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Preface: Painlevé's Problem

Let K be a compact subset of the complex plane. Call K *removable for bounded analytic functions*, or more concisely *removable*, if for each open superset U of K in the complex plane, each function that is bounded and analytic on $U \setminus K$ extends across K to be analytic on the whole of U. In 1888 Paul Painlevé became the first to seriously investigate the nature of removable sets in his thesis [PAIN]. Because of this the removable subsets of the complex plane are often referred to as *Painlevé null sets* and the task of giving them a "geometric" characterization has come to be known as *Painlevé's Problem*. In addition to being an academic, Painlevé was also a politician and statesman who served as War Minister and Prime Minister of France at various times in his life. For more on this interesting and multifaceted individual see Section 6 of Chapter 5 of [PAJ2].

The notion of "geometric" here is unavoidably vague and intuitive. On the one hand, a necessary but not sufficient condition for such a characterization is that it should make no reference to analytic functions. On the other hand, a sufficient but not necessary condition for such a characterization is that it be couched in terms of the cardinality of K or the topological, metric, or rectifiability properties of K. At the very end of this book the following question will command our attention: Should a characterization involving totally arbitrary measures be counted as "geometric"?

The goal of this book is to present a complete proof of the recent affirmative resolution of a special case of Painlevé's Problem known as *Vitushkin's Conjecture*. This conjecture states that a compact set with finite linear Hausdorff measure is removable if and only if it intersects every rectifiable curve in a set of zero arclength measure. We note in passing that arclength measure here can be replaced by linear Hausdorff measure since the two have the same zero sets among subsets of rectifiable curves. More importantly, we note that the forward implication of Vitushkin's Conjecture is equivalent to an earlier conjecture about a still more special case of Painlevé's Problem known as *Denjoy's Conjecture*. This conjecture states that a compact subset of a rectifiable curve with positive arclength measure is non-removable. So to prove Vitushkin's Conjecture, we must also prove Denjoy's Conjecture.

To understand this book a prospective reader should have a firm grasp of the first 14 chapters of Walter Rudin's *Real and Complex Analysis*, 3$^{\text{rd}}$ Edition (hereafter referred to as [RUD]). Indeed, the author has somewhat eccentrically sought to make this book, when used in conjunction with [RUD], entirely self-contained. Thus any standard result of analysis which is needed but is not contained in [RUD] is proved in this book (e.g., Besicovitch's Covering Lemma), and conversely, any standard result of analysis which is needed and is contained in [RUD] is always given a citation from [RUD] (e.g., Lebesgue's Dominated Convergence Theorem). Another eccentricity of the book is a deliberate exclusion of figures but an equally deliberate inclusion of verbal descriptions precise enough to enable an attentive reader to reconstruct the excluded figures. To a great extent the author wrote this book to convince himself of the truth of Vitushkin's Conjecture "beyond a reasonable doubt" and so has elected to err on the side of too much detail rather than too little. Finally, the author believes his notation is fairly standard or obvious but has nevertheless spelled out the meaning of a number of symbols upon first use and appended a symbol glossary and list to the back of the book for the reader's convenience.

We now turn to detailing the contents of the book, chapter by chapter.

Chapter 1 introduces and then proves various standard elementary results about the notions of removability and analytic capacity. The analytic capacity of a compact subset K of the complex plane is a nonnegative number $\gamma(K)$ which can be thought of as a quantitative measure of removability/nonremovability since K is removable if and only if $\gamma(K) = 0$. This result does *not* solve Painlevé's Problem since $\gamma(K)$ is *not* a geometric quantity – its definition (see Section 1.2) involves suping over a space of bounded analytic functions!

Chapter 2 introduces the notions of s-dimensional Hausdorff measure $\mathcal{H}^s$ and Hausdorff dimension $\dim_{\mathcal{H}}$ – these are not dealt with in [RUD] – and then relates them to removability. It turns out that a result of Painlevé implies that a compact K is removable whenever $\dim_{\mathcal{H}}(K) < 1$ and a result of Frostman implies that a compact K is nonremovable whenever $\dim_{\mathcal{H}}(K) > 1$. So Painlevé's Problem is reduced to determining the removability of those compact K for which $\dim_{\mathcal{H}}(K) = 1$. At the end of this chapter a natural conjecture presents itself which would finish off Painlevé's Problem if true. It is couched in terms of $\mathcal{H}^1$ but is summarily slain by a counterexample!

Chapter 3 proves a special case of Garabedian duality needed for our proof of Denjoy's Conjecture. Analytic capacity, whose definition involves suping over a space of bounded analytic functions, is an L^∞ object. It has an L^2 analog and Garabedian duality asserts that these two capacities, one L^∞ and the other L^2, are related in a manner that makes it clear that they vanish for the same sets. The importance of Garabedian duality is that it thus allows us to use Hilbert space methods to study an L^∞ problem – it is frequently easier to estimate an L^2 norm than it is to estimate an L^∞ norm.

Chapter 4 introduces the notion of the Melnikov curvature of a measure and the notion of a measure with linear growth. Garabedian duality is then used to prove a result called Melnikov's Lower Capacity Estimate. Given a compact set supporting a nontrivial positive Borel measure with finite Melnikov curvature and linear growth, this estimate gives a positive lower bound on the analytic capacity of the set in terms of the Melnikov curvature, the linear growth bound, and the mass of the measure. Of course this quantitative result trivially implies a qualitative one: a compact set which supports a nontrivial positive Borel measure with finite Melnikov curvature and linear growth is nonremovable. A Fourier transform argument due to Mark Melnikov and Joan Verdera is then given that shows that Lipschitz graphs support many such measures. After some preliminaries dealing with arclength and arclength measure, these two results combine to give a nice proof of Denjoy's Conjecture. At the end of this chapter a natural conjecture presents itself which would finish off Painlevé's Problem if true. It is couched in terms of rectifiable curves but meets the same fate as the earlier conjecture, i.e., it is summarily slain by a counterexample!

Chapter 5 is a grab bag of the measure theory needed to carry us forward. Amazingly, up to this point in the book it has sufficed to just know that s-dimensional Hausdorff measure is an outer measure defined on all subsets of the complex plane! Not so for what follows where we must know that it is an honest-to-god measure on a σ-algebra of subsets containing the Borel sets. The chapter has more in it than one would expect. The reason is that measures in [RUD] are typically obtained via the Riesz Representation Theorem and, in consequence, always put finite mass on any compact set. This is a property that s-dimensional Hausdorff measure on the complex plane has only when $s = 2$. So we cannot simply rely on [RUD] here for our measure theory.

Chapter 6 has a proof of Vitushkin's Conjecture *modulo* two difficult results. The next two chapters, comprising roughly half the book, are taken up with proving these results.

Chapter 7 has a proof of the first difficult result, a $T(b)$ theorem due to Fedor Nazarov, Sergei Treil, and Alexander Volberg for measures that need not satisfy a doubling condition. The complexity of this proof precludes us from saying anything enlightening about it just now.

Chapter 8 has a proof of the second difficult result, a curvature theorem for arbitrary measures due to Guy David and Jean-Christophe Léger. The complexity of this proof precludes us from saying anything enlightening about it just now.

With the end of Chapter 8, the goal of this book, the presentation of a complete proof of Vitushkin's Conjecture, has been achieved. But Vitushkin's Conjecture, although a big part of Painlevé's Problem, is not all of it. With the affirmative resolution of Vitushkin's Conjecture, Painlevé's Problem has been reduced to determining the removability of those compact sets K for which $\dim_{\mathcal{H}}(K) = 1$ but $\mathcal{H}^1(K) = \infty$. A Postscript following Chapter 8 seeks to shed some light on these sets. This Postscript deals with two items: first, the extension of Vitushkin's Conjecture to compact sets that are σ-finite for $\mathcal{H}^1$, and second, a conjecture due to Melnikov which essentially says that the qualitative consequence of Melnikov's Lower Capacity Estimate mentioned a few paragraphs ago is reversible. Both of these matters are resolved affirmatively with the aid of a quite recent and deep theorem, which we state but do not prove, due to Xavier Tolsa.

In writing this book the author has found three useful sources on Hausdorff measure and dimension: [ROG], [FALC], and [MAT3]. These items have been listed in order of increasing depth. For the purposes of this book [FALC] proved to be ideal. The author was also helped by several excellent survey articles dealing with the status of Painlevé's Problem and the various subproblems it has spawned. These are, in chronological order: [MARSH], [VER], [MAT5], [MAT6], [DAV2], [DAV3], [TOL4], and [PAJ3]. The author is also indebted to two books that are of a much more comprehensive scope than this one but deal with Painlevé's Problem: [GAR2], from the pre-Melnikov-curvature era, and [PAJ2], from the post-Melnikov-curvature era. Finally, it should be noted that [MAT3], a very comprehensive and deep book on Hausdorff measure and rectifiability that appeared at the cusp between the two eras, has an excellent chapter devoted to the status of Painlevé's Problem at that time. These sources also have superb and complete bibliographies. The bibliography of this book, being restricted solely to those articles and books that the author found necessary to cite, is spare by comparison.

The author would like to express his gratitude to the University of Tennessee at Knoxville where, as a visitor during the 2002–2003 academic year, he was able to present some of the material that made its way into this book in a faculty seminar. At various times during the composition of Chapter 8, Jean-Christophe Léger kindly responded to email inquiries about fine points of the proof of the curvature theorem bearing his and David's name. His responses were prompt, gracious, and, most importantly, very helpful, thus earning the author's heartfelt thanks. Last but certainly not least, the author would like to thank his wife for many things, just one of which is her making possible a leave of absence from teaching duties in order to engage in the last push to the finish line with this book!

East Lansing, MI James J. Dudziak

Chapter 1
Removable Sets and Analytic Capacity

1.1 Removable Sets

For now and forevermore, let K be a compact subset of the complex plane $\mathbb{C}$. This will be restated for emphasis many times in what follows but just as often will be tacitly assumed and not mentioned. For the sake of those readers who skip prefaces, we repeat a definition: K is *removable for bounded analytic functions*, or more concisely *removable*, if for each open superset U of K in $\mathbb{C}$, each function that is bounded and analytic on $U \setminus K$ extends across K to be analytic on the whole of U. These analytic extensions must be bounded on the whole of U since they are continuous and so also bounded on K. Thus the definition may be equivalently restated as follows: K is *removable* if for each open superset U of K in $\mathbb{C}$, each element of $H^\infty(U \setminus K)$ extends to an element of $H^\infty(U)$. Of course, for any open set V of $\mathbb{C}$, $H^\infty(V)$ denotes the Banach algebra of all functions bounded and analytic on V. What can we say about a removable K?

First, a removable K must have no interior. For if there were a point z_0 in the interior of K, then the function $z \mapsto 1/(z - z_0)$ would be a function nonconstant, bounded, and analytic on $\mathbb{C} \setminus K$ which, by Liouville's Theorem [RUD, 10.23], would not extend analytically to all of $\mathbb{C}$.

An immediate consequence of this first observation is that the analytic extension of any element of $H^\infty(U \setminus K)$ to an element of $H^\infty(U)$ is unique and of the same supremum norm, i.e., $H^\infty(U \setminus K)$ and $H^\infty(U)$ are isometrically isometric. This explains the terminology: "removing" K from U has made no difference to $H^\infty(U)$.

Second, a removable K must have connected complement. For if $\mathbb{C} \setminus K$ had more than one component, then the function which is one on the unbounded component and zero on all the bounded components would be a function nonconstant, bounded, and analytic on $\mathbb{C} \setminus K$ which, by Liouville's Theorem [RUD, 10.23], would not extend analytically to all of $\mathbb{C}$.

Third, a removable K must be *totally disconnected*, i.e., a removable K can contain no nontrivial connected subset. To see this we suppose otherwise and deduce a contradiction. So let C be a nontrivial connected subset of a removable K. Replacing C with its closure, we may assume that C is closed. Let U_∞ be the component of $\mathbb{C}^* \setminus C$ containing ∞. Of course, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane (also known as the Riemann sphere). By our second observation, U_∞ contains all

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of $\mathbb{C}^* \setminus K$. It is an exercise in point-set topology, which we leave to the reader, to show that $\mathbb{C}^* \setminus U_\infty$ is a nontrivial connected subset of $\mathbb{C}^*$. Fix $z_0 \in \mathbb{C}^* \setminus U_\infty$ and set $g(z) = 1/(z - z_0)$. Then $g(U_\infty)$ is a proper subregion of $\mathbb{C}$ for which $\mathbb{C}^* \setminus g(U_\infty) = g(\mathbb{C}^* \setminus U_\infty)$ is connected. Hence, by the many equivalences to simple connectivity and the Riemann Mapping Theorem [RUD, 13.11 and 14.8], there exists a one-to-one analytic mapping f of $g(U_\infty)$ onto the open unit disc at the origin. It follows that $f \circ g$ is a bounded analytic function on $\mathbb{C} \setminus K$. By the removability of K and Liouville's Theorem [RUD, 10.23], $f \circ g$ is constant on $\mathbb{C} \setminus K$, and so too on $U_\infty \setminus \{\infty\}$ by the uniqueness property of analytic functions [RUD, 10.18]. Clearly then, f does *not* map $g(U_\infty)$ onto the open unit disc at the origin. With this contradiction we are done.

As an aside, we note that the third observation subsumes the first two since any totally disconnected K must have no interior and a connected complement. While the first part of this assertion is trivial, the reader may find verifying the second part one of those exercises in "mere" point-set topology that is a wee bit frustrating!

Turning to concrete examples, any single point is removable since bounded analytic functions can be analytically continued across isolated singularities. This assertion is simple [RUD, 10.20], but it is instructive to reproduce its proof. So suppose f is bounded and analytic on int $B(z_0; r) \setminus \{z_0\}$, a punctured open disc about z_0. Using the boundedness of f, one easily sees that $(z - z_0)^2 f(z)$ extended to be zero at z_0 is differentiable there with derivative zero. Thus $(z - z_0)^2 f(z)$ is analytic on int $B(z_0; r)$ and so has a power series expansion there [RUD, 10.16]:

$$(z - z_0)^2 f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots.$$

Since this extension and its derivative vanish at z_0, $a_0 = 0$ trivially and $a_1 = 0$ by [RUD, 10.6]. Thus we may divide out a factor of $(z - z_0)^2$ in the above to conclude that f, extended to be a_2 at z_0, has the following power series expansion on int $B(z_0; r)$:

$$f(z) = a_2 + a_3(z - z_0) + a_4(z - z_0)^2 + a_5(z - z_0)^3 + \cdots.$$

But then, by [RUD, 10.6] again, our extended f is differentiable at z_0 with derivative a_3 there, and so f has been extended analytically across z_0.

Of course it follows that any finite set of points is removable. A little more nontrivially, any countable compact set K is removable. One proof involves a transfinite process which starting at $K_0 = K$ and any $f_0 \in H^\infty(U \setminus K)$, generates transfinite sequences $\{K_\alpha\}$ and $\{f_\alpha\}$ by stripping an isolated point off K_β after extending f_β across the isolated point when one is at a successor ordinal $\alpha = \beta + 1$, while intersecting all previous sets K_β, $\beta < \alpha$, and patching together all previous extensions f_β, $\beta < \alpha$, when one is at a limit ordinal α. Note that we always have $f_\alpha \in H^\infty(U \setminus K_\alpha)$. This process must break down at some ordinal α since K is not a proper class. A little thought shows that this can only happen at a successor ordinal $\alpha = \beta + 1$ and only then when K_β has no isolated points. Since any nonempty countable compact subset of the plane must have an isolated point by the Baire

Category Theorem [RUD, 5.7], it follows that we must have some $K_\beta = \emptyset$. But then f has been extended to $f_\beta \in H^\infty(U)$ and so K is removable. We note in passing that this ordinal β must be less than ω_1, the first uncountable ordinal, since K is countable. Those readers uncomfortable with transfinite induction/recursion will be relieved to know that the removability of countable compact sets is a simple corollary of Proposition 1.7 below whose proof does not use these tools from set theory.

This is as far as one can go with topology and cardinality alone since the next proposition shows that some uncountable, totally disconnected, compact subsets of the complex plane are removable while others are not. The simplest such sets are the linear Cantor sets. The standard middle-third's Cantor set is removable, but not all linear Cantor sets are. The next proposition shows this and indeed settles the question of removability for all linear compact sets. It also shows that a nonanalytic characterization of removability, if one is to be had, must involve metric notions which measure the "size" of the set. Later we shall see that metric "size" alone does not always suffice and that in certain situations the "rectifiability structure" of the set is decisive.

Proposition 1.1 *Let K be a linear compact subset of $\mathbb{C}$. Then K is removable if and only if the linear Lebesgue measure of K is zero.*

Proof Without loss of generality, let the line in which K lies be the real line $\mathbb{R}$.

Suppose that the linear Lebesgue measure of K is zero. Let U be any open superset of K in $\mathbb{C}$ and consider any $f \in H^\infty(U \setminus K)$. A very useful general fact, which we shall employ many times in this book, is that given any open superset U in $\mathbb{C}$ of any compact subset K of $\mathbb{C}$, there always exists a cycle Γ in $U \setminus K$ with winding number 1 about every point of K and 0 about every point of $\mathbb{C} \setminus U$ [RUD, proof of 13.5]. Letting Γ be such a cycle for the U and K now under consideration, set V equal to the union of the collection of components of $\mathbb{C} \setminus \Gamma$ which intersect K. Then V is an open superset of K. Since the winding number of Γ about every point of V is 1 [RUD, 10.10], V is a subset of U. Define a function g on V by

$$g(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Clearly g is analytic on V, so to show that f extends analytically across K it suffices to show that $f = g$ on $V \setminus K$.

Fixing $z \in V \setminus K$, let $\varepsilon > 0$ be smaller than the distance of K to $(\mathbb{C} \setminus V) \cup \{z\}$. Then, since K is compact and has linear Lebesgue measure zero, K can be covered by a finite number of open intervals of the real axis whose lengths sum to less than ε. By amalgamating intervals which intersect one another and then discarding any intervals which miss K, we may assume that these intervals are pairwise disjoint and intersect K. Upon each such interval describes the counterclockwise circle having that interval as a diameter. Let Γ_ε be the cycle consisting of the circles so produced. Clearly the length of Γ_ε is less than $\pi\varepsilon$. Applying Cauchy's Integral Theorem [RUD, 10.35] to the cycle $\Gamma - \Gamma_\varepsilon$ in $U \setminus K$, one has that

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma-\Gamma_\varepsilon}\frac{f(\zeta)}{\zeta - z}\,d\zeta = g(z) - \frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

The absolute value of the second integral above is at most

$$\frac{1}{2\pi}\cdot\frac{\|f\|_\infty}{\operatorname{dist}(z, K) - \varepsilon}\cdot\pi\varepsilon$$

which converges to zero as ε does. Thus $f(z) = g(z)$, and so K is removable.

Now suppose that the linear Lebesgue measure of K, denoted l, is positive. Define a function h on $\mathbb{C}\setminus K$ by

$$h(z) = \frac{1}{2}\int_K \frac{1}{z-t}\,dt.$$

(As an aside, the factor of a half in the definition of h is not necessary for this proof; however, we will be reusing h in the proof of Proposition 1.19 below and there it will be necessary!) Clearly h is analytic on $\mathbb{C}\setminus K$. Since as $z\to\infty$, $h(z)\to 0$ yet $zh(z)\to l/2\neq 0$, h is nonconstant. For $z = x+iy$ with $y\neq 0$,

$$|\operatorname{Im} h(z)| \le \frac{1}{2}\int_K \frac{|y|}{(x-t)^2+y^2}\,dt < \frac{1}{2}\int_{-\infty}^{+\infty}\frac{|y|}{u^2+y^2}\,du = \frac{\pi}{2}.$$

In consequence, $\exp(ih)$ is a nonconstant element of $H^\infty(\mathbb{C}\setminus K)$. By Liouville's Theorem [RUD, 10.23] such an element cannot be extended analytically to all of $\mathbb{C}$. Hence K is nonremovable. □

Our next goal is to state and prove a number of equivalences for removability. To do this however, we first need a few words about *analyticity at* ∞ and a lemma.

Consider a function f analytic and bounded on $\{z\in\mathbb{C} : |z| > R\}$, a punctured neighborhood of ∞. Note that $g(w) = f(1/w)$ is bounded and analytic on $\{w\in\mathbb{C} : 0 < |w| < 1/R\}$, a punctured neighborhood of 0. Since we have shown single points to be removable, g extends analytically to $\{w\in\mathbb{C} : |w| < 1/R\}$. By [RUD, 10.16],

$$g(w) = a_0 + a_1 w + a_2 w^2 + a_3 w^3 + \cdots$$

with the convergence being absolute and uniform for all w with $|w|\le 1/R'$ for any $R' > R$. Clearly then,

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots$$

with the convergence being absolute and uniform for all z with $|z|\ge R'$ for any $R' > R$. Set

$$f(\infty) = g(0) = a_0 = \lim_{z\to\infty} f(z)$$

and

$$f'(\infty) = g'(0) = a_1 = \lim_{z\to\infty} z\{f(z) - f(\infty)\}.$$

We describe this situation by saying that f extends to be "analytic" at ∞ with value $f(\infty)$ and "derivative" $f'(\infty)$ there. Another typical example of our use of language would be to say that f is "analytic at ∞" since it can be written as a "power series about ∞".

The last paragraph applies to any element of $H^\infty(\mathbb{C} \setminus K)$ and extends it to a function bounded and "analytic" on $\mathbb{C}^* \setminus K$. These extentions form a Banach algebra which we denote $H^\infty(\mathbb{C}^* \setminus K)$. Of course, it is isometrically isomorphic to $H^\infty(\mathbb{C} \setminus K)$. We will frequently be cavalier about the distinction between these two algebras.

Let us now back up a bit and consider any function f analytic on $\mathbb{C} \setminus K$ and bounded on a deleted neighborhood of ∞. Then for Γ a cycle in $\mathbb{C} \setminus K$ with winding number 1 about every point of K and C a counterclockwise circular path centered at the origin encircling K and Γ, one has

$$f'(\infty) = \frac{1}{2\pi i}\int_C f(\zeta)\, d\zeta = \frac{1}{2\pi i}\int_\Gamma f(\zeta)\, d\zeta.$$

The first equality follows by plugging the power series about ∞ above into the integral and then integrating term-by-term while the second inequality follows from Cauchy's Integral Theorem [RUD, 10.35]. If, in addition, z_0 is any point of K, then

$$f(\infty) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta.$$

To see this, apply the integral representations for $f'(\infty)$ above to the function $g(z) = f(z)/(z - z_0)$, noting that $g(\infty) = \lim_{z\to\infty} f(z)/(z - z_0) = f(\infty)\cdot 0 = 0$ and so $g'(\infty) = \lim_{z\to\infty} zf(z)/(z - z_0) = f(\infty)\cdot 1 = f(\infty)$.

Given U a punctured neighborhood of ∞, let $\{f_n\}$ be a sequence of functions analytic and uniformly bounded on U that converges uniformly on compact subsets of U to a function f. Of course f is then analytic and bounded on U [RUD, 10.28] and so all functions here are analytic at ∞. Our integral representations now make it clear that $f_n(\infty) \to f(\infty)$ and $f_n'(\infty) \to f'(\infty)$.

Lemma 1.2 *Let U be an open superset of a compact subset K of $\mathbb{C}$. Then any analytic function f on $U \setminus K$ can be written uniquely as $g + h$ where g is analytic on U and h is analytic on $\mathbb{C}^* \setminus K$ with $h(\infty) = 0$. Moreover, if f is bounded on $U \setminus K$, then g and h are bounded on U and $\mathbb{C}^* \setminus K$ respectively.*

Proof Given any $z \in U$, choose a cycle $\Gamma_g(z)$ in $U \setminus (K \cup \{z\})$ with winding number 1 about every point of $K \cup \{z\}$ and 0 about every point of $\mathbb{C} \setminus U$. Define

$$g(z) = \frac{1}{2\pi i}\int_{\Gamma_g(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Cauchy's Integral Theorem [RUD, 10.35] implies that this integral is independent of the particular $\Gamma_g(z)$ chosen. Thus g is well defined and, given any $\Gamma_g(z)$, we have

$$g(w) = \frac{1}{2\pi i} \int_{\Gamma_g(z)} \frac{f(\zeta)}{\zeta - w}\, d\zeta$$

for all w in the component of $\mathbb{C} \setminus \Gamma_g(z)$ containing z since $\Gamma_g(z)$ will serve as $\Gamma_g(w)$ for such w [RUD, 10.10]. We may thus differentiate under the integral in the last displayed equation to conclude that g is analytic on U.

Given any $z \in \mathbb{C} \setminus K$, choose a cycle $\Gamma_h(z)$ in $(U \setminus \{z\}) \setminus K$ with winding number one about every point of K and zero about every point of $\mathbb{C} \setminus (U \setminus \{z\})$. Define

$$h(z) = -\frac{1}{2\pi i} \int_{\Gamma_h(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Cauchy's Integral Theorem [RUD, 10.35] implies that this integral is independent of the particular $\Gamma_h(z)$ chosen. Thus h is well defined and an argument similar to that just given for g shows that h is analytic on $\mathbb{C} \setminus K$. Let Γ be a cycle in $U \setminus K$ with winding number one about every point of K and zero about every point of $\mathbb{C} \setminus U$. Then

$$h(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all z in the unbounded component of $\mathbb{C} \setminus \Gamma$ since Γ serves as $\Gamma_h(z)$ for such z [RUD, 10.10]. The last displayed equation makes it clear that h is bounded in a deleted neighborhood of ∞ and so analytic at ∞. Moreover, letting $z \to \infty$ in this equation, we see that $h(\infty) = 0$.

Cauchy's Integral Theorem [RUD, 10.35] applied to the cycle $\Gamma_g(z) - \Gamma_h(z)$ implies that $f = g + h$ on $U \setminus K$, so existence has been shown. Consider another representation $f = \tilde{g} + \tilde{h}$ as desired. Liouville's Theorem [RUD, 10.23] applied to $g - \tilde{g} = \tilde{h} - h$ gives us uniqueness.

To finish, suppose f is bounded on $U \setminus K$. Choose an open set V with compact closure such that $K \subseteq V \subseteq \operatorname{cl} V \subseteq U$. Since h is bounded on $\mathbb{C} \setminus V$, $g = f - h$ is bounded on $U \setminus V$. But then, since g is bounded on $\operatorname{cl} V$, g is bounded on U. Finally, $h = f - g$ is bounded on $U \setminus K$ and so also on $\mathbb{C} \setminus K$. □

Now to all but one of the promised equivalences for removability.

Proposition 1.3 *For a compact subset K of $\mathbb{C}$, the following are equivalent*:

(a) *K is removable.*
(b) *There exists an open superset U of K in $\mathbb{C}$ such that each function that is bounded and analytic on $U \setminus K$ extends across K to be analytic on the whole of U.*

(c) *The only elements of* $H^\infty(\mathbb{C}^* \setminus K)$ *are the constant functions.*
(d) *For every* $f \in H^\infty(\mathbb{C}^* \setminus K)$, *one has* $f'(\infty) = 0$.

Proof (a) $\Rightarrow$ (b): Trivial.

(b) $\Rightarrow$ (c): Given $f \in H^\infty(\mathbb{C}^* \setminus K)$, clearly one also has $f \in H^\infty(U \setminus K)$ and so f extends analytically across K. Since the extension is still bounded, Liouville's Theorem [RUD, 10.23] now implies that f is constant.

(c) $\Rightarrow$ (d): Trivial.

(d) $\Rightarrow$ (c): We suppose that there exists a nonconstant $g \in H^\infty(\mathbb{C}^* \setminus K)$ and construct a function $f \in H^\infty(\mathbb{C}^* \setminus K)$ with $f'(\infty) \neq 0$. Since g is nonconstant, there exists a point $z_0 \in \mathbb{C} \setminus K$ such that $g(z_0) \neq g(\infty)$. Set $f(z) = \{g(z) - g(z_0)\}/(z - z_0)$. Then $f \in H^\infty(\mathbb{C}^* \setminus K)$ and $f(\infty) = \lim_{z\to\infty} f(z) = 0$. In consequence, $f'(\infty) = \lim_{z\to\infty} zf(z) = g(\infty) - g(z_0) \neq 0$.

(c) $\Rightarrow$ (a): Given any open superset U of K in $\mathbb{C}$ and any $f \in H^\infty(U \setminus K)$, get g and h as in the previous lemma. Then h is constant, and since $h(\infty) = 0$, this constant must be 0. Thus g is an analytic extension of f to U. □

Our last equivalence for removability is a surprising apparent strengthening of the way we have defined the concept. It has been stated separately because it requires a finicky bit of topology which is the content of ...

Lemma 1.4 *Let* X *be a totally disconnected, compact, Hausdorff space. Suppose* C_1 *and* C_2 *are disjoint closed subsets of* X. *Then* X *can be written as the disjoint union of two closed subsets* X_1 *and* X_2 *such that* $C_1 \subseteq X_1$ *and* $C_2 \subseteq X_2$.

Proof Recall that a *clopen* subset of a topological set is one that is both closed and open.

First Claim. Given any point $x \in X$, let E_x denote the intersection of all clopen subsets of X containing x. Then E_x is connected. (Note: The total disconnectedness of X is not used here!)

Indeed, supposing E_x is the disjoint union of two closed subsets E_1 and E_2 with x contained in E_1 say, it suffices to show that E_2 is empty. Since X is compact and Hausdorff, there exist two disjoint open sets U_1 and U_2 such that $E_1 \subseteq U_1$ and $E_2 \subseteq U_2$. Since $E_x \subseteq U_1 \cup U_2$ and E_x is an intersection of clopen sets, finitely many of these sets must have intersection $\widetilde{E}$ contained in $U_1 \cup U_2$ by compactness. But then $\widetilde{E}$, being a finite intersection of clopen sets, is itself clopen. Hence $\widetilde{E} \cap U_1 = \widetilde{E} \setminus U_2$ is also clopen and it clearly contains x. Thus by the definition of E_x, $E_x \subseteq \widetilde{E} \setminus U_2$ and so $E_2 \cap U_2 = \emptyset$. Since $E_2 \subseteq U_2$ also, E_2 must be empty.

Second Claim. For any distinct points $x, y \in X$, there exists a clopen subset E of X such that $x \in E$ and $y \notin E$. (Note: The total disconnectedness of X is used here!)

This claim can be rephrased as $E_x = \{x\}$ where E_x is as in the first claim. It thus follows immediately from the first claim since the total disconnectedness of X just means that the only nonempty connected subsets of X are the singletons.

Third Claim. For any closed subset C_1 of X and any point $y \in X \setminus C_1$, there exists a clopen subset E of X such that $C_1 \subseteq E$ and $y \notin E$.

The proof of this claim is a simple compactness argument using the second claim.

Fourth Claim. For C_1 and C_2 disjoint closed subsets of X, there exists a clopen subset E of X such that $C_1 \subseteq E$ and $C_2 \cap E = \emptyset$.

The proof of this claim is a simple compactness argument using the third claim.

The proposition now follows by setting $X_1 = E$ and $X_2 = X \setminus E$ where E is as in the fourth claim. □

Proposition 1.5 *A compact subset K of $\mathbb{C}$ is removable if and only if for each open subset U of $\mathbb{C}$, each function that is bounded and analytic on $U \setminus K$ extends across K to be analytic on the whole of U.*

Note that the backward implication is trivial. The forward implication is also trivial if K is totally contained in or totally disjoint from U. Thus the meat of the proposition is when K is removable and "half" in/out of U.

Proof Suppose K is removable, U is open, and $f \in H^\infty(U \setminus K)$.

Given $\varepsilon > 0$, set $C_1(\varepsilon) = \{z \in K : \mathrm{dist}(z, \mathbb{C} \setminus U) \geq \varepsilon\}$ and $C_2 = K \setminus U$. By Lemma 1.4, K can be written as the disjoint union of two closed subsets $K_1(\varepsilon)$ and $K_2(\varepsilon)$ such that $C_1(\varepsilon) \subseteq K_1(\varepsilon)$ and $C_2 \subseteq K_2(\varepsilon)$. The equivalence of (a) with (d) in Proposition 1.3 makes it clear that any compact subset of a removable set is removable. Thus $K_1(\varepsilon)$ is removable. Then, since $f \in H^\infty(\{U \setminus K_2(\varepsilon)\} \setminus K_1(\varepsilon))$ and $U \setminus K_2(\varepsilon)$ is an open superset of $K_1(\varepsilon)$, f extends to a function f_ε that is analytic on $U \setminus K_2(\varepsilon)$.

Given any $z \in U$, note that $f_\varepsilon(z)$ is defined whenever $0 < \varepsilon < \mathrm{dist}(z, \mathbb{C} \setminus U)$. Since K is removable, it has no interior. Because of this, all the values of $f_\varepsilon(z)$ for these various values of ε are equal since they are uniquely determined as the limit of $f(w)$ as $w \in U \setminus K$ approaches z. Thus we may properly define an extension f_0 of f to U by setting $f_0(z) = f_\varepsilon(z)$ for any ε such that $0 < \varepsilon < \mathrm{dist}(z, \mathbb{C} \setminus U)$. Clearly f_0 is analytic on all of U and so we are done. □

The next result establishes the existence of a *nonremovable kernel* so-to-speak for compact sets with no interior.

Lemma 1.6 *Suppose that K is a compact subset of $\mathbb{C}$ with no interior. Then there exists a compact subset K^* of K with the following property: for each compact subset J of K, every element of $H^\infty(\mathbb{C}^* \setminus K)$ extends to an element of $H^\infty(\mathbb{C}^* \setminus J)$ if and only if $K^* \subseteq J$.*

Proof Call a subset J of K *good* if it is compact and every element of $H^\infty(\mathbb{C}^* \setminus K)$ extends to an element of $H^\infty(\mathbb{C}^* \setminus J)$. Let K^* be the intersection of all good subsets of K. Note that we have just made the forward implication of the equivalence we wish to prove true by definition. Also note that a compact subset of K which contains a good subset is clearly good. Thus to prove the backward implication it suffices to show that K^* itself is good.

So let $f \in H^\infty(\mathbb{C}^* \setminus K)$ and $z \notin \mathbb{C}^* \setminus K^*$. Then $z \notin J$ for at least one and possibly many good subsets J of K. There exist extensions $f_J \in H^\infty(\mathbb{C}^* \setminus J)$ of f for these J. Now take note of our assumption that K has no interior. Because of this, all the values of $f_J(z)$ for these various subsets J are equal since they are uniquely determined as the limit of $f(w)$ as $w \in \mathbb{C}^* \setminus K$ approaches z. Thus we may properly define an extension f_* of f to $\mathbb{C}^* \setminus K^*$ by setting $f_*(z) = f_J(z)$ for any good subset J of K that does not contain z. Clearly $f_* \in H^\infty(\mathbb{C}^* \setminus K^*)$ and so we are done. □

The final result of this section follows. Since single points are removable it has as a corollary that every countable compact subset of $\mathbb{C}$ is removable. It will also come in useful in the Postscript when we consider whether Vitushkin's Conjecture extends to compact subsets of $\mathbb{C}$ with infinite, but σ-finite, linear Hausdorff measure.

Proposition 1.7 *Let $\{K_n\}$ be a sequence of removable compact subsets of $\mathbb{C}$ whose union K is also compact. Then K is removable.*

Proof Each K_n, being removable, has no interior in $\mathbb{C}$. An argument by contradiction using the Baire Category Theorem [RUD, 5.7] now shows that K also has no interior in $\mathbb{C}$. Thus the last lemma applies and it suffices to show that K^*, the nonremovable kernel of K, is empty.

So we suppose that K^* is nonempty and get a contradiction. Since K^* is the countable union of $K_n \cap K^*$, some $K_n \cap K^*$ must have nonempty interior in K^* by the Baire Category Theorem [RUD, 5.7]. Thus there exists an open subset U of $\mathbb{C}$ such that $U \cap K^* \neq \emptyset$ and $U \cap K^* \subseteq K_n \cap K^*$.

Consider now any $f \in H^\infty(\mathbb{C}^* \setminus K)$. Via the last lemma extend it to an element of $H^\infty(\mathbb{C}^* \setminus K^*)$ which we will also denote by f. Note that $K_n \cap K^*$, being a compact subset of the removable set K_n, is itself removable. Thus by Proposition 1.5, f restricted to $U \setminus K^* = U \setminus \{K_n \cap K^*\}$ has an analytic extension g to U ... which is also bounded since f is and K^* has no interior. Let h denote the function on $\mathbb{C}^* \setminus \{K^* \setminus U\} = \{\mathbb{C}^* \setminus K^*\} \cup U$ which is equal to f on $\mathbb{C}^* \setminus K^*$ and g on U. This function is well defined since $f = g$ on $\{\mathbb{C}^* \setminus K^*\} \cap U = U \setminus K^*$. Clearly h is a bounded analytic extension of f to $\mathbb{C}^* \setminus \{K^* \setminus U\}$.

The last paragraph has shown that every element of $H^\infty(\mathbb{C}^* \setminus K)$ extends to an element of $H^\infty(\mathbb{C}^* \setminus \{K^* \setminus U\})$. Thus by the last lemma we must have $K^* \subseteq K^* \setminus U$, i.e., $U \cap K^* = \emptyset$. This contradiction finishes the proof. □

1.2 Analytic Capacity

For K a compact subset of $\mathbb{C}$, the number

$$\gamma(K) = \sup\{|f'(\infty)| : f \in H^\infty(\mathbb{C}^* \setminus K) \text{ with } \|f\|_\infty \leq 1\}$$

is called the *analytic capacity* of K. Thus (d) of Proposition 1.3 could just as well have been phrased as "$\gamma(K) = 0$" and Painlevé's Problem formulated as the task of giving a geometric characterization of the compact sets K for which $\gamma(K) = 0$. Indeed, for the rest of the book we shall treat the two phrases "$\gamma(K) = 0$" and "K is removable" as synonymous. The notion of analytic capacity was introduced in 1947 by Lars Ahlfors in [AHL] where he proved the equivalence of removability with analytic capacity zero [(a) ⇔ (d) of Proposition 1.3 above] and also introduced the Ahlfors function (see Proposition 1.14 below). Beyond this, analytic capacity has turned out to be of great importance for rational approximation theory (see Chapter VIII of [GAM1], [VIT2], and/or [ZALC]). Unfortunately it is difficult to work with, a fact to which this book is an indirect testimonial. The last decade has seen progress in understanding it. This section is devoted to an exposition of the classical elementary properties and estimates of analytic capacity.

The proofs of the first three propositions below are simple and left to the reader.

Proposition 1.8 *Analytic capacity is monotone, i.e., $\gamma(K_1) \leq \gamma(K_2)$ whenever $K_1 \subseteq K_2$.*

Proposition 1.9 *For α and β complex numbers, $\gamma(\alpha K + \beta) = |\alpha|\gamma(K)$.*

Proposition 1.10 *The analytic capacity of K depends only on the unbounded component of $\mathbb{C}^* \setminus K$. Thus, letting $\widehat{K}$ denote the union of K with all the bounded components of $\mathbb{C}^* \setminus K$,*

$$\gamma(K) = \gamma(\partial K) = \gamma(\widehat{K}) = \gamma(\partial \widehat{K}).$$

The slightly different characterization of capacity provided by the next proposition will be used without mention in what follows.

Proposition 1.11 *Suppose $g \in H^\infty(\mathbb{C}^* \setminus K)$ satisfies $\|g\|_\infty \leq 1$. Then there exists an $f \in H^\infty(\mathbb{C}^* \setminus K)$ with $\|f\|_\infty \leq 1$, $f(\infty) = 0$, and $|f'(\infty)| \geq |g'(\infty)|$. Consequently,*

$$\gamma(K) = \sup\{|f'(\infty)| : f \in H^\infty(\mathbb{C}^* \setminus K) \text{ with } \|f\|_\infty \leq 1 \text{ and } f(\infty) = 0\}.$$

Proof On the one hand, if $|g(\infty)| = 1$, then g is constant on the unbounded component of $\mathbb{C}^* \setminus K$ by the Maximum Modulus Principle [RUD, 12.1 modified to take into account regions containing ∞]. Clearly $f \equiv 0$ works in this case.

On the other hand, if $|g(\infty)| < 1$, then the function

$$f(z) = \frac{g(z) - g(\infty)}{1 - \overline{g(\infty)}g(z)}$$

is in $H^\infty(\mathbb{C}^* \setminus K)$ with $\|f\|_\infty \leq 1$ and $f(\infty) = 0$ [RUD, 12.4]. Moreover,

$$|f'(\infty)| = \lim_{z \to \infty} |zf(z)| = \frac{|g'(\infty)|}{1 - |g(\infty)|^2} \geq |g'(\infty)|,$$

so we are done. □

Proposition 1.12 *Suppose K is a nontrivial continuum in $\mathbb{C}$. Let f denote the one-to-one analytic mapping of the unbounded component of $\mathbb{C}^* \setminus K$ onto the open unit disc at the origin for which $f(\infty) = 0$ and $f'(\infty) > 0$. Then $\gamma(K) = f'(\infty)$.*

Existence here is a consequence of the Riemann Mapping Theorem [RUD, 14.8] which we are assuming to have been modified in the obvious way to encompass simply connected regions in $\mathbb{C}^*$. Recall that a *continuum* is a connected compact set.

Proof Extending f to be identically 0 on the bounded components of $\mathbb{C}^* \setminus K$, we clearly have $f \in H^\infty(\mathbb{C}^* \setminus K)$ with $\|f\|_\infty = 1$. Thus $f'(\infty) = |f'(\infty)| \le \gamma(K)$.

For the reverse inequality, consider any $g \in H^\infty(\mathbb{C}^* \setminus K)$ with $\|g\|_\infty \le 1$ and $g(\infty) = 0$. It suffices to show that $|g'(\infty)| \le |f'(\infty)|$. Note that the function $g \circ f^{-1}$ is an analytic mapping of the open unit disc at the origin into itself which fixes the origin. Thus by Schwarz's Lemma [RUD, 12.2], $|g(f^{-1}(w))| \le |w|$ whenever $|w| < 1$, so $|g(z)| \le |f(z)|$ whenever $|z|$ is big enough. To finish, multiply this inequality by $|z|$, let $z \to \infty$, and then take the supremum over all the gs. □

Corollary 1.13 *The analytic capacity of a closed ball is its radius and the analytic capacity of a closed line segment is a quarter of its length.*

Proof Considering closed balls first, by Proposition 1.9 we need only consider the special case of K equal to the closed unit ball at the origin. Clearly the function $f(z) = z^{-1}$ is a one-to-one analytic mapping of $\mathbb{C}^* \setminus K$ onto the open unit disc at the origin for which $f(\infty) = \lim_{z\to\infty} f(z) = 0$ and $f'(\infty) = \lim_{z\to\infty} zf(z) = 1$. Thus f is the function of Proposition 1.12 for this K and so $\gamma(K) = f'(\infty) = 1$.

Turning to closed line segments, by Proposition 1.9 we need to only consider the special case of $K = [-2, 2]$. Set $g(w) = w + w^{-1}$ for $w \in \mathbb{C} \setminus \{0\}$. An application of the quadratic formula shows that for any $z \in \mathbb{C}$, $z = g(w)$ for, and only for, the values $w_\pm = \{z \pm \sqrt{z^2 - 4}\}/2$. Note that these values are reciprocals of one another. Thus either both $|w_+|$ and $|w_-|$ are equal to 1 or exactly one of $|w_+|$ and $|w_-|$ is strictly smaller than 1. In the first case, $w_\pm = e^{\pm i\theta}$ for some real θ and so $z = g(e^{\pm i\theta}) = 2\cos\theta \in [-2, 2] = K$. Hence for $z \in \mathbb{C} \setminus K$ we must be in the second case. But then clearly $z = g(w)$ for exactly one w satisfying $|w| < 1$. We conclude that g is a one-to-one analytic mapping of the punctured unit disc at the origin onto $\mathbb{C} \setminus K$. By [RUD, 10.33], g restricted to the punctured unit disc at the origin has a one-to-one analytic inverse f that maps $\mathbb{C} \setminus K$ onto the punctured unit disc at the origin. By our discussion of behavior at ∞ just after Proposition 1.1, f extends to be analytic at ∞ with $f(\infty) = \lim_{z\to\infty} f(z) = \lim_{w\to 0} w = 0$ and $f'(\infty) = \lim_{z\to\infty} zf(z) = \lim_{w\to 0} g(w)\, w = 1$. Thus f is the function of Proposition 1.12 for this K and so $\gamma(K) = f'(\infty) = 1$. □

Proposition 1.14 *Let K be a compact subset of $\mathbb{C}$. Then there is a unique function f analytic and bounded by one on the unbounded component of $\mathbb{C}^* \setminus K$ such that $f(\infty) = 0$ and $f'(\infty) = \gamma(K)$.*

The unique function whose existence is guaranteed by this proposition is called the *Ahlfors function* of K. Note that if K is a nontrivial continuum, then the Ahlfors function of K is just the Riemann map of Proposition 1.12. We will frequently consider the Ahlfors function of K to be an element of $H^\infty(\mathbb{C}^* \setminus K)$ by canonically extending it to be zero on any bounded components $\mathbb{C}^* \setminus K$ may have. The very clever and elegant proof of uniqueness given below is taken from [FISH].

Proof We may choose a sequence of functions $\{f_n\}$ analytic and bounded by one on the unbounded component of $\mathbb{C}^* \setminus K$ such that each $f_n(\infty) = 0$ and $f_n'(\infty) \to \gamma(K)$. Since these functions form a normal family [RUD, 14.6], by extracting a subsequence we may assume that $\{f_n\}$ converges uniformly on compact subsets of the unbounded component of $\mathbb{C} \setminus K$ to a function f analytic and bounded by one on this same unbounded component. Then $f_n(\infty) \to f(\infty)$ and $f_n'(\infty) \to f'(\infty)$. Hence $f(\infty) = 0$ and $f'(\infty) = \gamma(K)$. This settles existence.

To establish uniqueness, suppose both f and g are analytic and bounded by one on the unbounded component of $\mathbb{C}^* \setminus K$ with values at ∞ equal to 0 and derivatives at ∞ equal to $\gamma(K)$. Setting $h = (f + g)/2$ and $k = (f - g)/2$, we have both functions analytic and bounded by one on the unbounded component of $\mathbb{C}^* \setminus K$ with $h(\infty) = 0$, $h'(\infty) = \gamma(K)$, $k(\infty) = 0$, and $k'(\infty) = 0$. Since $f = h + k$ and $g = h - k$, it suffices to show that $k = 0$.

Noting that $|h|^2 + |k|^2 \pm 2\,\mathrm{Re}\, h\overline{k} = |h \pm k|^2 \leq 1$, we must have $|h|^2 + |k|^2 \leq 1$. Thus

$$|h| + \frac{1}{2}|k|^2 \leq |h| + \frac{1}{2}(1 - |h|^2) \leq |h| + \frac{1}{2}(1 + |h|)(1 - |h|) \leq |h| + (1 - |h|) = 1.$$

If k is nonzero, then

$$\frac{1}{2}k^2 = \frac{a_n}{z^n} + \frac{a_{n+1}}{z^{n+1}} + \cdots$$

where $a_n \neq 0$. Since $k(\infty) = 0$, $n \geq 2$. Choose $\varepsilon > 0$ sufficiently small so that $\varepsilon|a_n||z|^{n-1} \leq 1$ on a neighborhood U of K. Set $\tilde{f} = h + \varepsilon\overline{a_n}z^{n-1}k^2/2$ and note that $\tilde{f}$ is analytic on the unbounded component of $\mathbb{C}^* \setminus K$. Then $|\tilde{f}| \leq |h| + |k|^2/2 \leq 1$ on $U \setminus K$ and so on this same unbounded component by the Maximum Modulus Principle [RUD, 12.1 modified to take into account regions containing ∞]. Hence $|\tilde{f}'(\infty)| \leq \gamma(K)$. However,

$$\tilde{f}'(\infty) = h'(\infty) + \varepsilon|a_n|^2 > \gamma(K).$$

Because of this contradiction, we must have $k = 0$. □

Scholium 1.15 *An inspection of the proof of Proposition 1.11 shows that one actually has the stronger conclusion* $|f'(\infty)| > |g'(\infty)|$ *when* $g(\infty) \neq 0$ *and* $g'(\infty) \neq 0$. *In consequence, when* $\gamma(K) > 0$, *one may drop the requirement that* $f(\infty) = 0$ *from the definition of the Ahlfors function and still retain its uniqueness. When* $\gamma(K) = 0$, *this however fails (consider the constant functions with modulus less than 1).*

Proposition 1.16 *Let* $\{K_n\}$ *be a decreasing sequence of compact subsets of* $\mathbb{C}$ *with intersection* K. *Then* $\gamma(K_n) \to \gamma(K)$.

Proof By Proposition 1.8, the sequence $\{\gamma(K_n)\}$ is decreasing and bounded below by $\gamma(K)$. Thus

$$\gamma(K) \le \lim_{n\to\infty} \gamma(K_n) = \liminf_{n\to\infty} \gamma(K_n).$$

Let f_n denote the Ahlfors function of K_n. Since these functions form a normal family on each $\mathbb{C} \setminus K_n$, we may apply [RUD, 14.6] and a diagonalization argument to conclude that a subsequence of $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{C} \setminus K$ to a function f analytic and bounded by one on $\mathbb{C} \setminus K$. Clearly then

$$\liminf_{n\to\infty} \gamma(K_n) = \liminf_{n\to\infty} f_n'(\infty) \le f'(\infty) = |f'(\infty)| \le \gamma(K).$$

Hence $\lim_{n\to\infty} \gamma(K_n) = \gamma(K)$. □

Scholium 1.17 *With a little more work, one can exploit the uniqueness of the Ahlfors function to show that in the situation above the Ahlfors functions of the sets K_n converge uniformly on compact subsets of $\mathbb{C}^* \setminus K$ to the Ahlfors function of K.*

We finish this section with a number of classical estimates of the analytic capacity of a set in terms of its diameter, length (when the set is linear), and area. Recall that the *diameter* of a subset E of $\mathbb{C}$ is

$$|E| = \sup\{|z - w| : z, w \in E\}.$$

Proposition 1.18 *For K a compact subset of $\mathbb{C}$,*

$$\gamma(K) \le |K|.$$

If K is also connected, and thus a continuum, then

$$\gamma(K) \ge \frac{|K|}{4}.$$

Proof Clearly any subset of the plane can be contained in a closed ball whose radius is the diameter of the set. Thus the first estimate follows from Proposition 1.8 and Corollary 1.13.

With regard to the second estimate, we may assume that K is nontrivial. Let f be the Riemann map of Proposition 1.12. Given $z_1 \in K$, set $g(w) = \gamma(K)/(f^{-1}(w) - z_1)$. Then g is a one-to-one analytic map on the open unit disc at the origin such that

$$g(0) = \lim_{w\to 0} g(w) = \lim_{w\to 0} \frac{\gamma(K)}{f^{-1}(w) - z_1} = \lim_{z\to\infty} \frac{\gamma(K)}{z - z_1} = 0$$

and

$$g'(0) = \lim_{w\to 0} \frac{g(w) - g(0)}{w} = \lim_{w\to 0} \frac{\gamma(K)}{w(f^{-1}(w) - z_1)} = \lim_{z\to\infty} \frac{\gamma(K)}{f(z)(z - z_1)}$$
$$= \frac{\gamma(K)}{f'(\infty) - f(\infty)\cdot z_1} = 1.$$

Hence, by the Koebe One-Quarter Theorem [RUD, 14.14], the range of g contains the open disc of radius one-quarter centered at the origin. For $z_2 \in K \setminus \{z_1\}$, $\gamma(K)/(z_2 - z_1)$ is not in the range of g. Thus $\gamma(K)/|z_2 - z_1| \geq 1/4$, i.e., $\gamma(K) \geq |z_2 - z_1|/4$. Since $z_1, z_2 \in K$ with $z_1 \neq z_2$ are otherwise arbitrary, $\gamma(K) \geq |K|/4$. □

Jung's Theorem states that any subset of the plane can be contained in a closed ball whose radius is $1/\sqrt{3}$ times the diameter of the set. Many proofs of this can be found in the delightful book [YB]. Thus the first estimate of the above proposition is not sharp and can be improved to $\gamma(K) \leq |K|/\sqrt{3}$. While the factor of $1/\sqrt{3}$ is known to be sharp in Jung's Theorem (consider an equilateral triangle), it is not sharp in the capacity estimate here. See the two paragraphs following the proof of Theorem 2.6 for a demonstration of this and an identification of the sharpest constant possible. With regard to the second estimate of the above proposition, consideration of a line segment and use of Corollary 1.13 show the factor of 1/4 to be sharp.

Given a linear subset E of $\mathbb{C}$, we denote the "length", i.e., *linear Lebesgue measure*, of E by $\mathcal{L}^1(E)$ (see Section 2.1 for our official definition of this).

Proposition 1.19 *Let K be a linear compact subset of $\mathbb{C}$. Then*

$$\frac{\mathcal{L}^1(K)}{4} \leq \gamma(K) \leq \frac{\mathcal{L}^1(K)}{\pi}.$$

Proof Without loss of generality, let the line in which K lies be the real line $\mathbb{R}$.

The function h in the proof of Proposition 1.1 is analytic off K with $h(\infty) = 0$ and $h'(\infty) = \mathcal{L}^1(K)/2$. While h is not bounded off K, its imaginary part is bounded by $\pi/2$ off K. So consider the one-to-one analytic map g of the horizontal strip $\{|\text{Im}\, w| < \pi/2\}$ onto the open unit disc at the origin given by

$$g(w) = \frac{e^w - 1}{e^w + 1}.$$

The function $f = g \circ h$ is then an element of $H^\infty(\mathbb{C} \setminus K)$ with norm at most one such that

$$f(\infty) = \lim_{z\to\infty} g(h(z)) = g(0) = 0$$

and

$$f'(\infty) = \lim_{z\to\infty} zg(h(z)) = \lim_{z\to\infty} zh(z) \cdot \frac{e^{h(z)} - 1}{h(z)} \cdot \frac{1}{e^{h(z)} + 1} = h'(\infty) \cdot 1 \cdot \frac{1}{2}$$
$$= \frac{\mathcal{L}^1(K)}{4}.$$

Hence $\gamma(K) \geq f'(\infty) = \mathcal{L}^1(K)/4$.

Now consider any $f \in H^\infty(\mathbb{C}^* \setminus K)$ with $\|f\|_\infty \le 1$. Given $\varepsilon > 0$, by the definition of linear Lebesgue measure and the compactness of K there exists a finite collection of pairwise disjoint, open intervals covering K whose lengths sum to less than $\mathcal{L}^1(K) + \varepsilon/2$. Upon each such interval describe a rectangle having that interval as a bisector. Let Γ_ε be the cycle consisting of the counterclockwise boundary paths of all the rectangles so produced. Clearly, if we make the thickness of our rectangles small enough, then the length of Γ_ε can be made less than $2\mathcal{L}^1(K) + 2\varepsilon$. Then

$$|f'(\infty)| = \left| \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} f(\zeta)\, d\zeta \right| \le \frac{2\mathcal{L}^1(K) + 2\varepsilon}{2\pi}.$$

Suping over all our fs and then letting $\varepsilon \downarrow 0$, we get $\gamma(K) \le \mathcal{L}^1(K)/\pi$. □

A result of Christian Pommerenke (see [POM], or Section 6 of Chapter I of [GAR2]) states that the analytic capacity of any linear compact set is exactly equal to a quarter of its length. We are content with the weaker estimates above since they suffice for removability and most other considerations while Pommerenke's proof would cost us too much effort.

Given a subset E of $\mathbb{C}$, we denote the "area," i.e., *planar Lebesgue measure*, of E by $\mathcal{L}^2(E)$ (see Section 5.3 for our official definition of this).

Lemma 1.20 *Let K be a compact subset of $\mathbb{C}$. Then for each $z \in \mathbb{C} \setminus K$,*

$$\left| \iint_K \frac{1}{\zeta - z}\, d\mathcal{L}^2(\zeta) \right| \le \sqrt{\pi \mathcal{L}^2(K)}.$$

Proof Fix $z \in \mathbb{C} \setminus K$. By translating K and then rotating the resulting set by a unimodular constant, we may assume that $z = 0$ and that the integral of the lemma is nonnegative. Thus

$$\left| \iint_K \frac{1}{\zeta - z}\, d\mathcal{L}^2(\zeta) \right| = \left| \iint_K \frac{1}{\zeta}\, d\mathcal{L}^2(\zeta) \right| = \mathrm{Re} \iint_K \frac{1}{\zeta} d\mathcal{L}^2(\zeta) = \iint_K \mathrm{Re} \frac{1}{\zeta} d\mathcal{L}^2(\zeta)$$

and so it suffices to show that

$$\iint_K \mathrm{Re}\, \frac{1}{\zeta}\, d\mathcal{L}^2(\zeta) \le \sqrt{\pi \mathcal{L}^2(K)}.$$

Without loss of generality, $\mathcal{L}^2(K) > 0$. Choose $a > 0$ such that $\pi a^2 = \mathcal{L}^2(K)$. Set $B = B(a; a)$. Then $\mathcal{L}^2(K \setminus B) = \mathcal{L}^2(B \setminus K)$. Writing $\zeta = re^{i\theta}$, we see that $\zeta \in B \Leftrightarrow r \le 2a \cos\theta \Leftrightarrow \mathrm{Re}(1/\zeta) = (\cos\theta)/r \ge 1/2a$. Thus

$$\iint_{K \setminus B} \mathrm{Re}\, \frac{1}{\zeta}\, d\mathcal{L}^2(\zeta) \le \frac{\mathcal{L}^2(K \setminus B)}{2a} = \frac{\mathcal{L}^2(B \setminus K)}{2a} \le \iint_{B \setminus K} \mathrm{Re}\, \frac{1}{\zeta}\, d\mathcal{L}^2(\zeta)$$

and so, adding the integral of $\mathrm{Re}(1/\zeta)$ over $K \cap B$ to both sides of this inequality, we get

$$\iint_K \mathrm{Re}\,\frac{1}{\zeta}\,d\mathcal{L}^2(\zeta) \le \iint_B \mathrm{Re}\,\frac{1}{\zeta}\,d\mathcal{L}^2(\zeta).$$

Finally, using polar coordinates we have

$$\iint_B \mathrm{Re}\,\frac{1}{\zeta}\,d\mathcal{L}^2(\zeta) = \int_{-\pi/2}^{+\pi/2}\int_0^{2a\cos\theta}\frac{\cos\theta}{r}\,rdrd\theta = \pi a = \sqrt{\pi\mathcal{L}^2(K)}.$$

□

Proposition 1.21 *For K a compact subset of $\mathbb{C}$,*

$$\gamma(K) \ge \sqrt{\frac{\mathcal{L}^2(K)}{\pi}}.$$

Proof Define a function f on $\mathbb{C}\setminus K$ by

$$f(z) = \iint_K \frac{1}{\zeta - z}\,d\mathcal{L}^2(\zeta).$$

Clearly f is analytic on $\mathbb{C}\setminus K$. The lemma implies that f is bounded on $\mathbb{C}\setminus K$ with modulus less than or equal to $\sqrt{\pi\mathcal{L}^2(K)}$ there. Lastly, $f(\infty) = 0$ and so $f'(\infty) = \lim_{z\to\infty} zf(z) = -\mathcal{L}^2(K)$. Hence

$$\mathcal{L}^2(K) = |f'(\infty)| \le \gamma(K)\|f\|_\infty \le \gamma(K)\sqrt{\pi\mathcal{L}^2(K)}$$

which leads to the desired inequality. □

Consideration of a closed ball and use of Corollary 1.13 show this estimate to be sharp.

If one wishes to show $\gamma(K) > 0$, i.e., K nonremovable, one must construct a nonconstant bounded analytic function on the complement of K. Propositions 1.1, 1.19, and 1.21 exhibit the most common technique for doing this: one considers the *Cauchy transform*

$$z \in \mathbb{C} \mapsto \hat{\mu}(z) = \int_{\mathbb{C}} \frac{1}{\zeta - z}\,d\mu(\zeta)$$

of an appropriately chosen nontrivial finite Borel measure μ supported on K. While not well defined at every point of $\mathbb{C}$, the Cauchy transform is always defined and analytic off the support of μ and so on the complement of K. It is nonconstant since its derivative at infinity is $-\mu(K) \ne 0$. The catch is that the Cauchy transform need not be bounded on the complement of K! The need to ensure boundedness, or to somehow get around unboundedness, accounts for the phrase "appropriately chosen" four sentences ago.

Another technique for showing nonremovability is by means of the Riemann maps introduced in Proposition 1.12. The lower estimate on capacity in Proposition 1.18 is an example of this. The author hazards to state that Cauchy transforms and Riemann maps are ultimately the only means known to mortals for showing nonremovability!

Chapter 2
Removable Sets and Hausdorff Measure

2.1 Hausdorff Measure and Dimension

At a fuzzy intuitive level, removable sets have small "size" and nonremovable sets big "size." A precise notion of "size" applicable to arbitrary subsets of $\mathbb{C}$ and appropriate to our problem is given by Hausdorff measure (and Hausdorff dimension). So in this section we will simply introduce Hausdorff measure as a gauge of the smallness of a set and as a necessary preliminary for another such gauge, Hausdorff dimension. Surprisingly, the assertions 2.1 through 2.4 below are enough to get us through to the end of Chapter 4. It is only after, in Section 5.1, that we shall need to take up the fact that Hausdorff measure is indeed a positive measure defined on a σ-algebra containing the Borel subsets of $\mathbb{C}$!

Given an arbitrary subset E of $\mathbb{C}$ and $\delta > 0$, a δ-*cover* of E is simply a countable collection of subsets $\{U_n\}$ of $\mathbb{C}$ such that $E \subseteq \bigcup_n U_n$ and $0 < |U_n| < \delta$ for each n. For any $s \geq 0$, define

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_n |U_n|^s : \{U_n\} \text{ is a } \delta\text{-cover of } E\right\}.$$

Clearly $\mathcal{H}^s_\delta(E)$ increases as δ decreases and so converges to a limit in $[0,\infty]$ as $\delta \downarrow 0$. This limit is called the s-*dimensional Hausdorff measure* (or s-*dimensional Hausdorff–Besicovitch measure*) of E and denoted $\mathcal{H}^s(E)$. Thus

$$\mathcal{H}^s(E) = \lim_{\delta\downarrow 0}\mathcal{H}^s_\delta(E) = \sup_{\delta>0}\mathcal{H}^s_\delta(E).$$

Since the diameters of a set, its convex hull, its closure, and its closed convex hull are the same, $\mathcal{H}^s(E)$ may be computed by restricting ones attention to δ-covers of E by convex, closed, or closed convex sets. Similarly, since any nonempty set U is contained in the open set $\{z : \text{dist}(z,U) < \varepsilon\}$ whose diameter is $|U| + 2\varepsilon$ and since $\{z : \text{dist}(z,U) < \varepsilon\}$ is convex whenever U is convex, $\mathcal{H}^s(E)$ may be computed by restricting ones attention to δ-covers of E by open or open convex sets. Lastly, when

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E is compact, $\mathcal{H}^s(E)$ may be computed by restricting ones attention to δ-covers of E by a finite number of open or open convex sets.

The following results are fairly immediate from the definition of Hausdorff measure.

Proposition 2.1 *For any $s \geq 0$, $\mathcal{H}^s$ is a set function defined on all subsets of $\mathbb{C}$ and taking values in $[0, \infty]$ such that*

$$\mathcal{H}^s(\emptyset) = 0,$$

$$\mathcal{H}^s(E) \leq \mathcal{H}^s(F) \textit{ whenever } E \subseteq F \subseteq \mathbb{C},$$

and

$$\mathcal{H}^s\left(\bigcup_n E_n\right) \leq \sum_n \mathcal{H}^s(E_n) \textit{ whenever } \{E_n\} \textit{ is a countable collection of subsets of } \mathbb{C}.$$

Proposition 2.2 *Let E be a subset of $\mathbb{C}$ and let f be a mapping of E into $\mathbb{C}$ such that there exists a constant $c \geq 0$ for which $|f(z) - f(w)| \leq c|z - w|$ whenever $z, w \in E$. Then $\mathcal{H}^s(f(E)) \leq c^s \mathcal{H}^s(E)$.*

Corollary 2.3 *For α and β complex numbers and E a subset of $\mathbb{C}$, $\mathcal{H}^s(\alpha E + \beta) = |\alpha|^s \mathcal{H}^s(E)$.*

If $t > s \geq 0$, $0 < \delta < 1$, and $\{U_n\}$ is a δ-cover of E, then $\sum_n |U_n|^t \leq \sum_n |U_n|^s$. By infing over all δ-covers and then letting $\delta \downarrow 0$, we see that $\mathcal{H}^t(E) \leq \mathcal{H}^s(E)$. Thus $\mathcal{H}^s(E)$ decreases as s increases. But more can be said in this situation: $\sum_n |U_n|^t \leq \delta^{t-s} \sum_n |U_n|^s$, and so by infing over all δ-covers,

$$\mathcal{H}^t_\delta(E) \leq \delta^{t-s} \mathcal{H}^s_\delta(E).$$

Letting $\delta \downarrow 0$, we see that if $\mathcal{H}^s(E) < \infty$, then $\mathcal{H}^t(E) = 0$ for all $t > s$, and that if $\mathcal{H}^t(E) > 0$, then $\mathcal{H}^s(E) = \infty$ for all $s < t$. In consequence, there exists a unique nonnegative number called the *Hausdorff dimension* (or *Hausdorff–Besicovitch dimension*) of E and denoted $\dim_{\mathcal{H}}(E)$ such that

$$\mathcal{H}^s(E) = \begin{cases} \infty \text{ when } s < \dim_{\mathcal{H}}(E) \\ 0 \text{ when } s > \dim_{\mathcal{H}}(E). \end{cases}$$

What happens at $s = \dim_{\mathcal{H}}(E)$? In this case, $\mathcal{H}^s(E)$ can be 0, ∞, or anything in between. When $\mathcal{H}^s(E) = \infty$, it can even be the case that E is non-σ-finite for $\mathcal{H}^s$, i.e., E cannot be expressed as a countable union of sets of finite $\mathcal{H}^s$-measure. An example of this is the Joyce–Mörters set at the end of Chapter 4 (see Proposition 4.34).