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In memory of Thérèse N'Guérékata, my mother

Preface

Since the publication of our first book [80], there has been a real resurgence of interest in the study of almost automorphic functions and their applications ([16, 17, 28, 29, 30, 31, 32, 40, 41, 42, 46, 51, 58, 74, 75, 77, 78, 79]). New methods (method of invariant subspaces, uniform spectrum), and new concepts (almost periodicity and almost automorphy in fuzzy settings) have been introduced in the literature. The range of applications include at present linear and nonlinear evolution equations, integro-differential and functional-differential equations, dynamical systems, etc...It has become imperative to take a bearing of the main steps of the theory.

That is the main purpose of this monograph. It is intended to inform the reader and pave the road to more research in the field. It is not a self contained book. In fact, [80] remains the basic reference and fundamental source of information on these topics.

Chapter 1 is an introductory one. However, it contains also some recent contributions to the theory of almost automorphic functions in abstract spaces.

Chapter 2 is devoted to the existence of almost automorphic solutions to some linear and nonlinear evolution equations. It contains many new results.

Chapter 3 introduces to almost periodicity in fuzzy settings with applications to differential equations in fuzzy settings. It is based on a work by B. Bede and S. G. Gal [40].

Finally in Chapter 4 the classical theory of almost automorphic vector-valued functions is extended to fuzzy settings. This chapter begins with the presentation of several "new" spaces in which the theory holds, called fuzzy-number type spaces. These spaces are more general than the Banach and Fréchet spaces, since they are not linear structures although they present nice metric properties. Their importance consists in the fact that they are very appropriate for situations where imprecision which appears in the modelization of real world problems by differential equations is due to uncertainty or vagueness (and not randomness). Applications to some fuzzy differential equations are also given. It is based on S. G. Gal and G. M. N'Guérékata's recent work [41].

At the end of each chapter, we recall some relevant bibliographical remarks and raise some open problems and/or potential research subjects for graduate students and begining researchers in the area. It is our hope that this monograhp be used to stimulate some seminars and graduate courses in Analysis, Dynamical Systems, Fuzzy Mathematics and other branches of Mathematics.

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Finally, this book would hardly have been possible without the emotional support and encouragement of my wife Béatrice.

Baltimore, MD- USA *Gaston M. N'Guérékata*

May 2004

Contents

1

Introduction and Preliminaries

This chapter has an introductory character to this monograph. We wish to recall briefly some concepts, results, methods and notations that will be used in the sequel. We will indicate in general some references where the reader can find more informations if necessary. Although for almost automorphy, our book [80] remains the main source of information, we give detailed proofs to some new results.

1.1 Measurable Functions

In this section we will recall some facts about measurable vector-valued functions and their integrals. We consider $(X, \|.\|)$ a Banach space and I an open interval in $\mathbb{R}$. We denote by $C_c(I; X)$ the Banach space of continuous functions $f : I \to X$ with compact support in I.

Definition 1.1. *A function $f : I \to X$ is said to be measurable if there exists a set $S \subset I$ of measure 0 and a sequence $(f_n) \subset C_c(I; X)$ such that $f_n(t) \to f(t)$ as $n \to \infty$, for all $t \in I \backslash S$.*

We observe that if $f : I \to X$ is measurable, then $\|f\| : I \to \mathbb{R}$ is measurable too.

Theorem 1.2. *Let $f_n : I \to X$, $n = 1, 2...$ be a sequence of measurable functions and suppose that $f : I \to X$ and $f_n(t) \to f(t)$ as $n \to \infty$, for almost all $t \in I$. Then f is measurable.*

Proof. We have $f_n \to f$ on $I \backslash S$, where S is a set of measure 0. Let $(f_{n,k})_{k\in\mathbb{N}}$ be a sequence of functions in $C_c(I; X)$ such that $f_{n,k} \to f_n$ almost everywhere on I as $k \to \infty$. By Egorov's Theorem (see [90, p.16]) applied to the sequence of functions $\|f_{n,k} - f_n\|$, there exists a set $S_n \subset I$ of measure less than $\frac{1}{2^n}$ such that $f_{n,k} \to f_n$ uniformly on $I \backslash S_n$, as $k \to \infty$.

Now let $k(n)$ be such that $\|f_{n,k(n)} - f_n\| < \frac{1}{n}$ on $I \backslash S_n$ and $F_n = f_{n,k(n)}$. Also let $B = S \bigcup (\bigcap_{m \geq 1} \bigcup_{n > m} S_n)$. Then it is clear that B is a subset of I of measure 0. Take $t \in I \backslash B$. So we get $f_n(t) \to f(t)$, as $n \to \infty$. On the other hand if n is large enough, $t \in I \backslash S_n$. It follows that $\|F_n - f_n\| < \frac{1}{n}$. Which means $F_n(t) \to f(t)$, as $n \to \infty$, and consequently, f is measurable. □

Remark 1.3. It is easy to observe that if $\phi : I \to \mathbb{R}$ and $f : I \to X$ are measurable, the $\phi f : I \to X$ is measurable too.

Theorem 1.4. *(Pettis' Theorem) A function $f : I \to X$ is measurable if and only if the following two conditions are satisfied:*

(a) f is weakly measurable (i.e. for every $x^* \in X^*$, the dual space of X, the function $(x^* f)(t) : I \to \mathbb{R}$ is measurable)
(b) There exists a set $S \subset I$ of measure 0 such that $f(I \backslash S)$ is separable.

Proof. See [90, p.131]. □

We also have the following

Theorem 1.5. *If $f : I \to X$ is weakly continuous, then it is measurable.*

Definition 1.6. *A measurable function $f : I \to X$ is said to be integrable on I if there exists a sequence of functions $f_n \in C_c(I;X),\ n = 1,2,...$ such that*

$$\int_I \|f_n(t) - f(t)\|dt \to 0, \quad as \quad n \to \infty.$$

Remark 1.7. If $f : I \to X$ is integrable, it can be shown that there exists a vector $x \in X$, such that if $f_n \in C_c(I;X),\ n = 1,2,..$ and $\int_I \|f_n(t) - f(t)\|dt \to 0$ as $n \to \infty$, then $\int_I f_n \to x$ as $n \to \infty$. Such x is called the integral of f on I and denoted $x := \int_I f$.

Moreover if $I = (a,b)$, then we denote $x := \int_a^b f$.

Theorem 1.8. *(Bochner's Theorem). Assume $f : I \to X$ is measurable. Then f is integrable if and only if $\|f\|$ is integrable. Moreover we have*

$$\|\int_I f\| \leq \int_I \|f\|.$$

Proof. Let $f : I \to X$ be integrable. Then by the definition, there exist $f_n \in C_c(I;X),\ n = 1,2,...$ such that $\int_I \|f_(t) - f(t)\|dt \to 0$ as $n \to \infty$.

We have $\|f\| \leq \|f_n\| + \|f_n - f\|$, for each n, so $\|f\|$ is integrable.

Conversely assume now $\|f\|$ is integrable. Let $F_n \in C_c(I;\mathbb{R}),\ n = 1,2,...$ be a sequence of functions such that $\int_I |F_n(t) - \|f(t)\||dt \to$

0 as $n \to \infty$ and $|F_n| \leq F$ almost everywhere for some $F : I \to \mathbb{R}$, with $\int_I \|F(t)|dt < \infty$.

Since f is measurable, there exist $f_n \in C_c(I; X)$, $n = 1, 2, ...$ such that $f_n \to f$ almost everywhere.

We now let

$$u_n = \frac{|F_n|}{\|f_n\| + \frac{1}{n}} f_n, \ n = 1, 2, ...,$$

then it is obvious that $\|u_n\| \leq F$ for each $n = 1, 2...$, and $u_n \to f$ almost everywhere on I. Therefore $\int_I \|u_n - f\|dt \to 0$ as $n \to \infty$ and so f is integrable.

Using Lebesgue-Fatou's Lemma (see [90]), we get

$$\begin{aligned} \| \int_I f \| &\leq \lim_{n \to \infty} \| \int_I u_n \| \\ &\leq \lim_{n \to \infty} \int_I \|u_n\| \\ &\leq \int_I \|f\|. \end{aligned}$$

The proof is complete. □

Theorem 1.9. *(Lebesgue's Dominated Convergence Theorem). Let $f_n : I \to X$, $n = 1, 2, ...$ be a sequence of integrable functions and $g : I \to \mathbb{R}$ be an integrable function. Let also $f : I \to X$ and assume that:*

(i) for all $n = 1, 2, ...$ $\|f_n\| \leq g$, almost everywhere on I, and
(ii) $f_n(t) \to f(t)$, as $n \to \infty$ for all $t \in I$.

Then f is integrable on I and

$$\int_I f = \lim_{n \to \infty} \int_I f_n.$$

Definition 1.10. *Let* $1 \leq p \leq \infty$. *We will denote by* $L^p(I;X)$ *the space of all classes of equivalence (with respect to the equality a.e on* I*) of measurable functions* $f : I \to X$ *such that* $\|f\| \in L^p(I)$.

If we define a norm on $L^p(I;X)$ *by*

$$\|f\|_p = \left(\int_I \|f(t)\|^p dt \right)^{\frac{1}{p}}, \quad if \;\; 1 \leq p < \infty$$

and

$$\|f\|_p =: \|f\|_\infty = ess \sup_I \|f(t)\|, \quad if \;\; p = \infty,$$

then $L^p(I;X)$ *is a Banach space.*

We shall denote by $L^p_{loc}(I;X)$ the space of all (equivalence classes of) measurable functions $f : I \to X$ such that the restriction of f to every bounded subinterval of I is in $L^p(I;X)$.

1.2 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset.

Definition 1.11. *A function* $g \in L^1_{loc}(\Omega)$ *is said to be the weak derivative of a function* $f \in L^1_{loc}(\Omega)$ *(or a derivative in the sense of distributions of order* α*), if*

$$\int_\Omega g \cdot \phi \, dx = (-1)^{|\alpha|} \int_\Omega f \cdot D^\alpha \phi \, dx, \quad for\ all \;\; \phi \in C_0^\infty(\Omega).$$

In this case, we write $g = D^\alpha f$.

Recall that $D^\alpha f$ denotes the α-derivative defined by:

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, α_i $(1 \leq i \leq n)$ is a nonnegative integer, and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

Definition 1.12. *Let k be a non-negative integer and let $1 \le p \le \infty$. We define the Sobolev space $W^{k,p}(\Omega)$ by*

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : \quad D^\alpha f \in L^p(\Omega) \quad \textit{for all } |\alpha| \le k\}.$$

In $W^{k,p}(\Omega)$, we define a norm by

$$(N)_{k,p} \qquad \|f\|_{k,p}^p := \int_\Omega \sum_{|\alpha|\le k} |D^\alpha f(x)|^p dx, \quad p < \infty,$$

$$(N)_{k,\infty} \qquad \|f\|_{k,\infty} := \max_{|\alpha|\le k} \|D^\alpha f\|_\infty,$$

and for $p = 2$, we define an inner product

$$(N)_{k,2} \qquad \langle f, g\rangle := \sum_{|\alpha|\le k} \int_\Omega \overline{D^\alpha f} \, . \, D^\alpha g \, dx.$$

We have the following

Theorem 1.13. *The space $W^{k,p}(\Omega)$ is a Banach space. If $p < \infty$, it is separable.*

Also we have

Definition 1.14. *By $W_0^{k,p}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. ($C_0^\infty(\Omega)$ denotes the space of functions of class C^∞ with compact support in Ω.*

Definition 1.15. *We define $H^k(\Omega) := W^{k,2}(\Omega)$ and $H_0^k(\Omega) := W_0^{k,p}(\Omega)$.*

Theorem 1.16. *$H^k(\Omega)$ and $H_0^k(\Omega)$ are Hilbert spaces when endowed with the inner product $(N)_{k,2}$.*

Theorem 1.17. *$C^\infty(\Omega) \cap H^k(\Omega)$ is dense in $H^k(\Omega)$, where $C^\infty(\Omega)$ is the space of functions defined on Ω of class C^∞.*

1.3 Semigroups of Linear Operators

Definition 1.18. *Let $A : X \to X$ be a linear operator with domain $D(A) \subset X$, in a Banach space $(X, \|\cdot\|)$. The family $T = (T(t))_{t\in\mathbb{R}^+}$ of bounded linear operators on X is said to be a C_0-semigroup if*

i) For all $x \in X$, the mapping $T(t)x : \mathbb{R}^+ \to X$ is continuous;
ii) $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}^+$ (semigroup property);
iii) $T(0) = I$, the identity operator.

The operator A is called the infinitesimal generator (or generator in short) of the C_0-semigroup T if

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}$$

where

$$D(A) = \left\{ x \in X / \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

It is observed that A commutes with $T(t)$ on $D(A)$. We define a C_0-group in a similar way, replacing $\mathbb{R}^+$ by $\mathbb{R}$.

For a bounded linear operator A, we have

$$T(t) := e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

We also have the exponential growth:

Proposition 1.19. *(see [90] page 232). If $T = (T(t))_{t\in\mathbb{R}^+}$ is a C_0-semigroup then there exist $K > 0$ and $\omega < \infty$ such that*

$$\|T(t)\| \leq Kc^{\omega t}, \text{ for all } t \in \mathbb{R}^+.$$

If $\omega < \infty$, we say that T is exponentially stable.

Proposition 1.20. *If $T = (T(t))_{t\in\mathbb{R}^+}$ is a C_0-semigroup, then:*

a) the function $t \to \|T(t))\|$, $\mathbb{R}^+ \to \mathbb{R}^+$ is measurable and bounded on any compact interval of $\mathbb{R}^+$.

b) the domain $D(A)$ of its generator is dense in X.

c) the generator A is a closed linear operator.

1.4 Fractional Powers of Operators

Let $(X, \|.\|)$ be a (complex) Banach space and let $C : D(C) \subset X \mapsto X$ be a densely defined closed unbounded linear operator acting in X. Assume that $-C$ is the infinitesimal generator of an analytic semigroup $(R(t))$ and that $0 \in \rho(C)$, where $\rho(C)$ is the resolvent of the operator C. Then one can define, for $0 < \alpha \leq 1$, the fractional powers of C^α.

It is well-known that $C^\alpha : D(C^\alpha) \subset X \mapsto X$ is a densely defined closed linear operator. Further, its domain $D(C^\alpha)$ is endowed with the norm defined as

$$\|x\|_\alpha = \|C^\alpha x\|, \quad \text{for } x \in D(C^\alpha).$$

Since C is closed, then it can be easily shown that $X_\alpha = (D(C^\alpha), \|.\|_\alpha)$ is also a Banach space.

Recall that if $-C$ is the infinitesimal generator of an analytic semigroup $(R(t))$ and that $0 \in \rho(C)$, for $\alpha > 0$, the fractional powers C^α of C are implicitly defined as

$$C^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} R(t)dt,$$

where $\Gamma(\alpha)$ is the classical Gamma function.

In the case where $0 < \alpha \leq 1$, since $0 \in \rho(C)$, then the operator $C^{-\alpha}$ is bounded, that is, there exists $K > 0$ such that $\|C^{-\alpha}\| \leq K$.

Theorem 1.21. *Under the above assumptions on the operator* C, *we have*

(i) $C^{-\alpha}C^{-\beta} = C^{-(\alpha+\beta)}$;

(ii) $\lim_{\alpha\to 0} C^{\alpha} = I$ *(strong operator topology).*

Proof. See [83] for instance.

We also recall the following.

Lemma 1.22. *Let* $-C$ *be the infinitesimal generator of an analytic semigroup* $R(t)$. *Assume that* $0 \in \rho(C)$.

Then for $\alpha > 0$, *we have the following:*

1. *for every* $u \in D(C^{\alpha})$, $R(t)C^{\alpha}u = C^{\alpha}R(t)u$. *Moreover* $C^{\alpha}R(t)$ *is bounded, with an estimate of the form*

$$\|C^{\alpha}R(t)\| \leq M_{\alpha}t^{-\alpha}e^{-\delta t}.$$

2. *If* $0 < \alpha \leq 1$ *and* $u \in D(C^{\alpha})$, *we have an estimate of the form*

$$\|R(t)u - u\| \leq C_{\alpha}t^{\alpha}\|C^{\alpha}u\|.$$

More details on fractional powers of operators can be found in the literature, especially in [83].

1.5 Evolution Equations

Unlike the finite dimensional case, the infinite dimensional theory of evolution equations has several notions of solutions. We will