QUANTUM DYNAMICS FOR CLASSICAL SYSTEMS

WITH APPLICATIONS OF THE NUMBER OPERATOR

FABIO BAGARELLO

WILEY

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Science never solves a problem
without creating ten more.
George Bernard Shaw

The most exciting phrase to hear in science,
the one who heralds new discoveries,
is not *Eureka* but *That's funny*...
Isaac Asimov

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Preface

In 2005 or so, I started wondering whether that particular part of quantum mechanics that theoretical physicists call *second quantization* could be used in the analysis of some particular, somehow *discrete*, classical system. In particular, I started considering stock markets, or a reasonably simplified version of these, since at that time this was a very fashionable topic: econophysics was in its early years, and the general feeling was that there was still room, and need, for many other contributions from physicists and mathematicians. I got the idea that the analysis of the huge amount of information going around a real market was only part of what was interesting to do. I was much more interested in considering the viewpoint of the single trader, who is more likely interested in having some control of his own portfolio. Therefore, I constructed a model of a simplified market, just to see if this *strange* approach could be interesting for such a hypothetical trader, and I suddenly realized that “yes, it might make sense to carry on in this line of research, but, wow, it is hard to have such a paper accepted in a good journal.” However, after a few weeks, I also realized that this topic seemed to be interesting not only for me, but also for a large community of scientists, and that this community was increasing very fast, producing more and more contributions on the *ArXiv*. People started citing my first paper, and I was contacted by people interested in what I was doing and who wanted to discuss my point of view. This pushed me in the direction of considering more sophisticated models for stock markets, using my knowledge of quantum mechanics for systems with infinite degrees of freedom in this other, and apparently completely different, field. I thought that this was essentially the end of the story: quantum versus

economics. Unexpectedly, a few years ago during a conference in Acireale where I gave a talk on my *quantum stock markets*, I had a discussion with an old friend of mine, Franco Oliveri, and he suggested using the same general strategy in a different classical context. I remember that in our first discussion, we were thinking of foxes and chicken, a topic that was not very exciting for me. After a while, we realized that what we were discussing could also have been used to describe something completely different: a love story. And that was the beginning of a long story that still continues. Since then, we have constructed several models for different classical systems, playing with our understanding of these systems and looking for some phenomenological description. It turned out that these models quite often produce nontrivial and, in my opinion, quite interesting features that are not fully explored yet. Moreover, what is also very intriguing to me is that the same general framework can be used in many different contexts, giving rise to a sort of unifying setting.

This book might be considered as a first attempt to summarize what I have done so far in this field. My idea was to make the book *reasonably simple* and self-contained. This is because I expect that some not necessarily mathematical-minded readers might be intrigued by the title, and I do not want to lose these readers. However, a complex system can be made easy only up to a certain extent, and this is exactly what I have tried to do in these notes. Even the love story I will consider in Chapter 3, which from the purely dynamic point of view is surely the simplest situation, is not simple at all. This is not a big surprise, as almost every lover knows very well from personal experience. I should also clarify that it is not my main interest to discuss the *psychological aspects* behind a love story, a migration process, or the choices of traders in a stock market. I am not even interested in giving any abstract, or too general, description

of these systems. Here I want to be quantitative! I want to deduce formulas from general ideas, and I want to see what these formulas imply for the system I have in mind, and if they have some predictive power. However, this ultimate goal implies some effort to the reader, who is required to create his own background on quantum mechanics (if needed) by reading Chapter 2. Dear reader, if you can understand Chapter 2, you can understand the rest! On the other hand, if Chapter 2 is too technical for you, do not worry: you could still try to read the book, simply jumping over this chapter. Of course, if you are not a physicist, you will lose a lot. But you can still get the feeling of what is going on. It is up to you! I really hope you enjoy reading this book!

Fabio Bagarello

Acknowledgments

It is always a pleasure to thank old friends of mine such as Camillo Trapani and Franco Oliveri for their very precious help in so many different situations that I cannot even remember myself. This could be seen as (the first) evidence of the death of my neurons. But, do not worry! I have still enough neurons left in my brain to remember that if something in functional analysis is not particularly clear, Camillo is the right one! And I also have a post-it on my desk that says “Any numerical problem? Call Franco!” What is more funny is that they still answer my questions, even when they are very, very busy! Franco is also responsible, in part, for what is discussed in these notes, and I also thank him for this scientific collaboration and for his enthusiasm.

It is also a pleasure to thank the various editors and referees whom I have *met* along these years, including the ones who referred this book (before it became a book!). Most of them have contributed significantly to the growth of my research, with many useful, and sometimes unexpected, suggestions. Particular thanks goes to Wiley, for the enthusiasm shown for my manuscript.

I dedicate this book to my beloved parents Giovanna and Benedetto, to my brother Vincenzo, and, *dulcis in fundo,* to Federico, Giovanna, and Grazyna. When I look at them I often ask myself the same question, a question for which I have no answer, yet: how can they resist with so much mathematics and physics going around?

Chapter 1: Why a Quantum Tool in Classical Contexts?

Of course, there is no other way to begin this book. In our experience, this is the first question that a referee usually raises when he receives a paper of ours. Hence this is the question that we try to answer in this chapter, to motivate our analysis.

Taking a look at the index, we see that the applications discussed here cover a large variety of problems, from love affairs to migration, from competition between species to stock markets. First of all, we have to stress that we are not claiming that, for instance, a love affair has a quantum nature! (Even though, as every lover knows very well, each love story is surely characterized by a strong stochastic ingredient that could be analyzed, for instance, using tools from probability theory. It is not surprising, then, that one could try to use quantum mechanics as well, in view of its probabilistic interpretation.) Therefore, we are not going to discuss any *quantum love affair*. Rather, we just claim that some quantum tools, and the number operator in particular, can be used, *cum grano salis*, in the analysis of several dynamical systems in which the variables are seen as operator-valued functions. The interesting fact is that the results we deduce using these tools describe very well the dynamics of the system we are considering. This is shown to be true for love affairs first, but this same conclusion apparently holds in other, completely different, contexts (migrations, stock markets, competition between species, etc.).

However, to answer in more detail the question raised in the title, we need a long introduction, and this is the main content of this chapter. We begin with a few useful facts on

(anti-)commutation rules, which are used to motivate our answer. Then, we describe briefly other appearances of quantum mechanics in the description of classical systems, proposed by several authors in recent years. We conclude the chapter with the plan of the book.

1.1 A First View of (Anti-)Commutation Rules

In many fields of quantum mechanics of systems with few or many degrees of freedom, the use of annihilation or creation operators, and of their related number operators, has been proved to be very useful. The first explicit application of the so-called canonical commutation relations (CCR) is usually found at the first level degree in physics while studying the one-dimensional quantum harmonic oscillator (Merzbacher, 1970). CCR are used to produce a purely algebraic procedure that helps in finding the eigenvalues and the eigenvectors of the energy operator $H = \frac{1}{2}\left(p^2 + x^2\right)$ of the oscillator, expressed here in convenient units. This procedure is much simpler than the one that returns the explicit solution of the Schrödinger equation in configuration space (i.e., in terms of the position variable x). The quantum nature of the system is reflected by the fact that p and x do not commute. Indeed, they satisfy the following rule: $[x, p] := xp - px = i1\!\!1$, $1\!\!1$ being the identity operator in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ where the oscillator lives. This means that x and p are not classical functions of time but, rather, operators acting on $\mathcal{H}$. As life is usually not easy, and science is even harder, x and p are unbounded operators. This produces a number of extra difficulties, mainly on the mathematical side, which we try to avoid as much as possible in these notes, but which is necessary at least to

mention and to have in mind.[1] Introducing $a := \frac{1}{\sqrt{2}}(x + ip)$, its adjoint $a^\dagger := \frac{1}{\sqrt{2}}(x - ip)$, and $N := a^\dagger a$, we can rewrite the Hamiltonian as $H = N + \frac{1}{2}11$, and, if φ_0 is a vector of $\mathcal{H}$, which is annihilated by a, $a\varphi_0 = 0$, then, calling $\varphi_n := \frac{1}{\sqrt{n!}}(a^\dagger)^n\varphi_0$, $n = 0, 1, 2, \ldots$, we have $H\varphi_n = (n + 1/2)\varphi_n$. In the literature, φ_0 is called *the vacuum* or *the ground state* of the harmonic oscillator. Then we find, avoiding the hard explicit solution of the Schrödinger differential equation

$$i\frac{\partial\Psi(x,t)}{\partial t} = \frac{1}{2}\left(-\frac{\partial^2\Psi(x,t)}{\partial x^2} + x^2\Psi(x,t)\right),$$

the set of eigenvalues ($E_n = n + 1/2$) and eigenvectors (φ_n) of H. In the derivation of these results, the crucial ingredient is the following commutation rule: $[a, a^\dagger] = 11$, easily deduced from $[x, p] = i11$ and from the definitions of a and $a^\dagger$, which have the useful consequence $[N, a^{\dagger n}] = na^{\dagger n}$, $n = 0, 1, 2, \ldots$. Using this result, $H\varphi_n = E_n\varphi_n$ follows immediately.

Standard quantum mechanical literature states a simple extension of $[a, a^\dagger] = 11$ is found soon after the one-dimensional harmonic oscillator, while moving to higher dimensional systems. In this case, the CCR look like $[a_l, a_n^\dagger] = 11\delta_{l,n}$, $l, n = 1, 2, 3, \ldots L$: we have L independent modes. It might be interesting to remind that Lth dimensional oscillators are usually the key ingredients to set up, both at a classical and at a quantum level, many perturbation schemes that are quite useful whenever the dynamics of the system cannot be deduced exactly. Some quantum perturbation approaches are quickly reviewed in Chapter 2 and used all along the book.

Studying quantum field theory, one is usually forced to consider mainly two different kinds of particles, which obey very different commutation rules and, as a consequence of

the spin-statistic theorem, different statistics: the *bosons* and the *fermions*. Bosons are particles with integer spin, such as the photons. Fermions are particles with half-integer spin, such as the electrons. This difference in the value of the spin has an important consequence: fermions satisfy the Pauli exclusion principle, whereas bosons do not. This is reflected, first of all, by the wave function that describes any set of identical fermions, which has to be antisymmetric with respect to the change of their variables, or by the wave function for the bosons, which has to be symmetric. Hence, if two indistinguishable fermions have exactly the same quantum numbers (e.g., they occupy the same position in space and they have the same energy), their wave function collapses to 0: such a configuration cannot occur! This is the Pauli exclusion principle, which, of course, does not hold for the bosons. In *second quantization*, (Roman, 1965), the bosons are created by the operators $a_l^\dagger$ and annihilated by their conjugate, a_l. Together, they satisfy the CCR above. Analogously, fermions are annihilated and created by similar operators, b_K and $b_k^\dagger$, but these satisfy a different rule, the so-called anticommutation relation (CAR): $\{b_l, b_k^\dagger\} = b_l b_k^\dagger + b_k^\dagger b_l = 11\delta_{l,k}$, with $\{b_l, b_k\} = \{b_l^\dagger, b_k^\dagger\} = 0$, l, k = 1, 2, 3, ... The main difference between these two commutation rules is easily understood. While the operator a_l^2 is different from 0, the square of b_l is automatically 0, together with all its higher powers. This is again an evidence of the Pauli principle: if we try to construct a system with two fermions with the same quantum numbers (labeled by l) using the language of second quantization, we should act twice with $b_l^\dagger$ on the vacuum φ_0, that is, on the vector annihilated by all the b_ls. But, as $b_l^{\dagger 2} = 0$, the resulting vector is 0: such a state has probability 0 to occur and, as a consequence, the Pauli principle is preserved!

In Chapter 2, we show, among other things, that the eigenvalues of $N_l^{(a)} := a_l^\dagger a_l$ are 0, 1, 2, ... , whereas those of $N_l^{(b)} := b_l^\dagger b_l$ are simply 0, 1. This is related to the fact that the *fermionic* and the *bosonic* Hilbert spaces differ as the first one is finite dimensional, whereas the second is infinite dimensional. Needless to say, this produces severe differences from a technical point of view. In particular, operators acting on a (finite modes) fermionic Hilbert space are automatically bounded, whereas those acting on a bosonic Hilbert space are quite often unbounded.

1.2 Our Point of View

In many classical problems, the relevant quantities we are interested in change discontinuously. For instance, if you consider a certain population $\mathcal{P}$, and its time evolution, the number of people forming $\mathcal{P}$ cannot change arbitrarily: if, at $t_0 = 0$, $\mathcal{P}$ consists of N_0 elements, at $t_1 = t_0 + \Delta t$, $\mathcal{P}$ may only consist of N_1 elements, with $N_1 - N_0 \in \mathbb{Z}$. The same happens if our system consists of two (or more) different populations, $\mathcal{P}_1$ and $\mathcal{P}_2$ (e.g., preys and predators or two migrating species): again, the total number of their elements can only take integer values.

Analogously, if we consider what in these notes is called *a simplified stock market* (SSM), that is, a group of people (the *traders*) with some money and a certain number of shares of different kind, which are exchanged between the traders who pay some cash for that, it is again clear that natural numbers play a crucial role: in our SSM, a trader may have only a natural number of shares (30, 5000, or 10^6, but not 0.75 shares), and a natural number of units of cash (there is nothing < 1 cent of euro, for instance). Hence, if two traders buy or sell a share, the number of shares in their portfolios

increases or decreases by one unit, and the amount of money they possess also changes by an integer multiple of the unit of cash.

In the first part of these notes, we also consider some quantities that change continuously but that can also still be measured, quite naturally, using discrete values: this is the case, for instance, of the love affair between Alice and Bob described in Chapter 3: in some old papers, see Strogatz (1988) and Rinaldi (1998a,b) for instance, the mutual affection between the two actors of the love affair is described by means of two continuous functions. However, it is not hard to imagine how a similar description could be given in terms of discrete quantities: this is what we have done, for instance, in Bagarello and Oliveri (2010, 2011): Bob's affection for Alice is measured by a discrete index, n_B, which, when it increases during a time interval $[t_i, t_f]$, from, say, a value 1 to the value 2, describes the fact that Bob's love for Alice increases during that particular time interval. Analogously, Alice's affection for Bob can be naturally measured by a second discrete index, n_A, which, when its value decreases from, say, 1 to 0, describes the fact that Alice's love for Bob simply disappears.

These are just a few examples, all described in detail in these notes, showing how the use of discrete quantities is natural and can be used in the description of several systems, in very different situations. Of course, at first sight, this may look as a simple discretization of a continuous problem, for which several procedures have been proposed along the years. However, this is not our point of view. We adopt here a rather different philosophy, which can be summarized as follows: the discrete quantities used in the description of the system $\mathcal{S}$ under analysis are closely related to the eigenvalues of some self-adjoint operator. Moreover, these operators can be quite often approximated with effective, finite dimensional, self-adjoint matrices,

whose dimensions are somehow fixed by the initial conditions; see, for instance, Chapter 3. Then the natural question is the following: how can we deduce the dynamical behavior of $\mathcal{S}$? This is, of course, the hard part of the job! Along all our work, we have chosen to use a Heisenberg-like dynamics, or its Schrödinger counterpart, which we believe is a good choice for the following reasons:

1. It is a natural choice when operators are involved. This is exactly the choice used in quantum mechanics, where the Heisenberg representation is adopted in the description of the dynamics of any closed microscopic system.

2. It is quite easy to write down an energy-like operator, the Hamiltonian H of the system $\mathcal{S}$, which drives the dynamics of the system. This is due to the fact that, following the same ideas adopted in particle physics, the Hamiltonian contains in itself the phenomena it has to describe. This aspect is clarified first in Sections 3.3 and 3.4 in a concrete situation, while in Chapter 6, we discuss the role and the construction of H in more detail and in a very general condition. Among the other criteria, the explicit definition of H will also be suggested by the existence of some conserved quantities of $\mathcal{S}$: if X is an operator, which is expected to be preserved during the time evolution of $\mathcal{S}$, for instance, the total amount of cash in a closed SSM, then, because of the definition of the Heisenberg dynamics, H must commute with X: $[H, X] = 0$. This gives some extra hints on how to define H explicitly, and then H can be used to find the time evolution of any observable A of $\mathcal{S}$ using the standard prescription: $A(t) = e^{iHt}A(0)e^{-iHt}$, $A(0)$ being the value of A at $t = 0$. We refer to Chapter 2 for many more details on the dynamics of $\mathcal{S}$.

3. It produces results which, at least for those systems considered in the first part of these notes, look quite

reasonable; that is, they are exactly those results, which one could expect to find as they reproduce what we observe in real life. This is a good indication, or at least gives us some hope, that the dynamics deduced for the systems discussed in Part II of the book, that is, for SSMs, reflect a reasonable time evolution for those systems.

This list shows that we have two technical and one a posteriori reason to use an energy-like operator H to compute the dynamics of $\mathcal{S}$. This is not, of course, the end of the story, but, in our opinion, it is already a very good starting point.

1.3 Do Not Worry About Heisenberg!

People with a quantum mechanical background know very well that, whenever incompatible (i.e., not commuting) observables appear in the description of a given physical system $\mathcal{S}$, some uncertainty results follow. Hence, one may wonder how our quantum-like description could be compatible with the classical nature of $\mathcal{S}$, whose observable quantities are not expected to be affected by any error, except, at most, by the error due to the experimental settings. This problem, actually, does not exist at all in the applications considered in these notes as all the observables we are interested in form a commuting subset of a larger nonabelian algebra. Therefore, they can be diagonalized simultaneously and a common orthonormal (o.n.) basis of the Hilbert space $\mathcal{H}$ used in the description of $\mathcal{S}$, made of eigenstates of these observables, can be indeed obtained, as we see several times in Chapters 3–9. This means that, in the complete description of $\mathcal{S}$, all the results that are

deduced using our approach are not affected by any uncertainty because all the relevant self-adjoint operators whose eigenvalues are relevant to us are compatible, that is, mutually commuting.

It should also be mentioned that, in some specific applications, the impossibility of observing simultaneously two (apparently) classical quantities has been taken as a strong indication of the relevance of a quantumlike structure in the description of that process, showing, in particular, the importance of noncommuting operators. This is what was proposed, for instance, in Segal and Segal (1998), which is based on the natural assumption that a trader in a market cannot know, at the same time, the price of a certain share and its forward time derivative. The reason is clear: if the trader has access to both these information with absolute precision, then he is surely able to earn as much as he wants! For this reason, Segal and Segal proposed to use two noncommuting operators to describe the price and its time derivative. Going back to the title of this section, although in this book we are happy to not deal with the uncertainty principle, in other approaches this is actually seen as the main motivation to use a quantum or noncommutative approach for a macroscopic system. For this reason, also in view of possible future applications, we describe in Section 2.4 a possible mathematical derivation of a rather general inequality for noncommuting operators, which, fixing the operators in a suitable way, gives back the Heisenberg uncertainty relation.

1.4 Other Appearances of Quantum Mechanics in Classical Problems

Going back to the crucial aspect of this book, which is surely the mixture of quantum and classical words, we want to stress again that this is surely not the first place in which such a mixture is extensively adopted. On the contrary, in the past few years, a growing interest in classical applications of quantum ideas appeared in the literature, showing that more and more people believe that there is not a really big difference between these two worlds or that, at least, some mathematical tool originally introduced in quantum mechanics may also play a significant role in the analysis of classical systems. These kinds of mixtures can be found in very different fields such as economics (Aerts et al., 2012; Segal and Segal, 1998; Schaden, 2002; Baaquie, 2004; Accardi and Boukas, 2006; Al, 2007; Choustova, 2007; Ishio and Haven, 2009; Khrennikov, 2010; Romero et al., 2011; Pedram, 2012), biology (Engel et al., 2007; Arndt et al., 2009; Pusuluk and Deliduman; Martin-Delgado; Panitchayangkoon et al., 2011; Ritz et al. 2004), sociology, and psychology (Shi, 2005; Jimenez and Moya, 2005; Busemeyer et al., 2006; Khrennikov, 2006; Aerts et al., 2009, 2010; Yukalov and Sornette, 2009a,b; Aerts, 2010; Mensky, 2010; Makowski and Piotrowski, 2011; Yukalov and Sornette), and also in more general contexts (Abbott et al., 2008; Khrennikov, 2010), just to cite a few. The number of scientific contributions having classical applications of quantum mechanics as their main subject is fast increasing. To have an idea of what is going on, it is enough to follow the *arXiv* at xxx.lanl.gov, where almost everyday new papers are submitted. This, of course, provides encouragement to pursue our analysis and to check how far we can go with our techniques and how our results can be used to explain some aspects of the real macroscopic world.

1.5 Organization of the Book

This book is essentially organized in three parts. In the first part, Chapter 2, we review some important aspects of quantum mechanics, which are used in the rest of the book. In particular, we discuss the dynamical problem in ordinary quantum mechanics using several representations and describing the relations between them. We also discuss in great detail CCR, CAR, and some perturbative approaches, which are used sometimes in the book, as well as other tools and aspects related to quantum mechanics of certain interest for us, such as the Green's functions, the states over the algebra of bounded operators, and the Heisenberg uncertainty principle.

In the second part, Chapters 3–5, we show how the CCR and the CAR can be used for classical systems with few *degrees of freedom*. In particular, we discuss our point of view on love relations, describing also the role of the environment surrounding the people involved in the love affair. Later, in Chapter 4, we show how the same general framework can be used in the description of competitions between species and for other biological systems. For instance, we describe a migration process involving two populations, one living in a rich area and the second one in a poor region of a two-dimensional lattice.

Chapter 5 is dedicated to the description of the dynamical behavior of a biological-like system (e.g., some kind of bacteria) coupled to two reservoirs, one describing the food needed by the system to survive, and the other mimicking the garbage that is produced by the system itself.

Chapter 6 is a sort of an interlude, useful to fix the ideas on the role of the Hamiltonian of the system $\mathcal{S}$ we are describing, and on how this Hamiltonian should be

constructed. More explicitly, we identify three main steps in the analysis of $\mathcal{S}$: the first step consists in understanding the main mechanisms taking place in $\mathcal{S}$, with a particular interest to the interactions between its constituents. Second, we deduce the Hamiltonian for $\mathcal{S}$, $H_{\mathcal{S}}$, following a set of rather general rules, which is listed and explained in detail. The final step in our analysis of $\mathcal{S}$ is the deduction, from $H_{\mathcal{S}}$, of its dynamics. This is usually the hardest part of the job.

The last part of the book, Chapters 7, 8, and 9, is concerned with systems with many *degrees of freedom* and in particular, with our closed SSM. We propose several models for an SSM, from very simple to more complicated ones, and we consider some of the related dynamical features. In particular, most of the times, we will be interested in the deduction of the time evolution of the portfolio of each single trader, but in Chapter 9, we also compute a *transition probability* between different states of the SSM.

We devote Chapter 10 to some final considerations and to possible generalizations and applications. In particular, we discuss several possible extensions of our main tools, the CCR and the CAR. This could be useful to describe a nonunitary time evolution, describing some decay, as well as systems with a finite, and larger than 2, number of energy levels.

Notes

[1] Along this book, we add some remarks concerning the unboundedness of some operators used in the description of the system under investigation.

Chapter 2: Some Preliminaries

In this chapter, we briefly review some basic facts in quantum mechanics. In particular, we focus on what physicists usually call *second quantization*, which is used in the rest of the book. This chapter is essentially meant to keep these notes self-contained and to fix the notation. Nevertheless, the reader with a background in quantum mechanics is surely in a better position to fully comprehend the material discussed in this book. Of course, people with such a background could safely skip this chapter. It might also be worth stressing that, sometimes, we discuss something more that what is really used, just to give a reasonably closed form to the arguments presented in this chapter.

2.1 The Bosonic Number Operator

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the set of all the bounded operators on $\mathcal{H}$. $B(\mathcal{H})$ is a so-called *C*-algebra*, that is, an algebra with involution that is complete under a norm, $||\cdot||$, satisfying the *C*-property*: $||A^*A|| = ||A||^2$, for all $A \in B(\mathcal{H})$. As a matter of fact, $B(\mathcal{H})$ is usually seen as a *concrete realization* of an abstract C*-algebra. Let $\mathcal{S}$ be our physical system and $\mathcal{A}$ the set of all the operators useful for a complete description of $\mathcal{S}$, which includes the observables of $\mathcal{S}$, that is, those quantities that are measured in a concrete experiment. For simplicity, it would be convenient to assume that $\mathcal{A}$ is a C*-algebra by itself, possibly coinciding with the original set $B(\mathcal{H})$, or, at least, with some closed

subset of $B(\mathcal{H})$. However, this is not always possible in our concrete applications. This is because of the crucial role of some unbounded operators within our scheme: unbounded operators do not belong to any C*-algebra. However, if X is such an operator, and if it is self-adjoint, then e^{iXt} is unitary and, therefore, bounded. The norm of e^{iXt} is 1, for all $t \in \mathbb{R}$, and X can be recovered by taking its time derivative in $t = 0$ and multiplying the result by $-i$. For this reason, C*-algebras and their subsets are also relevant for us when unbounded operators appear.

A concrete situation where these kinds of problems arise is the description of the time evolution of $\mathcal{S}$, which is driven by a self-adjoint operator $H = H^\dagger$, which is called *the Hamiltonian* of $\mathcal{S}$ and which, in standard quantum mechanics, represents the energy of $\mathcal{S}$. In most cases, H is unbounded. In the so-called *Heisenberg representation*, see Section 2.3.2, the time evolution of an observable $X \in \mathcal{A}$ is given by

2.1 $$X(t) = e^{iHt} X e^{-iHt}$$

or, equivalently, by the solution of the differential equation

2.2 $$\frac{dX(t)}{dt} = ie^{iHt}[H, X]e^{-iHt} = i[H, X(t)],$$

where $[A, B] := AB - BA$ is the *commutator* between A and B. In view of what we have discussed before, $e^{\pm iHt}$ are unitary operators, hence they are bounded. Time evolution defined in this way is usually a one parameter group of automorphisms of $\mathcal{A}$:[1] for each X, $Y \in \mathcal{A}$, and for all $t, t_1, t_2 \in \mathbb{R}$, $(XY)(t) = X(t)Y(t)$ and $X(t_1 + t_2) = (X(t_1))(t_2)$. An operator $Z \in \mathcal{A}$ is a *constant of motion* if it commutes with H. Indeed, in this case, Equation 2.2 implies that $\dot{Z}(t) = 0$, so that $Z(t) = Z(0)$ for all t. It is worth stressing that, in Equations 2.1 and 2.2, we are assuming that H does not depend explicitly on time, which is not always true. We give

a rather more detailed analysis of time evolution of a quantum system, under more general assumptions, in Section 2.3.

As already discussed briefly in Chapter 1, a special role in our analysis is played by the CCR: we say that a set of operators $\{a_l, a_l^\dagger, l = 1, 2, \ldots, L\}$, acting on the Hilbert space $\mathcal{H}$, satisfy the CCR if

2.3 $$\left[a_l, a_n^\dagger\right] = \delta_{ln}\mathbb{1}, \quad [a_l, a_n] = \left[a_l^\dagger, a_n^\dagger\right] = 0,$$

holds for all *l*, *n* = 1, 2, ... , *L*, $\mathbb{1}$ being the identity operator on $\mathcal{H}$. These operators, which are widely analyzed in any textbook on quantum mechanics (see Merzbacher, 1970; Roman, 1965), for instance, are those that are used to describe *L* different *modes* of bosons. From these operators, we can construct $\hat{n}_l = a_l^\dagger a_l$ and $\hat{N} = \sum_{l=1}^{L} \hat{n}_l$, which are both self-adjoint. In particular, $\hat{n}_l$ is the *number operator* for the *l*th mode, while $\hat{N}$ is the *number operator* for $\mathcal{S}$.

An orthonormal (o.n.) basis of $\mathcal{H}$ can be constructed as follows: we introduce the *vacuum* of the theory, that is, a vector $\varphi_{\mathbf{0}}$ that is annihilated by all the operators a_l: $a_l\varphi_{\mathbf{0}} = 0$ for all *l* = 1, 2, ... , *L*. Then we act on $\varphi_{\mathbf{0}}$ with the operators $a_l^\dagger$ and with their powers

2.4 $$\varphi_{n_1,n_2,\ldots,n_L} := \frac{1}{\sqrt{n_1! n_2! \ldots n_L!}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_L^\dagger)^{n_L} \varphi_{\mathbf{0}}$$

n_l = 0, 1, 2, ... , for all *l*, and we normalize the vectors obtained in this way. The set of the $\varphi_{n_1,n_2,\ldots,n_L}$'s forms a complete and o.n. set in $\mathcal{H}$, and they are eigenstates of both $\hat{n}_l$ and $\hat{N}$:

$$\hat{n}_l \varphi_{n_1,n_2,\ldots,n_L} = n_l \varphi_{n_1,n_2,\ldots,n_L}$$

and

$$\hat{N}\varphi_{n_1,n_2,\ldots,n_L} = N\varphi_{n_1,n_2,\ldots,n_L},$$

where $N = \sum_{l=1}^{L} n_l$. Hence, n_l and N are eigenvalues of $\hat{n}_l$ and $\hat{N}$, respectively. Moreover, using the CCR we deduce that

$$\hat{n}_l \, (a_l \varphi_{n_1, n_2, \ldots, n_L}) = (n_l - 1)(a_l \varphi_{n_1, n_2, \ldots, n_L}),$$

for $n_l \geq 1$ whereas if $n_l = 0$, a_l annihilates the vector, and

$$\hat{n}_l \left(a_l^\dagger \varphi_{n_1, n_2, \ldots, n_L} \right) = (n_l + 1)(a_l^\dagger \varphi_{n_1, n_2, \ldots, n_L}),$$

for all l and for all n_l. For these reasons, the following interpretation is given in the literature: if the L different modes of bosons of $\mathcal{S}$ are described by the vector $\varphi_{n_1, n_2, \ldots, n_L}$, this means that n_1 bosons are in the first mode, n_2 in the second mode, and so on.[2] The operator $\hat{n}_l$ acts on $\varphi_{n_1, n_2, \ldots, n_L}$ and returns n_l, which is exactly the number of bosons in the lth mode. The operator $\hat{N}$ counts the total number of bosons. Moreover, the operator a_l destroys a boson in the lth mode, whereas $a_l^\dagger$ creates a boson in the same mode. This is why in the physical literature a_l and $a_l^\dagger$ are usually called the *annihilation* and the *creation* operators, respectively.

The vector $\varphi_{n_1, n_2, \ldots, n_L}$ in Equation 2.4 defines a *vector (or number) state* over the set $\mathcal{A}$ as

2.5 $$\omega_{n_1, n_2, \ldots, n_L}(X) = \langle \varphi_{n_1, n_2, \ldots, n_L}, X \varphi_{n_1, n_2, \ldots, n_L} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the Hilbert space $\mathcal{H}$. These states are used to *project* from quantum to classical dynamics and to fix the initial conditions of the system under consideration, in a way that is clarified later. Something more concerning states is discussed later in this chapter.

For the sake of completeness, it is interesting to now check explicitly that the operators introduced so far, a_l, $a_l^\dagger$,

$\hat{n}_l$, and $\hat{N}$, are all unbounded. This can be easily understood as, for instance,

$$\|\hat{n}_l\| = \sup_{0\neq\varphi\in\mathcal{H}} \frac{\|\hat{n}_l\varphi\|}{\|\varphi\|} \geq \sup_{\{n_j\geq 0, j=1,2,\ldots,L\}} \|\hat{n}_l \varphi_{n_1,\ldots,n_l,\ldots,n_L}\|$$
$$= \sup_{\{n_j\geq 0, j=1,2,\ldots,L\}} n_l = \infty,$$

and it is clearly related to the fact that $\mathcal{H}$ is infinite dimensional. It is well known that unbounded operators have severe domain problems, as they cannot be defined in all of $\mathcal{H}$, (Reed and Simon, 1980), but only on a dense subset of $\mathcal{H}$. However, this is not a major problem for us here for two reasons: first, each vector $\varphi_{n_1,n_2,\ldots,n_L}$ belongs to the domains of all the operators that are relevant to us, even when raised to some power or combined among them. Second, at least in the numerical calculations performed in Chapter 3, $\mathcal{H}$ is replaced by an *effective* Hilbert space, $\mathcal{H}_{\text{eff}}$, which becomes *dynamically* finite dimensional because of the existence of some conserved quantity and because of the initial conditions, which impose some constraints on the levels accessible to the *members* of the system. This aspect is discussed in more detail later.

2.2 The Fermionic Number Operator

Given a set of operators $\{b_l, b_l^\dagger, \ell = 1, 2, \ldots, L\}$ acting on a certain Hilbert space $\mathcal{H}_F$, we say that they satisfy the CAR if the conditions

2.6 $\{b_l, b_n^\dagger\} = \delta_{l,n}\mathbb{1}, \qquad \{b_l, b_n\} = \{b_l^\dagger, b_n^\dagger\} = 0$

hold true for all *l*, *n* = 1, 2, … , *L*. Here, {*x*, *y*}: = *xy* + *yx* is the *anticommutator* of *x* and *y* and $\mathbb{1}$ is now the identity