the case $\Phi(x) = x^p$, $p \geq 2$, and P. W. Millar [120], using results of [29], extended these to all $p > 1$. Also A. A. Novikov [130], working independently of [56], used stochastic calculus to study questions raised by Millar's paper and proved some interesting results related to those of [56].

The growth condition on Φ involving K of Theorem 2.1 is necessary for the truth of either of the inequalities in (2.1), in the sense that if Φ is a continuous nondecreasing function which does not satisfy this condition for any K there are stopping times τ for B such that (either) one of $E\Phi(B_\tau^*)$, $E\Phi(\sqrt{\tau})$ is finite and the other is infinite.

Next we turn to discrete time martingales. After a very brief history of martingales before Doob we provide an overview of the work of the late sixties and seventies involving the martingale square function. More general results, with proofs and extensive references, may be found in Burkholder's Wald Memorial Lecture paper [31]. We have tried to be true to the spirit if not the letter of the papers we describe.

Paul Lévy defined martingales without the name, which was given by Doob. Before martingales were formally defined, several probabilists other than Lévy, and several analysts, worked on objects that were martingales. For example, R.E.A.C. Paley [138] proved an inequality for the Haar system which is a special case of the results of Burkholder in his 1966 paper. (See [42] for a sharp version of the Paley result.) Although the definition of martingales was made by a probabilist, there is no reason it couldn't have come from an analyst instead. Sequences of piecewise constant functions on the Lebesgue unit interval which are martingales seem now a natural generalization of Haar series, and are in a distributional sense all of the discrete (as described in the next paragraph) martingales. Of course, there's nothing like hindsight to clarify thinking. In another direction Courant, Fredricks, and Lewy in 1928 [70] used ideas related to martingale ideas, although without randomness, to study harmonic functions, in the paper which introduced the finite element method for numerical approximation of solutions of partial differential equations.

We begin with a description of martingales when time is discrete and the random variables which compose them are discrete, that is, have a discrete distribution. A sequence of discrete random variables $\{D_i, i \geq 0\}$, is a martingale difference sequence if each D_i has finite expectation and if for $n > 0$,

$$E(D_n | D_i = a_i, 0 \leq i < n) = 0, \text{ if } P(D_i = a_i, 0 \leq i < n) > 0. \tag{2.6}$$

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