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Transient Chaos

Complex Dynamics on Finite-Time Scales

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Ying-Cheng Lai · Tamás Tél

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Ying-Cheng Lai
Department of Electrical Engineering
Arizona State University
Tempe Arizona
USA
Ying-Cheng.Lai@asu.edu

Tamás Tél
Department of Theoretical Physics
Institute of Physics
Eötvös University
1117 Budapest
Hungary
tel@general.elte.hu

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Preface

In a dynamical system, transients are temporal evolutions preceding the asymptotic dynamics. Transient dynamics can be more relevant than the asymptotic states of the system in terms of the observation, modeling, prediction, and control of the system. As a result, transients are important to dynamical systems arising from a wide range of disciplines such as physics, chemistry, biology, engineering, economics, and even social sciences. Research on nonlinear dynamical systems has revealed that sustained chaos, as characterized by a random-like yet structured dynamics with sensitive dependence on initial conditions, is ubiquitous in nature. A question is, then, can chaos be transient?

A common perception, as conveyed in many existing books on nonlinear dynamics, is that chaos is an asymptotic property that manifests itself only after a long observation. Indeed, standard characteristics of chaos, such as the Lyapunov exponents that measure the exponential separation rates of nearby trajectories and hence quantify the degree of the sensitivity to initial conditions, are defined in the infinite time limit. These features seem to be incompatible with the possibility of chaotic transients.

Research on nonlinear dynamics has shown, however, that the essential feature of chaos is the existence of so-called *chaotic sets* in the phase space, and quantitative characterization of chaos is meaningful with respect to the dynamics on such sets only. Since this does not imply that trajectories from random initial conditions would necessarily approach these sets asymptotically, transient chaos can arise. Transient chaos is associated with the existence of *nonattracting* chaotic sets. Research has also revealed that transient chaos is in fact more common and possibly richer than sustained or permanent chaos, since the latter can be regarded merely as a limit of transient chaos when the average lifetime of the underlying chaotic set becomes infinite. Transient chaos thus plays a similar role in the realm of complex dynamics to that of a weakly unstable equilibrium state in regular dynamics. In fact, transient chaos can be regarded as a kind of metastable state. The concept of transient chaos is ideally suited to the description of nonequilibrium processes.

The aim of this book is to give an overview, based on the results of nearly three decades of intensive research, of transient chaos. One belief that motivates us to write this book is that transient chaos may not have been appreciated even within the nonlinear-science community, let alone other scientific disciplines. During the course of research and interactions with various scientific communities, we have become increasingly convinced that knowledge of transient chaos can be particularly important and useful as we witness a proliferation of applications in various branches of science and engineering based on or motivated by nonlinear dynamics.

We shall show in this book that the basic concepts required to understand transient chaos are actually fairly easily generalized from concepts of standard nonlinear dynamics. One special emphasis will be on the fact that certain interesting dynamical phenomena can be understood only in the framework of transient chaos.

That transient chaos can arise in a broad array of fields can be illustrated by the following examples:

- Chemical reactions in closed containers can lead to thermal equilibrium only. However, the transients can be chaotic if one begins sufficiently far from equilibrium states.
- Certain epidemiological data, e.g., on the spread of chickenpox, can be consistently and meaningfully interpreted only in terms of transient chaos.
- The so-called shimmy (an irregular dancing motion) of the front wheels of motorcycles and airplanes, which can lead to disastrous incidents, turns out to be a manifestation of transient chaos.
- Satellite encounters and the escapes from major planets are chaotic transients.
- The trapping of advected material or pollutant around obstacles, often seen in the wake of pillars or piers, is a consequence of transient chaos.
- In nanostructures, today a cutting-edge field of science and engineering, the classical dynamics of electrons bear the signature of transient chaos.

This book should be regarded as a research monograph and is intended for graduate students and researchers in science and engineering who are interested in understanding and applying this extended concept of chaotic dynamics to their respective areas of research. Preliminary knowledge of sustained chaos, e.g., chaotic attractors, Lyapunov exponents, fractals, periodic orbits, stable and unstable manifolds, is assumed. These concepts can be found in almost any existing textbook on chaotic dynamics.[1]

Our Book not only gives an introduction to the novel concepts needed for understanding and for properly treating transient chaos, but also provides an overview of various transient-chaos-related phenomena. The book is organized as follows.

Part I: Basics of Transient Chaos. The first part covers the basic concepts, notions, ideas, theories, and algorithms required for understanding transient chaos.

- *Chapter 1: Introduction to Transient Chaos.* This chapter is devoted to a preliminary acquaintance with transient chaos, where basic properties of nonattracting chaotic sets are presented. To underline the relevance of transient chaos, a brief presentation of a number of experiments is given, which also illustrate different aspects of the applicability.

[1] The textbooks [564, 773] also provide an elementary treatment of transient chaos.

- *Chapter 2: Transient Chaos in Low-Dimensional Systems.* Dynamics from a one-dimensional map mimic those along the unstable manifold of, for example, a two-dimensional invertible map associated with a three-dimensional flow. Many fundamental insights into transient chaos can be gained by investigating one- and two-dimensional dynamical systems.
- *Chapter 3: Crises.* Transient chaos often precedes the birth of permanent chaos. Attractor destructions, explosions, and merging are often accompanied by transient chaos. Dynamical properties of transient chaos are partially inherited by the enlarged attractor. Transient chaos thus provides the *backbone* of the motion on composed attractors. Periodic windows, in spite of their name, are in fact parameter regions in which transient chaos is typically present.
- *Chapter 4: Noise and Transient Chaos.* In systems subject to external random forces, the attractor and the associated dynamics depend on the noise intensity. The phenomenon that a dynamical system with simple periodic attractors becomes chaotic in the presence of noise is noise-induced chaos. It is due to the transient chaotic dynamics coexisting with the periodic attractors in the noise-free system, which become stabilized by noise. This chapter presents an extensive treatment of the effects of noise on dynamical systems exhibiting transient chaos, which is physically important because noise is inevitable in any realistic dynamical systems.

Part II: Physical Manifestations of Transient Chaos. This part presents physical manifestations of transient chaos in various natural systems. A striking aspect of transient chaos is that it can lead to fundamental difficulties in predictability. Chaotic scattering, the manifestation of transient chaos in open Hamiltonian systems, will also be described both in classical and in quantum mechanics.

- *Chapter 5: Fractal Basin Boundaries.* If two or more periodic or chaotic attractors coexist, a trajectory may wander for a long time before approaching one of the attractors asymptotically. When there is transient chaos on the boundaries separating the basins of attraction, prediction of the final (asymptotic) state of the system may not be possible. There can also be situations in which the boundaries are severely interwoven (riddled basins), so that the motions on the boundaries dominate the dynamics. Fractal basin boundaries or riddled basins cause a fundamental difficulty in predicting the asymptotic state of the system.
- *Chapter 6: Chaotic Scattering.* For scattering processes in open conservative systems the only way chaos can appear is in the form of transients, as a consequence of the asymptotic freedom of the incoming and outgoing motions. Physical trajectories are usually trapped in a scattering region of the configuration space for a finite amount of time before leaving the system. Applications range from chemical reactions to celestial mechanics.
- *Chapter 7: Quantum Chaotic Scattering and Conductance Fluctuations in Nanostructures.* This chapter deals with signatures of chaotic scattering when the same system is treated quantum-mechanically in the semiclassical regime. Scattering-matrix elements exhibit random fluctuations as some physical parameters of the system change. Depending on whether the classical scattering

is hyperbolic or nonhyperbolic, statistical properties of the fluctuations can be quite distinct. One area in which quantum chaotic scattering finds significant applications is electronic transport in semiconductor nanostructures.

Part III: High-Dimensional Transient Chaos. Although low-dimensional transient chaos for which the underlying nonattracting chaotic sets have only one positive Lyapunov exponent is relatively well understood, high-dimensional transient chaos generated by chaotic sets with multiple positive Lyapunov exponents remains a forefront area of research in nonlinear dynamics. This part summarizes what is known so far about high-dimensional transient chaos.

- *Chapter 8: Transient Chaos in Higher Dimensions.* The increase in the unstable dimension from dimension one represents a highly nontrivial extension in terms of what has been known about transient chaos. Topics treated include the dimension formulas, algorithms for computing high-dimensional chaotic saddles, and chaotic scattering in physical systems with three degrees of freedom. In high-dimensional dynamical systems, transients can differ from those in low-dimensional systems in that the average lifetime is often extremely long before the system settles into a final attractor, which is usually nonchaotic. The presence of such transients implies that observation of the actual attractors of the system is practically impossible. The basic scaling law characterizing the so-called superpersistent chaotic transients and the effect of noise are treated.
- *Chapter 9: Transient Chaos in Spatially Extended Systems.* In a spatially extended system, transient lifetime often grows with the system size, and this growth can be as fast as exponential, or even faster. The presence of such superlong transients implies that the observed spatiotemporal behavior is not related to chaotic attractors. Certain phenomena such as pipe turbulence may thus turn out to exist on finite time scales only. An overview of transient chaos in spatially extended dynamical systems and open issues is presented in this chapter.

Part IV: Applications of Transient Chaos. This part focuses on different aspects of applications of transient chaos in physical, chemical, biological, and engineering systems. A physical context in which transient chaos is ubiquitous is fluid systems. Another broad area of application is control and maintenance of transient chaos for desirable system performance. The collection and analysis of transient chaotic time series for probing the underlying system are also applicable in many areas of science and engineering.

- *Chapter 10: Chaotic Advection in Fluid Flows.* The passive advection of tracer particles (e.g., small dye droplets) in open hydrodynamical flows with uniform inflow and outflow velocities turns out to be an appealing application of chaotic scattering. The unstable manifold of the nonattracting chaotic set becomes a direct physical observable in such cases as this manifold is traced out by particles or pollutants while being advected downstream. These manifolds form the backbone of possible chemical and biological reactions taking place in the flow. The transient-chaos-based approach to advection in fluid flows can have significant applications in engineering and environmental sciences.

- *Chapter 11: Controlling Transient Chaos and Applications.* We demonstrate in this chapter that transient chaos can be controlled by small perturbations. As in the control of permanent chaos, an unstable orbit on the chaotic set can be stabilized. A different form of control is to convert transient chaos into permanent chaos. Applications presented include voltage collapse in electrical-power systems and prevention, population control in ecology, and digital-information encoding.
- *Chapter 12: Transient Chaotic Time Series Analysis.* For transient chaos, only short time series are available, which makes the application of the methods developed in data analysis nontrivial. This chapter is devoted to basic issues in transient chaotic time series analysis, which include delay-coordinate embedding, and estimation of fractal dimension and Lyapunov exponents.

The main text is closed by a few final remarks. In the appendices, we treat a number of technical issues such as multifractal spectra, open random baker maps, semiclassical theory of chaotic scattering, and scattering cross sections.

To preview the applicability of the subject, we give in Table 1 a list of applications of transient chaos in various disciplines, all of which will be treated (although not in the same depth) in different chapters, including those outside of Part IV of this book.

We try to give as broad as possible an overview. The field is, however, actively developing, and full coverage of the literature is hardly possible by now. The selection of the material is therefore unavoidably biased, influenced by the authors' own experience.

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Phoenix and Budapest, 2009

Ying-Cheng Lai
Tamás Tél

Table 1 Applications of transient chaos in different disciplines

Discipline	Subject	Chapters
Mathematics	Continued fraction	2
	Transfer operators	2
	Almost invariant sets	2,10
	Snapshot attractors and saddles, random maps	4,10
	Leaked dynamics	2,10
Astronomy	Escape of celestial bodies	6
Statistical physics	Poincaré recurrences	2,6,7
	Random systems and noise	2,4,11
	Lobe dynamics	6,10
	Transport processes	6,7,10
Optics	Dielectric cavities	6,7
	Lasers	6,12
Quantum mechanics	Open quantum systems	7
	Quantum echoes	7
	Fractal Weyl law	7
Nanoscience	Quantum dots	7
	Graphene	7
	Microfluidics	10
Fluid dynamics	Stirring and mixing	10
	Vortex dynamics	10
	von Kármán vortex street	8,10
	Turbulence	9
Engineering	Shimmying wheels	2
	Voltage collapse	11
	Encoding digital information	11
Chemistry	Classical molecular reactions	6,8
	Reactions in open flows	10
	Reaction–diffusion systems	9
Biology	Population and plankton dynamics	6,10,11
	Epidemiology and ecology	4
	Food chains	11
	Species extinction	11
Environmental sciences	Spreading of pollutants	10
	Lagrangian coherent structures	10
	Convection in the Earth's mantle	10
	Advection of finite-size particles	10

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Part III High-Dimensional Transient Chaos

Part I
Basics of Transient Chaos

Chapter 1
Introduction to Transient Chaos

In numerical or experimental investigations one never has infinitely long time intervals at one's disposal. In fact, what is needed for the observation of chaos is a well-defined *separation of time scales*. Let t_0 denote the internal characteristic time of the system. In continuous-time problems, t_0 can be the average turnover time of trajectories on a Poincaré map in the phase space. In a driven system, it is the driving period. In discrete-time dynamics, t_0 can be the time step itself.

Suppose one observes signals that appear random for an *average lifetime* τ. Since chaos is characterized by a sensitive dependence on initial conditions, which is meaningful only on sufficiently long time scales, the appearance of chaotic signals requires that τ be much greater than the internal characteristic time:

$$\tau \gg t_0 \,. \tag{1.1}$$

The difference between sustained and transient chaos lies in the actual value of τ: for the former, τ is infinite, but it is finite for the latter. As a matter of practicality, one cannot exclude the possibility that a system apparently exhibiting a chaotic attractor may turn out to be transiently chaotic if a much longer period of observation is allowed. It is therefore useful to consider an additional time scale: the observation time T_O. The sustained or transient nature of chaos then depends on how τ is compared with T_O. We can speak of transient chaos if

$$\tau < T_O \,. \tag{1.2}$$

In the numerical investigation of attractors, a general habit is to discard a long sequence of the trajectory in order to concentrate on the asymptotic properties. A much richer dynamics may be observed, however, if one follows the trajectories from the beginning, i.e., if transients are not thrown out. One often finds then complex dynamics over some time, different from the dynamics governed by the attractor. The lifetime of a chaotic transient depends on the initial condition. An example can be seen in Fig. 1.1, where transiently chaotic trajectories are shown from the Hénon map [325, 564] at a parameter set where the attractor is a limit cycle.

Such signals can also be observed in experiments. An example is shown in Fig. 1.2, where the measured quantity is the temperature difference between two

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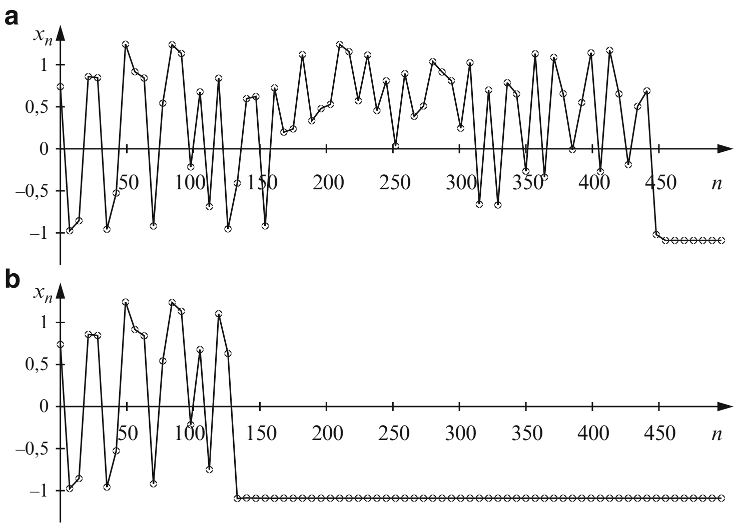

Fig. 1.1 Transient chaotic signals from the Hénon map $x_{n+1} = 1 - ax_n^2 + by_n$, $y_{n+1} = x_n$ for parameters $a = 1.25$ and $b = 0.3$, with a period-7 attractor. For clear visualization, only every seventh iterate is shown. (**a**) Trajectory initiated at $x_0 = 0.738816$, $y_0 = 0.893088$ exhibits chaotic behavior over 441 iterates. (**b**) The initial condition is shifted by $2 \cdot 10^{-19}$ in the x direction and the length of the chaotic transient is only 126

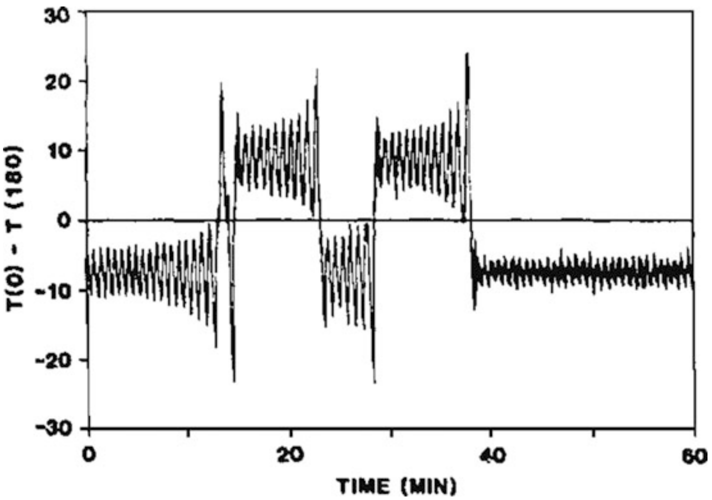

Fig. 1.2 Transient chaotic signal of the temperature difference observed between two points of an experimental loop of fluid heated from below with a constant heat flux (see Sect. 1.3 for more details). In this run, chaotic oscillations last up to nearly 40 min [823] (with kind permission from Elsevier Science)

points in a fluid loop. Over some time chaotic temperature oscillations are observed, which are accompanied by chaotic velocity oscillations of the laminar flow in the loop, and then, rather suddenly, a crossover takes place towards a nearly constant temperature difference corresponding to a uniform rotation of the fluid motion. (For a list of other representative experiments, see Sect. 1.3.)

Based on these and many other examples, one concludes that transiently chaotic signals (whose precise characterization will be discussed in Sect. 1.2) have the following characteristic properties:

1. For a fixed initial condition the signal appears chaotic up to certain time and then switches over, often quite abruptly, into a different, often nonchaotic, behavior that governs all the rest of the signal. The average lifetime, τ, can be obtained from an ensemble of such observations, although for individual observations, the actual lengths of transients depend sensitively on initial conditions: nearby trajectories typically have drastically different lifetimes.
2. The probability *distribution*, $P(t)$, of finding lifetimes longer than t is a smooth function, which satisfies $P(t) \to 0$ for $t \to \infty$.
3. There exist infinitely long transients. Mathematically, however, the set of initial conditions leading to infinite transients has zero volume in the phase space (has Lebesgue measure zero). Physically, this means that such infinite transients cannot be realized by initial conditions chosen randomly. In fact, for a typical (i.e., randomly chosen) initial condition, the transient lifetime is finite. Nonetheless, it is the presence of the measure-zero set of the initial conditions with infinite transients which causes the random distribution of the transient lifetimes for typical initial conditions.
4. It is known [564] that in a parameter region where chaotic attractors arise, periodic windows are dense. That is, for a specific parameter value that leads to a chaotic attractor, an arbitrarily small perturbation in the parameter can lead to a periodic attractor. In this sense, chaotic attractors are not structurally stable. Transient chaos is, however, robust against small parameter perturbations.

Similar to the fact that sustained chaotic signals are due to chaotic attractors in the phase space, there exist chaotic invariant sets that are responsible for transiently chaotic signals. Globally, such a chaotic set does not attract trajectories from its neighborhood, and hence it is *nonattracting*. Nonattracting chaotic sets (chaotic saddles or repellers; see Sect. 1.1.2) are therefore the *phase-space objects* that underly transient chaos. We thus accept the following definition: *transient chaos is the form of chaos due to nonattracting chaotic sets in the phase space*.

This chapter serves as a "first acquaintance" with transient chaos. The basic properties of nonattracting chaotic sets will be described. The average lifetime and the *escape rate* from these sets will then be introduced. Different methods for numerically constructing nonattracting chaotic sets will be given. The construction of the *natural* probability distribution on these sets will also be discussed, and an important related distribution, the *conditionally invariant measure* (c-Measure), will be introduced, from which characterizing quantities such as the Lyapunov exponents of the transients and dimensions of the nonattracting chaotic sets can be defined and calculated. To underline the scientific relevance of transient chaos, a list of experiments taken from different disciplines will be presented, which also illustrate different aspects of transient chaos. Finally, a brief history of transient chaos will be given.

1.1 Basic Notions of Transient Chaos

1.1.1 Dynamical Systems

Dynamical systems are usually described by a set of ordinary differential equations:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}, p), \tag{1.3}$$

where $\mathbf{x}(t)$ is the vector characterizing the state of the system at time t and p represents a set of parameters. Alternatively, discrete-time dynamical systems, or maps, of the form

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p) \tag{1.4}$$

can be investigated, where $\mathbf{x}_n$ is the state vector at discrete time n. Unless otherwise stated, the map is assumed to be autonomous, i.e., $\mathbf{f}$ does not depend explicitly on n. Maps can always be deduced from flows (1.3) by taking an appropriately defined Poincaré surface of section or stroboscopic map [564], the latter corresponding to repeatedly taking snapshots of the system at the multiples of some characteristic time t_0. Using such maps, the phase-space dimension is reduced effectively by one, facilitating visualization and analysis. In fact, Poincaré or stroboscopic maps have been used commonly in numerical and laboratory experiments on transient chaos (see Sect. 1.3). In order to have a consistent terminology, maps will be used for the rest of the chapter to illustrate the basic dynamical properties of transient chaos, but the main results apply also to flows (see also [398]).

1.1.2 Saddles and Repellers

The actual form of a nonattracting chaotic set depends on whether the dynamics is invertible. A dynamical system is invertible if its motion can be uniquely followed when time is reversed. This does not imply, however, that the time-reversed dynamics can actually occur in reality (although this is true for Hamiltonian systems, which are invariant under time reversal if no external magnetic field or Coriolis effect is present). Dynamical systems described by differential equations are typically invertible due to the uniqueness of solutions. Invertible dynamical systems are thus physically relevant. Noninvertible systems such as those described by one-dimensional maps can, however, be quite useful models for understanding specific features of transient chaos, and we shall consider them as well.

In an invertible dynamical system, a typical nonattracting chaotic set repels trajectories only along some special hypersurface in the phase space, which is called the *unstable manifold*. Along a different invariant hypersurface, or the *stable manifold*, the set can actually attract nearby trajectories. Usually, the local phase space at a point in the chaotic set can be decomposed into the stable and the unstable subspaces. For this reason, nonattracting chaotic sets in invertible dynamical systems are called *chaotic saddles*. Because differential equations are, in general, invertible, and many real-life phenomena are described by differential equations, *transient chaos in experiments is typically related to chaotic saddles*.

In contrast, for noninvertible dynamical systems in which the inverse is not unique, nonattracting chaotic sets are often *chaotic repellers*, objects that are repellent in all possible directions of the phase space. Chaotic repellers possess only unstable manifolds. These considerations are summarized in Table 1.1. The geometrical appearances of chaotic saddles and chaotic repellers can be quite different, as Fig. 1.3 illustrates.

The dynamical difference between chaotic repellers and saddles is that long-lived trajectories can start only from a neighborhood of the repeller, but for saddles

Table 1.1 Types of typical nonattracting chaotic sets in nonlinear dynamical systems

Dynamics	Nonattracting chaotic set
Invertible	Chaotic saddle
Noninvertible	Chaotic repeller

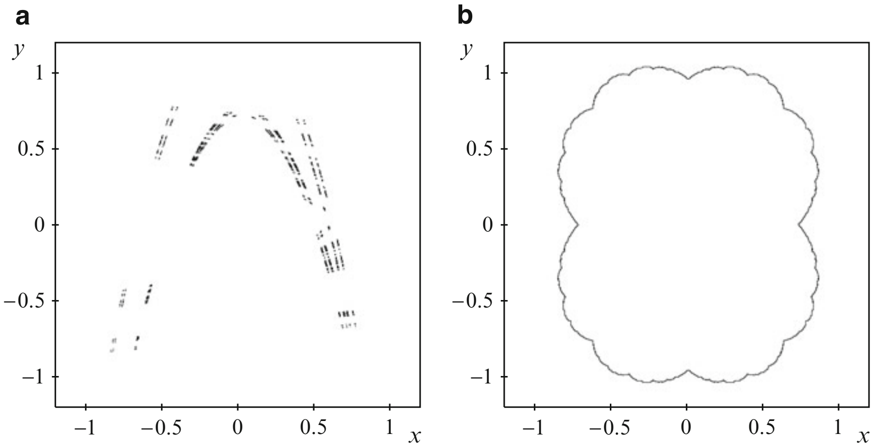

Fig. 1.3 Comparison of a chaotic saddle and a chaotic repeller. (**a**) A chaotic saddle from a periodically kicked harmonic oscillator. On a stroboscopic plane the position x_n and the velocity y_n of the oscillator evolve according to the map [773] $x_{n+1} = y_n$, $y_{n+1} = 1 - 3.2y_n^2 - 0.49x_n$. (**b**) A chaotic repeller of the quadratic map $z_{n+1} = z_n{}^2 + 0.2$ in the complex plane $z = x + iy$, which is in fact a Julia set [824]. The saddle in (**a**) appears as a fractal set of points, which is in fact the direct product of two Cantor-like sets, while the repeller in (**b**) is a complicated but nonetheless continuous curve in the plane

they can also start from a neighborhood of the stable manifold, a typically much larger set. If a chaotic repeller and saddle coexist,[1] transient chaos is primarily governed by the chaotic saddle.

Because a nonattracting chaotic set is invariant, trajectories starting from points on the set *never* leave the set and in fact exhibit chaotic motion for infinitely long time. However, because the Lebesgue measure of the set is zero, the probability that a randomly chosen point of the phase space is in the set is zero. What is *observable* is not the nonattracting set but a a *small neighborhood* of it. In particular, trajectories can originate from points in the vicinity of the set and can then stay in the neighborhood of the set for a long but finite amount of time, and they eventually leave the nonattracting chaotic set. These are the trajectories that generate transiently chaotic signals. The phenomenon of transient chaos thus illustrates that the existence of a set of Lebesgue measure zero can be observed via *finite-time properties*. As a consequence, we shall also see that the fractal features of a nonattracting chaotic set are different from those of a chaotic attractor.

A related point is that the natural measure, a special invariant distribution characterizing the dynamics on a nonattracting chaotic set, not only exists but can be obtained approximately in numerical or actual experiments. In particular, the distribution can be approximately specified on a small neighborhood of the set. The approximate natural measure can then be used to perform *ensemble averages* of physical quantities of interest, similar to the situation with chaotic attractors. Since the distribution is only approximate, any ensemble average will contain errors, but they can be controlled.

Transient chaotic dynamics can also be classified according to whether the process is dissipative or conservative. In a strictly dissipative system where the local phase-space volume contracts everywhere, the asymptotic states of the system are attractors that may be regular, but transient chaos provides a "platform" for approaching the attractors. In such a case the transient dynamics before the final state of the system is reached is chaotic. In *dissipative* systems, transient chaos appears in the form of *chaotic transients*. In conservative or Hamiltonian systems, the phase-space volume is constant under time evolution. As a result, there are no attractors, but some simple asymptotic states of the system can still be defined. Consider, for example, a particle-scattering experiment in which the underlying dynamics is Hamiltonian. Particles coming from far away approach the scattering region, and after a finite amount of time, they leave the region and escape to "infinity." There can, however, be qualitatively different exit routes to infinity. In this case, the different exit routes can be regarded as asymptotic states (but not attractors) of the system. The dynamics in the scattering region can, however, be regular or chaotic, where the latter, i.e., transient chaos in *Hamiltonian systems*, defines the phenomenon of *chaotic scattering*. Hamiltonian systems are invertible, so the nonattracting set underlying chaotic scattering is typically a saddle.

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[1] For instance, in the time-reversed dynamics of an invertible system possessing a chaotic attractor and a coexisting chaotic saddle.

It should be noted that *nonchaotic transients* may also exist in dynamical systems. An example is provided by trajectories that approach an attractor but are far away from any nonattracting chaotic set. These transients are typically short and do not exhibit chaotic features, although the actual asymptotic state may be chaotic. Thus, *transients to chaos* can be quite different from chaotic transients, since the latter, but not the former, are due to an underlying nonattracting chaotic set.

1.1.3 Types of Transient Chaos

According to the type of attractor(s) with which a nonattracting chaotic set coexists, we can distinguish two main types of transient chaos. The first type is for the case in which the coexisting attractor is simple, e.g., a periodic attractor. While the asymptotic behavior of the system is relatively simple, the transients are chaotic. Transient chaos arising in situations in which there is an attractor at infinity, and in open Hamiltonian systems in which attractors are replaced by different exit routes also exhibit this type of transient chaos.

The second type occurs when a nonattracting chaotic set coexists with a chaotic attractor. In this case, there are two distinct forms of chaotic behavior. A signal from the system typically exhibits one form of chaotic behavior, the one due to the nonattracting set, on time scale τ, and then switches over to another form of chaos asymptotically. A common situation is that the motion determined by the nonattracting set is more chaotic than that due to the chaotic attractor (for more detail see Fig. 1.16 and Chap. 3). Thus, focusing on the asymptotic properties will "miss" the dominant chaotic part of the full complex dynamics that contains important information about the underlying dynamical system.

1.2 Characterizing Transient Chaos

Having introduced the basic concepts of transient chaos in a qualitative manner, we now discuss its quantitative characterization. A natural question is whether there is actually chaos in the seemingly chaotic signals observed over finite time scales. There are different levels of characterization of increasing complexity, as follows:

1. Measurement of the lifetime distribution, the escape rate, and the average lifetime.
2. Construction of nonattracting chaotic sets in the phase space.
3. Construction of invariant measures on the chaotic set.
4. Determination of dynamical invariants such as the Lyapunov exponents and the fractal dimensions of the nonattracting chaotic set and its natural measure.

Following this hierarchy, one can find criteria to address the question of whether the system is indeed chaotic and if so, to calculate some measure of the strength of chaoticity. In the following we discuss these levels of characterization.

1.2.1 Escape Rate

In transient chaos, typical trajectories, i.e., trajectories initiated from random initial conditions, escape any neighborhood of the nonattracting chaotic set. A quantity measuring how quickly this occurs is the *escape rate* [824]. To define the escape rate, imagine distributing a large number N_0 of initial points according to some initial density ρ_0 in a phase-space region R that does not contain any attractor or asymptotic state of the system. The density ρ_0 is often chosen to be uniform, and the geometry of R can be chosen to be simple, e.g., a rectangle in a two-dimensional phase space. Many trajectories from the initial points may come close to the nonattracting chaotic set at some later time. We define a *restraining region* Γ as a bounded, compact region containing the nonattracting set. Once a point leaves the restraining region, it cannot return to it. After visiting a neighborhood of the set, almost all trajectories eventually leave Γ. Let $N(n)$ denote the number of trajectories remaining inside Γ after n steps, and choose N_0 to be sufficiently large that $N(n) \gg 1$. As n is increased, one observes in general an exponential decay in the number of trajectory points that are still in Γ (surviving points) [373, 596, 843]:

$$N(n) \sim \mathrm{e}^{-\kappa n} \quad \text{for} \quad n \gg 1, \tag{1.5}$$

where κ is called the escape rate.[2] A small value of κ implies weak "repulsion" of typical trajectories by the nonattracting chaotic set. The escape rate turns out to be *independent* of the distribution ρ_0 of the initial conditions, of its support R, and of the choice of the restraining region Γ. The escape rate κ is thus a property *solely* of the nonattracting chaotic set. However, the prefactor of the exponential form in (1.5), and the behavior of the system preceding the exponential decay do depend on details such as the choices of ρ_0, R, and Γ.

A practical issue concerns about the choice of the support R of the initial density. In a noninvertible system, R should overlap with the chaotic repeller, while in an invertible system it is sufficient to choose R so that it overlaps with the stable manifold of the chaotic saddle. In any case, if an exponential decay is found, its rate should be given by the escape rate κ. In practice, the initial density is often distributed on the restraining region, implying $R = \Gamma$.

In a realistic physical system, the exponential decay can be observed with high accuracy after a finite, often short, time n^*, i.e.,

$$N(n) = N\mathrm{e}^{-\kappa n} \quad \text{for} \quad n \geq n^*, \tag{1.6}$$

[2] There are situations in which the decay follows a power law for certain types of nonhyperbolic chaotic sets, which will be treated in Sect. 2.4 and Chap. 6. Such decays cannot be characterized by escape rates.

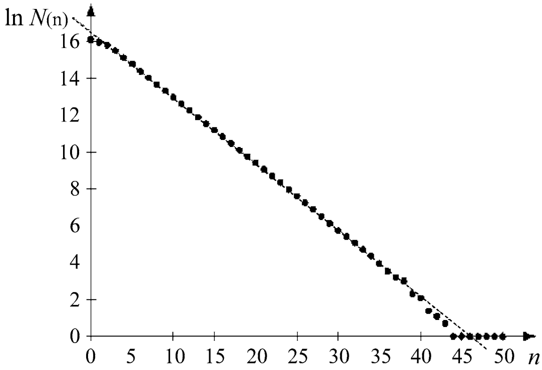

Fig. 1.4 Survival in the Hénon map $x_{n+1} = 1 - ax_n^2 + by_n$, $y_{n+1} = x_n$ for parameters $a = 2.0$ and $b = 0.3$. Number $N(n)$ of surviving trajectory points in the square defined by $\Gamma : |x_n|, |y_n| \leq 1.0$, obtained from $N_0 = 10^6$ initial points distributed uniformly in the same square ($R = \Gamma$). The fitted *dashed line* has slope approximately -0.36, giving $\kappa \approx 0.36$. The value of n^* is approximately 4. The survival probability $P(n)$ is approximately $N(n)/N_0$

where the value of n^* and the prefactor N may also depend on ρ_0, R, and Γ.[3] An example is shown in Fig. 1.4, where we see that the value of n^* is relatively small.

The definition of the escape rate indicates that the number of surviving points is decreased by a factor of $1/e$ after about $1/\kappa$ time steps. This implies that most trajectories do not live longer than $1/\kappa$ in the restraining region. It is thus reasonable to *estimate* the average lifetime τ of the chaotic transient as

$$\tau \approx \frac{1}{\kappa}. \tag{1.7}$$

Since the escape rate can be obtained by following the decay law over a finite time interval, cf. (1.5), transient chaos of short average lifetime may be difficult to identify. A condition for the practical observability of transient chaos is thus that κ be small.

In a more general context, for any initial distribution on R and choice of Γ, we can define the *probability* $P(n)$ of finding survival times larger than $n \geq 1$. The *survival probability* $P(n)$ is thus the probability of finding initial points that have not escaped Γ up to time n, which can be approximated by $N(n)/N_0$ for large N_0. In view of (1.6), the decay of $P(n)$ is exponential:

$$P(n) = g\mathrm{e}^{-\kappa n} \quad \text{for} \quad n \geq n^*. \tag{1.8}$$

[3] The prefactor N yields what the number of initial points would be if the decay were exponential from the very beginning. Therefore N is different from N_0.

A related probability is the *escape-time distribution*, $p(n)$, the probability that a particle escapes region Γ exactly in the nth iterate. This quantity can be estimated as $[N(n-1)-N(n)]/N_0$ and is therefore the "density" of the cumulative distribution $P(n)$. We have

$$P(n) = \sum_{n'=n+1}^{\infty} p(n'). \tag{1.9}$$

Being the "derivative" of an exponential function, the long-time behavior of $p(n)$ is also exponential and can be written in the form of (1.8) (with a different n^*, but the *same* escape rate).[4]

The average lifetime τ is *defined* as the average escape time, i.e.,

$$\tau \equiv \bar{n} = \sum_{n=1}^{\infty} np(n). \tag{1.10}$$

Since the distribution is not exponential for $n < n^*$, the exact average lifetime τ *does* depend on the choices of ρ_0, R, and Γ. Note that the estimate (1.7) does not reflect this property.[5] Since the average lifetime depends on many details, the escape rate κ is a *more appropriate characteristic* of the decay process than τ. The escape rate is a unique property of the underlying nonattracting chaotic set, in contrast to the average lifetime, which also contains information about, e.g., the initial distribution of particles. While the values of τ and $1/\kappa$ are typically different even for slow decays, their *scaling properties* in terms of, for example, parameter changes are usually the same.

There can be situations in which two (or more) nonattracting chaotic sets coexist with different escape rates κ_1 and κ_2. In such a case, the number of surviving trajectory points in a given restraining region Γ is the sum of two exponentials for large n:

$$N(n) \sim N_1 e^{-\kappa_1 n} + N_2 e^{-\kappa_2 n}, \tag{1.11}$$

and the prefactors N_i depend on the choices of ρ_0, R, and Γ.

It should be emphasized that the existence of a positive escape rate κ for transients does not at all imply their chaoticity. One should also measure, for example, the Lyapunov exponents on time scale $1/\kappa$ [714] and check whether at least one of the exponents is positive. A complication is that even simple nonattracting sets, for instance a *single, regular saddle point* (also called a hyperbolic point) are at least partially repelling. Trajectories deviate from them exponentially. Regular

[4] For continuous-time systems, (1.5)–(1.8) remain valid under the transform $n \to t$. The escape-time distribution becomes then a probability density, and the sum in (1.9) is replaced by an integral. The escape rate in the corresponding continuous-time system is κ/t_0, where t_0 denotes the internal characteristic time mentioned in the introduction to this chapter. Analogously, the average lifetime can be estimated as t_0/κ.

[5] Equation (1.7) is a rough estimate, since even in the ideal case of $n^* = 1$, when $p(n) = (\exp(\kappa)-1)\exp(-\kappa n)$, we obtain $\tau = (1-\exp(-\kappa))^{-1}$ from (1.10) [147], which is consistent with (1.7) for $\kappa \ll 1$ only.

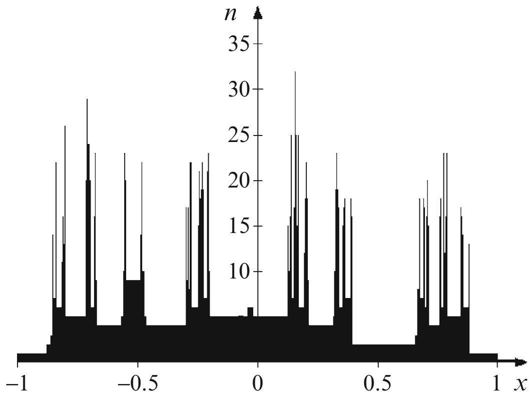

Fig. 1.5 Lifetime function: dependence of the lifetime n on the initial position x along the interval defined by $y = -1.5$ and $|x| \leq 1$ in the Hénon map at the parameters of Fig. 1.4. (For the corresponding phase-space patterns, see Figs. 1.7 and 1.9.) The fractal irregularity of this lifetime function is a sign of transient chaos

nonattracting sets are therefore characterized by a positive Lyapunov exponent, although the dynamics about them are not chaotic. The positivity of at least one Lyapunov exponent is thus not sufficient for the chaotic behavior of transients. This is why we accept the definition, used throughout the book, that transient chaos is the dynamics associated with nonattracting chaotic sets.

To determine whether the transients are truly chaotic, one therefore needs more information than the mere positivity of the Lyapunov exponent. Qualitatively, the visual appearance of the signal can be helpful: about chaotic nonattracting sets trajectories should be complicated. This is, nonetheless, only a hint. A property uniquely indicating the chaotic nature of the transients is the *irregular* dependence of lifetimes on initial conditions, as illustrated by Fig. 1.5. Suppose one starts trajectories along a smooth curve in the phase space that intersects a chaotic repeller or the stable manifold of a chaotic saddle. One then finds that for some points the lifetimes are large. In principle, points of infinitely large lifetimes belong to a *fractal* subset of initial conditions, since these must be points of the chaotic repeller or of the saddle's stable manifold. A fingerprint in a finite-accuracy numerical simulation is large lifetimes separated by small values in between.

1.2.2 Constructing Nonattracting Chaotic Sets

Repellers are straightforward to construct, since they are the attractors of the inverted dynamical systems. Noninvertibility is generally due to the existence of more than one inverted branch. When following the time-reversed dynamics, all possible inverses should be taken into account.

For an invertible dynamical system, the calculation of chaotic saddles is more delicate. While such a system can be inverted, the inverted dynamics still results in a chaotic saddle. This feature can in fact be viewed as an illustration of the robustness of the *hyperbolic structure* that is often seen for chaotic saddles. Roughly, a chaotic saddle is the set of intersections between the stable and the unstable manifolds, and in hyperbolic cases, the angles at the intersecting points are bounded away from zero. In what follows, we will describe an intuitive numerical procedure for calculating chaotic saddles, which serves to further illustrate their dynamical structures. More practical numerical methods will then be introduced.

1.2.2.1 Horseshoe Construction

The intuitive method is based on the observation that a chaotic saddle has typically embedded within itself a dense set of unstable periodic orbits, a property of any chaotic set. Imagine that we choose an unstable periodic orbit in an invertible two-dimensional map and plot its stable and unstable manifolds, which are the curves along which the orbit is attracting in the direct and in the inverted dynamics, respectively. If these curves cross each other once at a point (a homoclinic point), they must do so infinitely many times, since the images and the preimages of such an intersection are of the same type. All the homoclinic points form a *homoclinic orbit*. Since it belongs simultaneously to the stable and the unstable manifolds of the original periodic orbit, a homoclinic orbit approaches asymptotically, but can never reach, the periodic orbit. As a result, the stable and unstable manifolds exhibit a complex, intertwined structure, as shown schematically in Fig. 1.6. The *horseshoe* structure of the manifolds and the existence of homoclinic orbits have been known since the works of Smale [300, 721]. Thus, mathematically, chaotic saddles are closed, bounded, and invariant sets with dense orbits. They are the "soul" of chaotic dynamics [721]. Similar to the formation of homoclinic orbits, the stable (unstable) manifold of a periodic orbit can intersect with the unstable (stable) manifold of a *different* orbit, forming a *heteroclinic orbit*. The stable and the unstable manifolds of different periodic orbits of a chaotic saddle are usually close to each other in the phase space, and all the resulting homoclinic and heteroclinic orbits belong to the chaotic saddle.

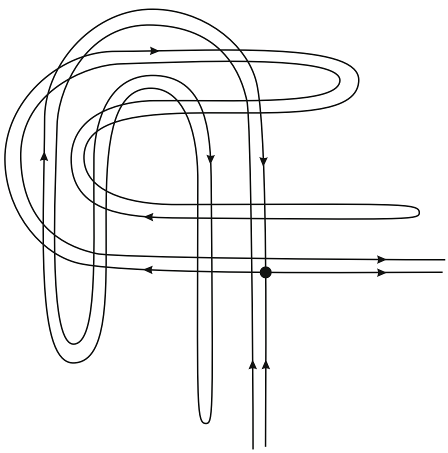

Fig. 1.6 Horseshoe structure: schematic illustration of horseshoes formed by the stable and the unstable manifolds of a fixed point (period-1 orbit) denoted by the *dot*. The set of intersection points (homoclinic points) between the manifolds belongs to the chaotic saddle

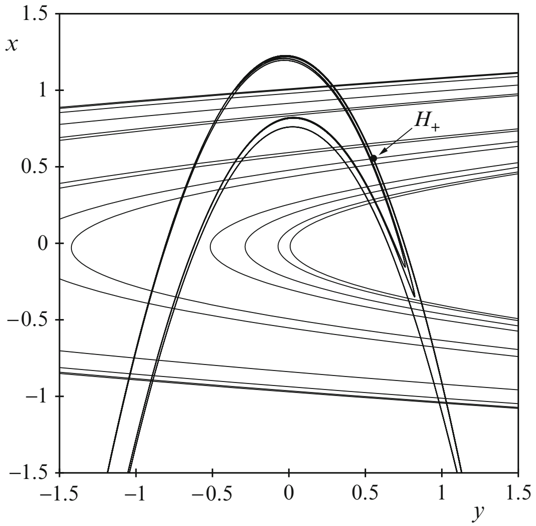

Fig. 1.7 A horseshoe construction: a few branches of the stable and the unstable manifolds of the fixed point H_+ of the Hénon map $x_{n+1} = 1 - 2.0x_n^2 + 0.3y_n$, $y_{n+1} = x_n$

The above discussion suggests the following procedure for numerically calculating a chaotic saddle. One first finds a simple hyperbolic orbit, such as a fixed point or a periodic orbit of low period, and then calculates its stable and unstable manifolds. In particular, the unstable (stable) manifold can be obtained by distributing a large number of initial points in a small neighborhood of the hyperbolic orbit and iterating them under the forward (inverted) dynamics. The set of intersecting points between the manifolds is part of the chaotic saddle. Since in practice, only a finite number of branches of the manifolds can be constructed, the intersections provide an approximate representation of the saddle. If the number of initial points used in the calculation is reasonably large, the fractal nature of the saddle and its stable and unstable manifolds can be revealed. An example is shown in Fig. 1.7. In general, the appearance of a fractal geometry along both the stable and the unstable manifolds and the existence of a horseshoe type of structure are indications that a chaotic saddle exists in the phase space of interest. Note that if the manifolds of the hyperbolic orbit chosen do not intersect each other, the orbit does not belong to a chaotic saddle. In this case, it is necessary to choose a different periodic orbit to start with.

1.2.2.2 Ensemble Method

The idea of this method, introduced by Kantz and Grassberger [380], is to follow an ensemble of trajectories and select the pieces that remain in the vicinity of the saddle. In particular, one first chooses a region R close to the suspected chaotic saddle but not containing any attractor, distributes uniformly a large number N_0 of points in R, and iterates these initial conditions under the forward dynamics. A criterion is needed for deciding when a trajectory is away from the saddle, which can simply be that the trajectory moves out of a restraining region Γ surrounding the saddle (regions R and Γ can be the same as the respective ones used for computing the escape rate). Another criterion can be [380] to calculate the effective Lyapunov exponents over a finite number of time steps and examine whether they are close to the corresponding exponents characterizing an attractor. In the case of a point attractor, it is simply the negativity of all local Lyapunov exponents that can be used as an indicator of the trajectory's having left the saddle. All trajectories leaving the saddle earlier than n_0 steps are discarded, and trajectories of lifetime longer than or equal to n_0 are kept. The choice of the value of n_0 can be somewhat arbitrary, but some large value should be chosen if the lifetime τ of the chaotic saddle is large. (Experience indicates that choosing n_0 a few multiples of $1/\kappa$ is proper.) One can then select *long-lived* trajectories in the neighborhood of the saddle to approximate it. For example, if the desirable number of trajectories whose lengths are not less than n_0 is M_0, the number N_0 of initial points should be of the order of $n_0 M_0 \exp(\kappa n_0)$, which can be a few orders of magnitude larger than M_0. To ensure that trajectories close to the saddle are selected, the long-lived trajectories need to be *truncated* at both the beginning and the end. For example, for a trajectory of length larger than n_0, one can discard the first n_1 and the last n_2 points so that the resulting trajectory is close to the saddle but not close to its stable and unstable manifolds, respectively, where n_1 and n_2 are each a fraction of n_0. A representative example is shown in Fig. 1.8.

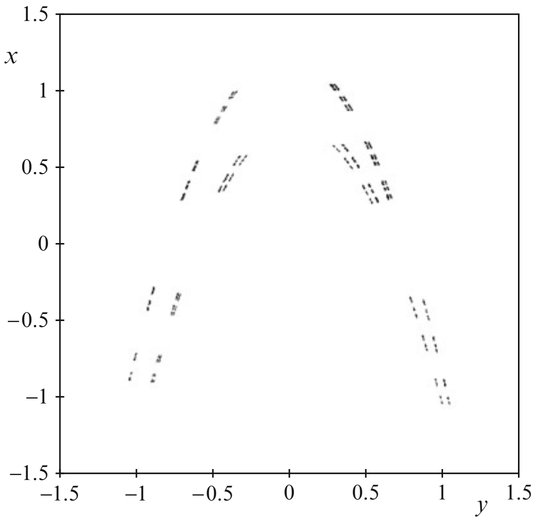

Fig. 1.8 Chaotic saddle in the Hénon map ($a = 2.0$, $b = 0.3$) obtained by the ensemble method, where $N_0 = 10^6$ initial points are distributed uniformly in the interval $R = (| y_0 |< 0.5,\ x_0 = 0)$. The restraining region is $\Gamma =| x_n |\leq 1.2$. The first 10 and the last 20 steps of long-lived trajectories are discarded ($n_0 = 30$). Observe that the pattern is practically the same as the one formed by the set of homoclinic points in Fig. 1.7. The direct product structure of two Cantor-like sets is a generic characteristic of chaotic saddles of two-dimensional maps