Applied and Numerical Harmonic Analysis

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Editors

Wavelets and Multiscale Analysis

Theory and Applications

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time–frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time–frequency and*
Numerical partial differential equations	*time-scale analysis*
	Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries, Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of “function”. Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor’s set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, e.g., by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener’s Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time–frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated

Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

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John J. Benedetto
Series Editor

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Preface

This book is a collection of papers written or co-authored by participants in the "Twenty Years of Wavelets" conference held at DePaul in May, 2009. The conference attracted almost a hundred participants from five different countries over three days. There were 13 plenary lectures and 16 contributed talks. The conference was envisioned to celebrate the twentieth anniversary of a one-day conference on applied and computational harmonic analysis held at DePaul in May 1989 and was organized by one of the editors, Jonathan Cohen. The 1989 DePaul conference was scheduled to supplement a two-day special session of a regional AMS meeting on computational harmonic analysis and approximation theory. Combined together, the three days of talks may have been the first conference in the United States which featured the subject of wavelets. Although the focus of that conference was computational harmonic analysis, wavelet theory, which was in its infancy at the time, played a central role in the three days of talks.

After two decades of extensive research activities, it was appropriate to pause and have a look back at what had been accomplished and ponder what lay ahead. This was exactly the aim of the 2009 conference. The conference had two subthemes, past and future. Some of the plenary speakers, including I. Daubechies and J. Kovačević, gave expository and survey talks covering the history and major accomplishments in the field and some speakers focused on new directions for wavelets, especially in the area of geometric harmonic analysis.

All conference speakers were invited to submit papers related to the themes of the conference. This was interpreted broadly to include articles in applied and computational harmonic analysis. Though many of the articles are based on conference presentations, this book was not envisioned as a proceedings and some of the articles represent material not presented at the conference. All the papers in this book were anonymously refereed.

The book is divided into three parts. The first is devoted to the mathematical theory of wavelets and features several papers on the geometry of sets and the development of wavelet bases. The second part deals with the underlying geometry of large data sets and how tools of harmonic analysis prove useful in extracting information from them. The third part is devoted to exploring some ways that harmonic analysis, and wavelet theory in particular, have been applied to study real-world problems.

The articles in this book are mostly written by mathematicians and are intended for mathematicians and engineers with some background in Fourier analysis and the theory of wavelets. The book should be accessible to workers in the field and to graduate students with an interest in working in related areas.

We gratefully acknowledge the National Science Foundation, NSF Grant DMS-0852170, and DePaul University, both of whom provided generous financial support for the conference. We would also like to express our appreciation to the authors for submitting their work and meeting the deadlines, to the referees for their help and cooperation, and to Thomas Grasso, the science editor for Birkhäuser, for his support throughout this project.

Finally, we note with sadness that one of the authors in this volume, Daryl Geller, passed away in late January. He was a very fine mathematician who will be missed by his colleagues and friends.

DePaul University, Chicago, Illinois *Jonathan Cohen*
February, 2011 *Ahmed I. Zayed*

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Chapter 1
An Introduction to Wavelets and Multiscale Analysis: Theory and Applications

Ahmed I. Zayed

Abstract The purpose of this introductory chapter is to give the reader an overview of the contents of the monograph and show how the chapters are tied together. We give a brief description of each chapter but with emphasis on how the chapters fit in the monograph and the general subject area. The descriptions are not meant to replace, but to supplement, the chapters' abstracts, which summarize the chapters' main results.

1.1 Introduction

This monograph is broadly based on talks given at an international conference on wavelets that was held at DePaul University, May 15–17, 2009, and was partially supported by grants from the National Science Foundation and DePaul University Research Council. The title of the conference was "Twenty Years of Wavelets" to commemorate the twentieth anniversary of another conference on wavelets that was held at the same university in 1989.

Since the introduction of wavelets in the early 1980s, the subject has undergone tremendous developments both on the theoretical and applied fronts. Myriads of research and survey papers and monographs have been published on the subject covering different areas of applications, such as signal and image processing, denoising, and data compression. This monograph not only contributes to this burgeoning subject, but also sheds light on new directions for wavelets, especially in the area of *geometric harmonic analysis* which aims at developing harmonic analysis techniques to deal with large data sets in high dimensions. This approach, which was pioneered by R. Coifman and his team at Yale University, has shown very promising results.

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R. Coifman and some of his collaborators and former students have contributed chapters to this book. These chapters may provide researchers and graduate students an opportunity to learn about recent developments in the area of multiscale harmonic analysis.

The purpose of this chapter is to give the reader an overview of the contents of the monograph and show how the chapters are connected together. We assume that the reader is familiar with the rudiments of wavelets and multiresolution analyses. The book is divided into three parts: the first part is the Mathematical Theory of Wavelets, the second is the Multiscale Analysis of Large Data Sets, and the third is Applications of Wavelets. The chapters in the first part are grouped together by common themes: wavelet sets and wavelet construction. Chapters 2 and 3 deal with wavelet sets, while Chaps. 4–6 discuss the construction of wavelets in different settings, such as wavelets on a torus, crystallographic composite dilation wavelets, and vector-valued (multichannel) wavelets. The second part comprises chapters on multiscale analysis of large data sets. The chapters in part three discuss applications of wavelets in three different fields: cosmology, atmospheric data analysis, and denoising speech signals for digital hearing aids.

Admittedly, the boundaries between the three parts are rather subjective. Chapter 12 would have also fit nicely in the first part of the book because it introduces a wavelet construction on compact Riemannian manifolds; however, it is placed in the third part because of the authors' emphasis on the applications of their work to cosmology.

The notation we use in this chapter is standard. We denote the sets of real numbers by $\mathbb{R}$, the integers by $\mathbb{Z}$, and the natural numbers by $\mathbb{N}$. Recall that the standard dyadic wavelets are functions of the form

$$\psi_{m,n}(x) = 2^{n/2}\psi(2^n x - m), m,n \in \mathbb{Z}, x \in \mathbb{R}, \tag{1.1}$$

that are generated from one single function ψ, called the mother wavelet, by translation and dilation. More generally, if we denote the translation, dilation, and modulation operators, respectively, by T_m, D^n, E_k where

$$T_m f(x) = f(x-m), \quad D^n f(x) = 2^{n/2} f(2^n x), \quad E_k f(x) = \mathrm{e}^{\mathrm{i}<k,x>} f(x),$$

where $x \in \mathbb{R}^d, m,k \in \mathbb{Z}^d, n \in \mathbb{Z}$, then the wavelet and Gabor systems on $\mathbb{R}^d$ can be written, respectively, as

$$\psi_{m,n}(x) = D^n T_m \psi(x), \quad g_{m,n}(x) = E_k T_m g(x).$$

A more general form of wavelets in $\mathbb{R}^d$ is given by

$$\left(\psi_j^A\right)_{m,n}(x) = |\det A|^{n/2} \psi_j(A^n x - m), \tag{1.2}$$

where A is a real expansive $d \times d$ invertible matrix and $j = 1, \ldots, J$, $n \in \mathbb{Z}$, $m \in \mathbb{Z}^d$.

A multiresolution analysis of $L^2(\mathbb{R})$ consists of a nested sequence of closed subspaces $\{V_n\}_{n=-\infty}^{\infty}$ of $L^2(\mathbb{R})$ and a function $\phi \in V_0$, called the scaling function, such that

(1) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$.
(2) $\bigcup_{i=-\infty}^{\infty} V_i$ is dense in $L^2(\mathbb{R})$.
(3) $\bigcap_{i=-\infty}^{\infty} V_i = \{0\}$.
(4) $f(x) \in V_0 \Leftrightarrow f(2^j x) \in V_j$.
(5) $\{T_n\phi(x) = \phi(x-n)\}_{n=-\infty}^{\infty}$ is an orthonormal (Riesz) basis for V_0.

It is well known that given a multiresolution analysis, one can construct an orthonormal wavelet basis of $L^2(\mathbb{R})$.

1.1.1 Mathematical Theory of Wavelets

The construction of dyadic orthonormal wavelet bases in $L^2(\mathbb{R})$ hinges on the construction of the mother wavelet ψ. The pioneering work of Y. Meyer and S. Mallat [5, 6] gave an algorithm for constructing the mother wavelet in the setting of multiresolution analysis (MRA). However, not every wavelet is generated from a multiresolution analysis as J. Journé in 1992 demonstrated by his celebrated example of a non-MRA wavelet basis for $L^2(\mathbb{R})$; see [3, p. 136]. In higher dimensions, the construction of orthonormal wavelet bases was more elusive. The most common construction of an orthonormal wavelet basis came from the theory of multiresolution analysis (MRA) which requires 2^d-1 functions $\psi_j, j=1,\ldots,2^d-1$, to generate the resulting orthonormal basis (ONB), $(\psi_j)_{m,n}$, of $L^2(\mathbb{R}^d)$; see [6, p. 90].

For some time there was doubt about the existence of a single dyadic orthonormal wavelet basis for $\mathbb{R}^d, d>1$. The work of Dai, Larson, and Speegle [1, 2] on operator theory proved the existence of such wavelets in $L^2(\mathbb{R}^d), d>1$. Their proof used the notion of wavelet sets and operator algebra methods. It turned out that the Fourier transform of such a mother wavelet ψ is the characteristic function χ_Ω of a measurable set Ω, which is called a wavelet set.

The construction of such a wavelet set is not obvious, but it is necessary that its translates by $\mathbb{Z}^d$ provide a tiling of $\mathbb{R}^d$. In fact, it is known that if $E \subset \mathbb{R}$ is a measurable set, then E is a wavelet set if and only if E is both a 2-dilation generator (modulo null sets) of a partition of $\mathbb{R}$ and a 2π-translation generator (modulo null sets) of a partition of $\mathbb{R}$. The simplest example of a wavelet set is given by the Shannon's wavelet ψ whose Fourier transform $\hat{\psi} = \chi_E$, where $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$. The structure of wavelet sets in higher dimensions is much more intricate.

In Chap. 2, John Benedetto and his son Robert give a general method for constructing single dyadic wavelets, which generate wavelet orthonormal bases (ONBs) for the space of square-integrable functions in two important antipodal cases. These cases are $L^2(\mathbb{R}^d)$, where $\mathbb{R}^d$ is the d-dimensional Euclidean space, and $L^2(G)$, where G belongs to the class of locally compact Abelian groups (LCAGs) which contain a compact open subgroup. The wavelets they construct are not derived from any MRA.

In the first five sections, the authors discuss the geometry and construction of Euclidean wavelet sets and then proceed to extend their results to the non-Euclidean cases, such as locally compact Abelian groups and p-adic fields, $\mathbb{Q}_p$, of p-adic rationals. One of the salient features of this well written chapter is that it is almost self contained. The authors give a historical account of the subject, as well as the needed definitions and mathematical tools.

It is a well known fact that if ψ is an orthonormal wavelet, i.e., it generates an orthonormal wavelet basis in $L^2(\mathbb{R})$, then $\sum_k |\hat{\psi}(\omega+2\pi k)|^2 = 1$, and hence $|\hat{\psi}(\omega)| \leq 1$. Furthermore,

$$2\pi = \int_{\mathbb{R}} |\hat{\psi}(\omega)|^2 \mathrm{d}\omega = \int_E |\hat{\psi}(\omega)|^2 \mathrm{d}\omega \leq |E| \leq |\overline{E}| = |\mathrm{supp}\ \hat{\psi}|,$$

where $E = \{\omega : \hat{\psi}(\omega) \neq 0\}$ and $\overline{E}$ is the closure of E. Thus, the minimal measure of the support of the Fourier transform of a wavelet ψ is 2π and clearly the support of the Fourier transform of ψ possesses the minimal measure if and only if

$$|\hat{\psi}| = \chi_E \text{ and } |E| = |\overline{E}| = 2\pi,$$

where χ_E is the characteristic function of E. Recall that in the latter case, where the Fourier transform of ψ is the a characteristic function of a set E, the set E is called a wavelet set. Many wavelet sets have been constructed where E is a finite union of intervals. Of course, in these cases $|E| = |\overline{E}| = 2\pi$ and even in the known cases where E is a union of infinitely many intervals, still $|E| = |\overline{E}| = 2\pi$. Hence, this raises the question of whether there exists a wavelet set E^* such that $|\overline{E^*}| > |E^*| = 2\pi$. In Chap. 3, Z. Zhang answers this question in the affirmative. He also answers the same question for the scaling function of the associated multiresolution analysis.

One of the key elements in the construction of a multiresolution analysis is the notion of shift-invariant spaces and the nested sequence of subspaces generated from them by dilation. These ideas are extended by K. Hoover and B. Johnson in Chap. 4 to construct a multiresolution analysis and hence a finite-dimensional system of orthonormal wavelets in $L^2(\mathbb{T})$, where $\mathbb{T}$ is the torus. In this setting, dilation and translation have different meanings. The dilation operation is achieved through a matrix A, called the Quincunx dilation matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix};$$

see (1.2), while translation is considered over a discrete subgroup of $\mathbb{T}^2$. More precisely, for a fixed integer $j > 0$, a lattice, Γ_j of order 2^j generated by A is a collection of 2^j distinct coset representatives of $A^{-j}\mathbb{Z}^2/\mathbb{Z}^2$. The translation operators are generated by elements of Γ_j. A shift-invariant space is defined as a space that consists of functions in $L^2(\mathbb{T}^2)$ that are invariant under translation by elements of Γ_j. Having introduced the basic tools for shift-invariant spaces, the authors go on to construct a MRA of order 2^j consisting of closed subspaces $\{V_k\}_{k=0}^{j}$ of $L^2(\mathbb{T}^2)$ satisfying similar properties to those of the standard MRA, such as the nested sequence property and the existence of a scaling function. They conclude their work by giving examples of wavelet systems on the torus that are analogs of the Shannon and Haar wavelets.

Another type of wavelets, called crystallographic Haar-type composite dilation wavelets, is discussed in Chap. 5 by J. Blanchard and K. Steffen. These wavelets are composite dilation wavelets which arise from crystallographic groups and are linear combinations of characteristic functions. To briefly explain some of the terminology, let $GL_d(\mathbb{R})$ be the group of $d \times d$ invertible matrices. For any $A \in GL_d(\mathbb{R})$, define the dilation by (1.2). A full rank lattice Γ is a subset of $\mathbb{R}^d$ with the property that there exists $A \in GL_d(\mathbb{R})$, such that $\Gamma = A\mathbb{Z}^d$. A group of invertible matrices G and a full rank lattice Γ are said to satisfy the crystallographic condition if Γ is invariant under the action of G, i.e. $G(\Gamma) = \Gamma$. The translation of f by $k \in \Gamma$ is defined as usual by $T_k f(x) = f(x-k)$. With these two unitary operators, an affine system,

$$\mathscr{U}_{C,\Gamma} = \left\{ D_c T_k \psi^\ell(x) : c \in C, k \in \Gamma, \ell = l, \ldots, L \right\},$$

is constructed from a countable set of invertible matrices, $C \subset GL_d(\mathbb{R})$, a full rank lattice Γ, and a set of generating functions, $\Psi(x) = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^d)$. An affine system with composite dilations are obtained when $C = AB$ is the product of two subsets of invertible matrices A and B.

In this chapter, it is assumed that $A = \left\{ a^j : j \in \mathbb{Z} \right\}$ is a group generated by integer powers of an expanding matrix, a, and B is a subgroup of invertible matrices so that the affine system is in the form

$$\mathscr{U}_{a,B,\Gamma}(\Psi) = \left\{ D_a^j D_b T_k \psi^\ell : j \in \mathbb{Z}, b \in B, k \in \Gamma, 1 \leq \ell \leq L \right\}.$$

The system of functions $\Psi = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^d)$ is called a composite dilation wavelet if $\mathscr{U}_{a,B,\Gamma}(\Psi)$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

When $B = I_d$, where I_d is the identity matrix, we obtain the standard multiwavelet definition, and when $d = 1, \Gamma = Z$, and $a = 2$, we obtain the standard dyadic wavelets. The authors discuss examples of Haar-type composite dilation wavelets that were introduced in [4] by Krishtal, Robinson, Weiss, and Wilson under the assumptions: (a) B is a finite group and (b) the lattice Γ is invariant under the action of B, i.e., $B(\Gamma) = \Gamma$. It is shown that assumption (b) implies assumption (a), i.e., B must be a finite group in a Haar-type composite dilation wavelet system. Assumption (b) is the crystallographic condition. The authors develop a systematic way to construct crystallographic Haar-type composite dilation wavelets for $L^2(\mathbb{R}^d)$ and then discuss in more details, with examples, the construction for $L^2(\mathbb{R}^2)$.

Chapter 6 by C. Conti and M. Cotronei deals with the construction of wavelets for the analysis of vector-valued functions. Such functions arise naturally in many applications where the data to be processed are samples of vector-valued functions. Multichannel signals, or vector-valued signals whose components come from different sources with possible intrinsic correlations, such as brain activity (EEG/MEG) data or colored images, may exhibit a high correlation which can be revealed and exploited by what is called multichannel wavelet analysis.

The most common signal processing technique to handle vector-valued data is to deal with the signal's components one by one and thus ignoring possible relationships between some components. The standard scalar wavelet analysis, including multiwavelets, does not take into account the correlations among components. Recently, matrix wavelets, multichannel wavelets (MCW), and multichannel multiresolution analyses (MCMRA) have been proposed for the analysis of matrix-valued and vector-valued signals. An underlying concept in these different schemes is the existence of a matrix refinable function which satisfies the so-called full rank condition.

To explain some of the terminology, let $r,s \in \mathbb{N}$, and $A \in \mathbb{R}^{r\times s}$ be an $r\times s$ matrix. Let $\mathscr{A} = (A_j \in \mathbb{R}^{r\times s}, j \in \mathbb{Z})$ be a bi-infinite sequence of such matrices such that

$$\|\mathscr{A}\|_2 = \left(\sum_{j\in\mathbb{Z}} |A_j|_2^2\right)^{1/2} < \infty,$$

where $|.|_2$ denotes the standard ℓ^2 norm of $r\times s$ matrices. Let $L_2^{r\times s}(\mathbb{R})$ denote the Banach space of $r\times s$ matrix-valued functions on $\mathbb{R}$ with components in $L^2(\mathbb{R})$ and norm

$$\|F\|_2 = \left(\sum_{k=1}^{s}\sum_{j=1}^{r}\int_{\mathbb{R}} |F_{j,k}(x)|^2 \mathrm{d}x\right)^{1/2}.$$

For a matrix function F and a matrix sequence $\mathscr{A}$, the convolution $*$ is defined as

$$F * \mathscr{A} = \sum_{k\in\mathbb{Z}} F(\cdot - k)A_k.$$

Fix a matrix sequence $\mathscr{A} = (A_j \in \mathbb{R}^{r\times r}, j \in \mathbb{Z})$ and for any bi-infinite vector sequence $c = (c_j \in \mathbb{R}^r, j \in \mathbb{Z})$ define the vector subdivision operator $S_{\mathscr{A}}$ based on the matrix mask $\mathscr{A}$, as

$$S_{\mathscr{A}}(c) = \left(\sum_{k\in\mathbb{Z}} A_{j-2k}c_k, j \in \mathbb{Z}\right).$$

A vector subdivision scheme is defined inductively as

$$c^0 = c, \quad c^n = S_{\mathscr{A}}(c^{n-1}).$$

A multichannel multiresolution analysis (MCMRA) in the space $L_2^r(\mathbb{R})$ of square integrable vector-valued functions can be defined as a nested sequence $\cdots V_{-1} \subset V_0 \subset V_1 \subset \cdots$ of closed subspaces of $L_2^r(\mathbb{R})$ with similar properties to those in the scalar case, but in which the role of the scaling function ϕ is now replaced by a vector function so that the space V_0 is generated by the integer translates of r function vectors, f^i. That is, there exist $f^i = (f_1^i, \ldots, f_r^i) \in L_2^r(\mathbb{R}), i = 1,..,r,$ so that any $h \in V_0$ can be written as $h = F * c$, where $F = (f^1, \ldots, f^r) \in L_2^{r\times r}(\mathbb{R})$ and $c = (c_k)$. The matrix F is called the matrix scaling function and it is required that F be stable, i.e., there exist K_1, K_2 such that

$$K_1 \|c\|_2 \leq \|F * c\|_2 \leq K_2 \|c\|_2 .$$

It is shown that the subspaces V_j form a MCMRA if there exists a full rank stable matrix refinable function $F \in L_2^{r\times r}(\mathbb{R})$ such that

$$V_j = \left\{F * c(2^j \cdot)\right\}, j \in \mathbb{Z}.$$

The reader who is familiar with MRA can now see the analogy with the scalar case and how one can associate a matrix wavelet to any orthogonal MCMRA.

In addition, the authors investigate full rank interpolatory schemes and show their connection to matrix refinable functions and multichannel wavelets. They show how to solve matrix quadrature mirror filter equations and give a constructive scheme that uses spectral factorization techniques and a matrix completion algorithm based on the solution of generalized Bezout identities.

In Chap. 7, D. Han and D. Larson use operator algebra techniques to study, in a unified way, wavelet and Gabor frames, or more generally, Bessel wavelets and Gabor–Bessel sets.

Recall that a sequence of vectors $\{x_n\}$ in a separable infinite dimensional Hilbert space $\mathscr{H}$ is said to be a frame if there exist $A, B > 0$ such that for all $x \in \mathscr{H}$, we have

$$A \|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B \|x\|^2 .$$

If only the right-hand inequality holds, the sequence is called a Bessel Sequence. A unitary system $\mathscr{U}$ is a set of unitary operators containing the identity operator I and acting on the Hilbert space $\mathscr{H}$. A vector $x \in \mathscr{H}$ is called complete wandering vector (respectively frame generator vector, or Bessel generator vector) for $\mathscr{U}$ if

$$\mathscr{U} x = \{Ux : U \in \mathscr{U}\}$$

is an orthonormal basis (respectively a frame, or a Bessel sequence). A bounded linear operator A on $\mathscr{H}$ that maps every frame (Bessel) generator for $\mathscr{U}$ to a frame (Bessel) generator for $\mathscr{U}$ is called a frame (Bessel) generator multiplier.

In this chapter, the authors characterize some special Bessel generator multipliers for unitary systems that are ordered products of two unitary groups. This includes the wavelet and the Gabor unitary systems since wavelets can be viewed as being generated by an ordered product $\{D^n T_m\}$ of two unitary groups, the dilation group $\{D^n\}$ and the translation group $\{T_k\}$, and acting on a separable infinite dimensional Hilbert space $L^2(\mathbb{R}^d)$. In the Gabor theory, the groups are the modulation group $\{E_k\}$ and the translation group $\{T_k\}$.

The authors point out that one can gain much additional perspective on wavelets and frame wavelets if one views them as special cases of Bessel wavelets. One reason for this is that, unlike the sets of wavelets and frame wavelets, the set of Bessel wavelets is a linear space. Some new results for wavelets and frame wavelets, as well as, some new proofs of previously known results, are obtained as special cases of the Bessel wavelet case. The same can be said about Gabor–Bessel generators. The chapter provides a detailed exposition of part of the history leading up to this work.

With the advent of wavelets and multiresolution analyses, shift-invariant spaces have become the focus of many research papers. Sampling spaces are special cases of shift-invariant spaces and they may be defined as

$$V = \left\{ f : f \in L^2(\mathbb{R}), \quad f(x) = \sum_k f(t_k)\phi(x - t_k), \quad \phi \in L^2(\mathbb{R}) \right\},$$

where $\{t_k\}$ is an increasing sequence of real numbers such that $\lim_{k\to-\infty} t_k = -\infty$, $\lim_{k\to\infty} t_k = \infty$, and $0 < \delta \le |t_{k+1} - t_k|$, and $\{\phi(x - t_k)\}$ is a frame, or Riesz basis, or an orthonormal basis in $L^2(\mathbb{R})$. In most cases of interest, $t_k = k \in \mathbb{Z}$, and in this case the prototype is the Paley–Wiener space of functions f bandlimited to $[-\pi, \pi]$ and which have the representation

$$f(x) = \sum_k f(k)\frac{\sin\pi(x-k)}{\pi(x-k)} = \sum_k f(k)\phi(x-k),$$

where $\phi(x) = \operatorname{sinc} x = \sin\pi x/\pi x$.

An important tool in the study of sampling spaces is the Zak transform, which is defined for $f \in L^2(\mathbb{R})$ by

$$(Zf)(x,w) = \sum_{k\in\mathbb{Z}} f(x+k)\mathrm{e}^{-2\pi\mathrm{i}kw}, \quad f \in L^2(\mathbb{R}). \tag{1.3}$$

In the last chapter of Part I, E. Hernández, H. Šikić, G. Weiss, and E. Wilson give a new insight into the Zak transform and its properties. Let $\phi(x,w)$ denote the function given by (1.3) and M be the space of all such functions. It is easy to see that ϕ is periodic in w with period one and that

$$\phi(x+\ell, w) = \mathrm{e}^{2\pi\mathrm{i}\ell w}\phi(x,w), \quad \ell \in \mathbb{Z}.$$

Moreover, it is known that $\phi \in L^2\left(T^2\right)$, where $T = [-1/2, 1/2)$ and that the linear operator Z maps $L^2(\mathbb{R})$ isometrically onto M.

Let us define $\tilde{\phi} \in L^2\left(T^2\right)$ by

$$\tilde{\phi}(x,w) = \sum_{k\in\mathbb{Z}} f(w+k)\mathrm{e}^{2\pi\mathrm{i}kx},$$

and denote the space of all such functions by $\tilde{M}$. It is easy to see that $\tilde{\phi}$ is periodic in x with period one and $\tilde{\phi}(x, w+\ell) = \mathrm{e}^{-2\pi\mathrm{i}\ell x}\tilde{\phi}(x,w)$. The authors introduce the unitary map

$$(U\phi)(x,w) = \mathrm{e}^{-2\pi\mathrm{i}xw}\phi(x,w)$$

on M, and also the Zak-like transform $\tilde{Z}$ by

$$(\tilde{Z}g)(x,w) = \sum_{\ell\in\mathbb{Z}} g(w+\ell)\mathrm{e}^{2\pi\mathrm{i}\ell x} = \tilde{\phi}(x,w), \quad g \in L^2(\mathbb{R}).$$

It is shown that

$$\tilde{Z}^{-1}UZf = \hat{f}, \quad Z^{-1}U^{*}\tilde{Z}g = \check{g},$$

where $\hat{f}$ denotes the Fourier transform of f and $\check{g}$ denotes the inverse Fourier transform of g. The authors also show how to use the Zak transform to obtain elementary proofs of some results in harmonic analysis, including the Plancherel theorem and the Shannon sampling theorem.

1.1.2 Multiscale Analysis of Large Data Sets

R. Coifman and M. Gavish in Chap. 9 introduce digital data bases represented by data matrices M using harmonic analysis techniques. The prototype matrix M, is a data matrix whose set of columns X may be interpreted as observations or data points and set of rows Y may be interpreted as variables measured on the data points. The classical tools of multivariate statistics do not apply well in this case because one of the basic assumptions in multivariate statistics, namely, that the observations are independent and identically distributed, is no longer valid because, in general, correlations exist among the rows and columns. In this work, the authors propose another approach using harmonic analysis technique.

The authors first introduce the notion of a geometry and a Haar-like orthonormal basis on an abstract set, such as a set of observations. They then proceed to construct two coupled geometries on the rows and columns of the data matrices, which is called the coupled geometry of the matrix M. The coupled geometry is constructed using what the authors call a partition tree.

Guided by a classical result of J. O. Strömberg [7] that the tensor product of Haar bases is very efficient in representing functions on product spaces, the authors construct a Haar-like basis on X and Y, denoted by ψ_i and ϕ_j, and then show that the tensor basis $\left\{\psi_i \times \phi_j\right\}$ is an efficient basis of M. The Haar-like basis is a multiscale, localized orthonormal basis induced by a partition tree. Once a coupled geometry that is compatible with the matrix is obtained, the data matrix is expanded in the tensor-Haar basis and the data set is processed in the coefficient domain. It is shown that ℓ_p entropy conditions on the expansion coefficients of the database, viewed as a function on the product of the geometries, imply both smoothness and efficient reconstruction.

The authors describe how a tensor-Haar basis, induced by a coupled geometry that is compatible with the data matrix, can be used to perform compression, as well as, statistical tasks such as denoising and learning on the data matrix. They illustrate their technique for finding the coupled geometry of matrix rows and columns by giving two examples; the first example involves potential interaction between two point clouds in three dimensions and the second involves organizing a matrix whose rows are samples of some manifold Laplacian eigenfunctions.

As an interesting concrete example of data analysis using the general procedure described above, the authors consider a term-document data matrix. The documents are abstracts of 1,047 articles obtained from the Science News journal website of different fields. One thousand words with the highest correlation to article subject classification are selected so that the $M_{i,j}$ entry of the data matrix M is the relative frequency of the ith word in the jth document, $i = 1,\dots 1{,}000$ and $j = 1$, $=1\dots 1{,}024$.

Chapter 10 by G. Chen, A. Little, M. Maggioni, and L. Rosasco, reports on some advances in multiscale geometric analysis for the study of large data sets that lie in high dimensional spaces but are confined to low-dimensional nonlinear manifolds. Symbolically, the data sets lie on a d-dimensional nonlinear manifold M embedded in a high dimensional space $\mathbb{R}^D$ and are corrupted by high-dimensional noise.

The data sets are represented as discrete sets in $\mathbb{R}^D$, where D can be as large as 10^6. Unlike classical statistics, which deals with large data sets of size n in a space of fixed dimension D, with at least $n \gg 2^D$, in the case under consideration here we have n of the same order as D, and oftentimes $n < D$.

One of the statistical tools used to analyze data sets is the Principle Component Analysis (PCA) which is based on the Singular Value Decomposition (SVD). Recall that if X is an $n \times D$ matrix, it can be decomposed as $X = UYV^t$, where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{D\times D}$ are orthogonal and $Y \in \mathbb{R}^{n\times D}$, is diagonal and semi positivedefinite. The diagonal elements $\{\lambda_i\}$ of Y are called singular values and are ordered in decreasing order of magnitude and called the SVD of X. The first d columns of V provide the d-dimensional least square fit to X. If the rows $\{x_i\}$ of X represent n data points in $\mathbb{R}^D$, and lie on a bounded domain in a d-dimensional linear subspace of $\mathbb{R}^D$, then the SVD method may be applied.

However, in this chapter, the authors focus on the case where the data lie on a *non-linear* d-dimensional manifold $\mathscr{M}$ in $\mathbb{R}^D$ with $d \ll D$, and where the SVD method does not work. They propose a new technique based on what they call *geometric wavelets* which aims at efficiently representing the data. Essentially, the authors construct a data-dependent dictionary with I elements using multiscale geometric analysis of the data such that every element in the data set may be represented, up to a certain precision ε, by m elements of the dictionary. The elements of the dictionary are called geometric wavelets. They share some similarities with the standard wavelets, but they are quite different from them in crucial ways. Geometric wavelets are not based on dilation and translation and their multiscale analysis is nonlinear. The authors develop fast but nonlinear algorithms for computing a fast geometric wavelet transform and discuss the problem of estimating the intrinsic dimension of a point cloud and give different examples to illustrate their techniques.

The last chapter that deals with data in high-dimensional spaces is Chap. 11 by Lieu and Saito. Many problems in pattern recognition often require comparison between ensembles of signals (i.e., points in a high-dimensional space) instead of comparing individual signals. Let

$$X^i = \left\{x_1^i, \dots, x_{m_i}^i\right\},$$

where x_j^i is a signal or a vector in $\mathbb{R}^d$. The set X^i is called a training ensemble, which is assumed to have a unique label among C possible labels, and m_i is the number of signals in the ensemble X^i. The collection of ensembles $X = \cup_{i=1}^{M} X^i \subset \mathbb{R}^d$ is called a collection of M training ensembles. Let $\mathscr{Y} = \cup_{j=1}^{N} \mathscr{Y}^j \subset \mathbb{R}^d$ be a collection of test ensembles, where $\mathscr{Y}^j = \left\{Y_1^j, \ldots, Y_{n_j}^j\right\}$ is the jth test ensemble of n_j signals. The goal is to classify each $\mathscr{Y}^j$ to one of the possible C classes given the training ensembles X. The authors propose an algorithm for doing just that.

The proposed algorithm consists of two main steps. The first step performs the dimensionality reduction without losing important features of the training ensembles, followed by constructing a compact representation, called a signature, that represents an essence of the ensemble in the reduced space. The second step embeds a given test ensemble into the reduced space obtained in the first step followed by classifying it to the label of the training ensemble whose signature is most similar to that of the test ensemble. How to define the similarity or the distance measure between signatures is the key issue discussed in this work.

For the first step the authors try to find an effective low-dimensional representation of the data. The most well-known linear embedding techniques are Principal Component Analysis (PCA) and Multidimensional Scaling (MDS). More recently, many nonlinear methods for dimensionality reduction, such as Laplacian eigenmaps and diffusion maps, have been proposed in order to improve the shortcomings of PCA/MDS. The authors compare the performance of these reduction methods and then propose an algorithm that extends a given test ensemble into the trained embedding space. They then measure the distance between the test ensemble and each training ensemble in that space, and classify it using the nearest neighbor method. The label of the training ensemble whose signature is *closest* to that of a given test ensemble is assigned to it. This raises the question of how you define the distance measure for the nearest neighbor classifier. There are many such measures for comparing two given signatures, such as the usual Euclidean distance. But the most robust one the authors choose is the so-called Earth Mover's Distance (EMD).

The chapter is concluded by describing the results of numerical experiments on two real data sets to illustrate how the proposed algorithm can be applied in practice. The first experiment is on underwater object classification using acoustic signals and the second is on a lip-reading problem, whose objective is to train a machine to automatically recognize the spoken words from the movements of the lips captured on silent video segments (no sound is involved).

1.1.3 Wavelet Applications

The third part of the monograph is dedicated to wavelets applications. Some of the applications are based on the continuous wavelet transform and its generalizations. Recall that the continuous wavelet transform of a function $f \in L^2(\mathbb{R})$ is defined as

$$W_\psi[f](a,b) = \frac{1}{\sqrt{|a|}} \int_{\mathbb{R}} f(x)\psi\left(\frac{x-b}{a}\right) \mathrm{d}x, \quad a,b \in \mathbb{R},\ a \neq 0,$$

where ψ is the mother wavelet which is assumed to satisfy the admissibility condition

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \mathrm{d}\omega < \infty. \tag{1.4}$$

Spherical harmonics have been, for a long time, the main tool for analyzing functions on spheres but in recent years spherical wavelets have become a competing tool. Chapter 12 by D. Geller and A. Mayeli deals with the construction of wavelets on a smooth compact (connected) Riemannian manifold M, of dimension n, in particular, on spheres. They first show how to construct nearly tight frames that are well-localized both in frequency and space on M and then introduce what they call spin wavelets. They then apply their construction to the analysis of cosmic microwave background radiation (CMB).

Starting with an appropriately chosen mother wavelet on the real line and replacing the scale parameter in the associated continuous wavelet transform by a positive self-adjoint operator T on $L^2(M)$, an analogue of the wavelet transform can be defined. Since the transition from the real line to compact Riemannian manifolds is not very obvious, we will sketch the procedure.

From (1.4), we obtain for a suitable smooth function h with $h(0) = 0$

$$0 < \int_0^\infty \frac{|h(t)|^2}{t} \mathrm{d}t = c < \infty,$$

which implies that for all $\xi > 0$

$$\int_0^\infty |h(t\xi)|^2 \frac{\mathrm{d}t}{t} = c < \infty. \tag{1.5}$$

Standard discretization of this formula yields for $a > 1, \xi > 0$ (see [3, p. 69])

$$0 < A_a \leq \sum_{j\in\mathbb{Z}} \left|h(a^{2j}\xi)\right|^2 \leq B_a < \infty.$$

Now, let $h(\xi) = \xi^2 \mathrm{e}^{-\xi^2}$, so that $\psi = \check{h}$ is a constant multiple of the Mexican hat function. By multiplying (1.5) by $|F(\xi)|^2$, where F is a square integrable function and integrating with respect to ξ, we have

$$\int_{\mathbb{R}} \|F * \psi_t\|_2^2 \frac{\mathrm{d}t}{t} = c\,\|F\|_2^2,$$

where $\psi_t(x) = (1/t)\psi(x/t)$. The authors show that by discretizing the last integral, one obtains

$$A\,\|F\|_2^2 \leq \sum \left|\langle F, \phi_{j,k}\rangle\right|^2 \leq B\,\|F\|_2^2,$$

where

$$\phi_{j,k} = \sqrt{ba^j}\,\psi_{a^j}\left(x - bka^j\right),$$

for sufficiently small b and a sufficiently close to 1.

Replacing ξ in (1.5) by a positive self-adjoint operator T on a Hilbert space $\mathscr{H}$, the authors show that

$$\int_0^\infty |h|^2(tT)\frac{\mathrm{d}t}{t} = c(I-P),$$

and

$$A_a(I-P) \leq \sum_{j\in\mathbb{Z}} |h|^2(a^{2j}T) \leq B_a(I-P),$$

where P is the projection onto the null space of T, I is the identity operator, A_α and B_α are constants.

As a special case, take $T = -d^2/\mathrm{d}x^2$ on the real line, so that

$$\left[h(t^2T)F\right](x) = \int_{\mathbb{R}} K_t(x,y)F(y)\mathrm{d}y, \quad F \in L^2,$$

where $K_t(x,y) = \psi_t(x-y)$. On a smooth compact (connected) Riemannian manifold M, T is taken as the Laplace–Beltrami operator. A wavelet theory can now be developed for $L^2(M)$. A characterization of the Besov spaces using the frames constructed earlier can be obtained. The authors proceed to introduce what they call spin wavelets on the sphere and discuss their applications.

One of the most interesting features of this work is its applications to physics, in particular, to cosmic microwave background radiation (CMB), which was emitted after the Big Bang and is regarded as one of the pieces of evidence for the Big Bang theory. In this application, spherical wavelets seem to work better than spherical harmonics since the former are well-localized.

P. Fisher and K. Tung in Chap. 13 present two applications of wavelets as a numerical tool for atmospheric data analysis. In the first application, they use the continuous wavelet transform to determine the local Quasi-Biennial Oscillation (QBO) period, which is a dominant oscillation of the equatorial stratospheric zonal wind. The period is irregular but averages to about 28 months. Because the continuous wavelet transform can determine the local intrinsic period of an oscillation, it gives a more objective method for calculating the period of the QBO than previously used subjective methods.

The second application employs a wavelet-based multifractal formalism to analyze multifractal signals and find their singularity spectra. The authors apply the wavelet-based multifractal approach to the analysis of two sets of atmospheric data.

In the last chapter of the monograph, N. Whitmal, J. Rutledge, and J. Cohen, present a wavelet based approach for denoising speech signals for digital hearing aids. They describe an algorithm to remove background noise from noisy speech signals for use in digital hearing aids. The algorithm, which is based on wavelets and wavelet packets, was the result of a series of successive experiments in which the algorithm was adjusted to overcome a number of problems, such as the selection of a basis that best distinguishes noise from speech and the choice of the optimal number of coefficients from which the denoised signal can be constructed. Experiments performed at a computer lab in the Northwestern University Department of

Electrical Engineering and Computer Science showed, via spectrograms and tables of signal to noise ratios, that their method provided better noise reduction than spectral subtraction or other subspace methods. Testing on hearing impaired subjects, however, yielded intelligibility results comparable with other methods. Subsequent research revealed, both theoretically and experimentally, that subspace methods that involve hard thresholding produce effects on normal hearing listeners that are similar to recruitment of loudness, a common hearing disorder.

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Part I
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