Principles of Discontinuous Dynamical Systems

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To my beloved mother, wife and daughters:
Zhanar and Laila

Preface

The main subject of this book is discontinuous dynamical systems. These have played an extremely important role theoretically, as well as in applications, for the last several decades. Still, the theory of these systems seems very far from being complete, and there is still much to do to make the application of the theory more effective. This is especially true of equations with trajectories discontinuous at moments that are not prescribed.

The book is written not only on the basis of research experience but also, importantly, on the basis of the experience of teaching the course of Impulsive differential equations for about 10 years to the graduate students of mathematics. It is useful for a beginner as we try not to avoid any difficult instants in delivering the material. Delicate questions that are usually ignored in a research monograph are thoroughly addressed. The standard material on equations with fixed moments of impulses is presented in a compact and definitive form. It contains a large number of exercises, examples, and figures, which will aid the reader in understanding the enigmatic world of discontinuous dynamics. The following peculiarity is very important: the material is built on the basis of close parallelism with ordinary differential equations theory. For example, even higher order differentiability of solutions, which has never been considered before, is presented with a full definition and detailed proofs. At the same time, the definition of the derivatives as coefficients of the expansion is fruitfully used, which is very rare in the theory of ordinary differential equations. Moreover, the description of stability, continuous and differentiable dependence of solutions on initial conditions, and right-hand side, chaotic ingredients is given on a more strong functional basis than that of ordinary differential equations.

The book is attractive to an advanced researcher, since a strong background for the future analysis of all theoretical and application problems is built. It will benefit scientists working in other fields of differential equations with discontinuities of various types, since it reflects the experience of the author in working on these subjects. We would like to emphasize that the basics of discontinuous flows are for the first time rigorously laid out so that all the attributes of dynamical systems are present. Hence, there is plenty of room for extending all the results of continuous, smooth and analytic dynamics to the systems with discontinuities.

The content of the book is a good background for the application in vibro-mechanisms theory, mechanisms with friction, biology, molecular biology, physiology, pharmacology, secure communications, neural networks, and other real world problems involving discontinuities.

Chapters 5–10 contain the core research contributions. Chapters 1–4 present preliminaries for the theory and elements of differential equations with fixed moments of discontinuity. Chapters 1–8 provide sufficient material for a standard one-semester graduate course. It is natural to finalize a general theory with more specific results. For this reason, in the last two chapters (9 and 10) we discuss Hopf bifurcation of periodic discontinuous solutions, Devaney's chaos, and the Shadowing property for discontinuous dynamical systems.

The author expresses his gratitude to his students who have contributed to the preparation of this book: Duygu Arugaslan, Cemil Buyukadali, Mehmet Turan, and Enes Yilmaz.

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Chapter 1
Introduction

Nowadays, many mathematicians agree that discontinuity as well as continuity should be considered when one seeks to describe the real world more adequately. The idea that, besides continuity, discontinuity is a property of motion is as old as the idea of motion itself. This understanding was strong in ancient Greece. For example, it was expressed in paradoxes of Zeno. Invention of calculus by Newton and Leibniz in its last form, and the development of the analysis adjunct to celestial mechanics, which was stimulating for the founders of the theory of dynamical systems, took us away from the concept of discontinuity. The domination of continuous dynamics, and also smooth dynamics, has been apparent for a long time. However, the application of differential equations in mechanics, electronics, biology, neural networks, medicine, and social sciences often necessitates the introduction of discontinuity, as either abrupt interruptions of an elsewhere continuous process (impulsive differential equations) or in the form of discrete time setting (difference equations). If difference equations may be considered as an instrument of investigation of continuous motion through, for example, Poincaré maps, impulsive differential equations seem appropriate for modeling motions where continuous changes are mixed with impact type changes in equal proportion. Recently, it is becoming clear that to discuss real world systems that (1) exist for a long period of time, or (2) are multidimensional, with a large number of dependent variables, researchers resort to differential equations with: (1) discontinuous trajectories (impulsive differential equations); (2) switching in the right-hand side (differential equations with discontinuous right-hand side); (3) some coordinates ruled by discrete equations (hybrid systems); (4) disconnected domains of existence of solutions (time scale differential equations), where these properties may be combined in a single model.

The theory of equations with discontinuous trajectories has been developed through applications [14, 16, 38, 41, 43, 50, 52, 53, 56, 57, 70, 71, 75, 79, 89, 91, 99, 101, 103, 107–109, 115, 121, 123, 125–127, 130, 144, 145, 155, 158–160, 162] and theoretical challenges [4–9, 19, 32–36, 65, 69, 75, 85, 95–97, 99–101, 103, 110, 111, 118–124, 135–142, 151–153].

We give a limited number of references, since this work was written as a textbook rather than a research monograph, and secondly, sources related to systems with nonfixed moments of discontinuity were preferentially presented.

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Our main objective is to present the theory of differential equations with solutions that have discontinuities either at the moments when the integral curves reach certain surfaces in the extended phase space (t, x), as time t increases (decreases), or at the moments when the trajectories enter certain sets in the phase space x. That is, the moments when the solutions have discontinuities are not prescribed. Notably, the systems with nonprescribed times of discontinuity were first introduced in [91, 123, 124], manuscripts in applied mathematics, which underscores the practical importance of the theory of equations with variable moments of impulses. Differential equations with fixed moments of impulses were the next to be studied. These serve as an auxiliary instrument for the study of the above named systems in the same way as nonautonomous equations play a role in the analysis of autonomous systems through linearization. For that reason, we provide a more extensive discussion of the theory of equations with fixed moments of impulses, than might otherwise seem necessary. It takes the first four chapters of the manuscript. We thoroughly describe the solutions of these equations, consider the existence and uniqueness of solutions and their dependence on parameters. The problem of extension of a solution for both increasing and decreasing time is investigated. For example, we prove the Gronwall–Bellman Lemma for piecewise continuous functions, and the integral representation formulas, for decreasing time, as well as for increasing time. This extension of the results is obviously necessary to explore dynamical systems' properties in the fullest form, as required for applications. Since the moments of discontinuities are different for different solutions, the equations are nonlinear. Equations with nonfixed moments of discontinuity create a great number of opportunities for theoretical inquiry, as well as theoretical challenges. This is due to the structure of these systems, namely the three components: a differential equation, an impulsive action, and the surfaces of discontinuity which are involved in the process of governing the motion. Therefore, in addition to the features of the ordinary differential equations, we may vary the properties of the maps, which transform the phase point at the moment of impulse, and try various topological and differential characteristics of the surfaces of discontinuity to produce one or another interesting theoretical phenomenon, or satisfy a desired application property.

Effective methods of investigation of systems with nonfixed moments of impulsive action can be found in [2–4, 32–36, 65, 69, 95–97, 123, 124, 135, 136, 142, 152, 153]. Theoretical problems of nonsmooth dynamics and discontinuous maps [17, 19, 38, 48, 51, 66, 68, 86, 92, 93, 146] are also close to the subject matter of our book.

The present book plays its own modest role in attracting the attention of scientists, first of all mathematicians, to the symbiosis of continuity and discontinuity in the description of a motion.

The book presented to the attention of the reader is to be viewed first and foremost as a textbook on the theory of discontinuous dynamical systems. There is some similarity between the content of this book and that of the monographs on ordinary differential equations. Accordingly, we deliver some standard topics: description of the systems, definition of solutions, local existence and uniqueness theorems, extension of solutions, dependence of solutions on parameters. It is our conviction that many results of the theory of equations with impulses (if not all), that at the moment appear as very specific, in fact, have their counterparts in the theory of ordinary differential equations. We take up the task of extending the parallels with the theory of continuous (smooth) dynamical systems. It seems appropriate to place the results on the existence of periodic solutions and Hopf bifurcation of periodic solutions in the final part of the book. The last chapter is devoted to complex motions, in whose description we use ingredients of Devaney chaos. It is noteworthy that the method of creation of chaos through impulses does not have analogs in continuous dynamics yet. We bring up only a few examples to illustrate the possibilities for application.

We use a powerful analytical tool of B-equivalence, which was introduced and developed in our papers. The method was created especially for the investigation of systems with solutions that have discontinuities at variable moments of time [1–4, 25–37]. But it can also be applied to differential equations with discontinuous right-hand-side [13, 15, 27, 34] and differential equations on variable time scales with transition conditions [20]. The method is effective in the analysis of chaotic systems [8–11], as well.

In the last decades, the exceptional role of differential equations with impulses at variable times in dealing with problems of mechanisms with vibrations has been perceived. Collision-bifurcations, oscillations, and chaotic processes in this mechanisms have been investigated in many papers and books [67, 79, 118, 119, 144, 151, 156]. We are very confident that the content of this book will give a strong push to the development of this field, as well as other related areas of research, where a discontinuity appears.

Let us consider the following examples, which highlight the modeling role of discontinuous dynamics.

Example 1.1. Consider a mechanical model consisting of a bead B bouncing on a massive, sinusoidally vibrating table P (see Fig. 1.1). Such a system has been investigated in [71, 79, 89, 133, 158]. We assume that the table is so massive that

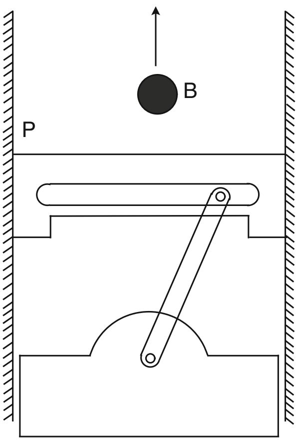

Fig. 1.1 A model consisting of a bead bouncing on a vibrating table

it does not react to collisions with the bead and moves according to the law $X = X_0 \sin \omega t$. The motion of the bead between collisions is given by the formula

$$x = \frac{-g(t - t_0)^2}{2} + x_0'(t - \phi) + x_0, \tag{1.1}$$

where x_0 and x_0' are, respectively, the values of the coordinate and the velocity of the bead at the instant $t = \phi$ immediately after collision, and $g = 9.8\, m/s^2$ is the gravitational constant. The change of the velocity of the bead at the moment of the hit is given by the following relation:

$$R = \frac{X_+' - x_+'}{x_-' - X_-'}. \tag{1.2}$$

Here R is the restitution coefficient $(0 < R \leq 1), X_-', x_-', X_+'$, and x_+' are the velocities of the table and the bead before and after the strike, repectively, $(X_+' = X_-')$.

Among the results of investigation of the model, one can mention those in [71], where the period-doubling bifurcation, as well as chaos emergence, is discussed. If we write $x_1 = x, x_2 = x', \tau_i(x_1) = \arcsin(x_1/X_0) + (\pi/\omega)i$, where i are integers, then, using (1.1) and (1.2), one can construct a suitable mathematical model in the form of the following nonlinear system of differential equations with impulsive actions:

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -g, \\ \Delta x_2|_{t=\tau_i(x_1)} &= (1 + R)[x_0\omega \cos(\omega\tau_i(x_1)) - x_2]. \end{aligned} \tag{1.3}$$

Example 1.2. Consider a mechanical model of the oscillator consisting of a cart C (see Fig. 1.2), which can impact against a rigid wall W, and is subjected to an external force $H \sin(\omega t + \gamma)$. There is an elastic element S. The wall is at the distance B from the origin of the coordinate system, which is placed at the equilibrium point. The change of velocity of the cart at the moment of the hit against the wall is given by the relation $x_+' = -Rx_-'$, where R is the restitution coefficient $(0 < R \leq 1)$, and x_-' and x_+' are the velocities of the cart before and after the strike, respectively. One can easily find a mathematical model of the system, which takes the form of the following differential equations with impulses:

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -cx_1 + H \sin(\omega t + \gamma), \\ \Delta x_2|_{x_1=-B} &= (1 + R)x_2, \end{aligned} \tag{1.4}$$

where $x_1 = x, x_2 = x'$.

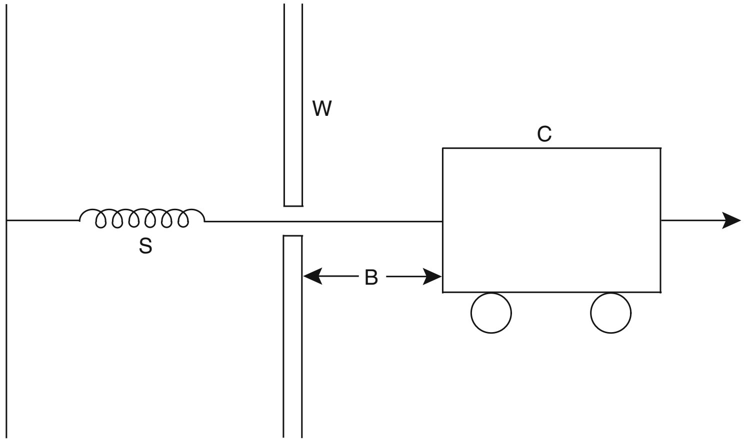

Fig. 1.2 A model of an oscillator consisting of a cart C, which can impact against a rigid wall W

Systems (1.3) and (1.4) are typical examples of differential equations with variable moments of impulses discussed in the book. The first system has solutions that exhibit discontinuity when they reach surfaces in the extended phase space. Solutions of the other system have jumps at the moments when they cross a set in the phase space of the equation.

The book is organized as follows:

We start with the description of differential equations with fixed moments of impulses in the second chapter. The characteristics of the sets of discontinuity moments are listed, and the spaces of piecewise continuous functions are introduced. The extension of solutions is presented in a very detailed manner. The theorems on local and global existence, and uniqueness of solutions are proved. The continuous dependence of solutions on initial conditions and the right-hand side are discussed.

The third chapter is devoted to the generalities of stability and periodic solutions of differential equations with fixed moments of impulses. Definitions of stability, the description of periodic systems, and illustrating examples are provided.

The basics of linear impulsive systems are the focus of the fourth chapter: Linear homogeneous systems; Linear nonhomogeneous systems; Linear periodic systems; Spaces of solutions; Stability of linear systems.

The next, fifth chapter is one of the main parts of the book. Nonautonomous differential equations with impulses at variable moments of time, whose solutions have jumps at the moments of intersection with surfaces in the extended phase space, are considered. In this chapter, we provide all conditions that make the investigation of these equations convenient. Namely, the conditions that guarantee the absence of beating of the solutions against the surfaces of discontinuity, and the conditions that preserve the ordering of the intersection with the surfaces. Moreover, we obtain conditions that allow the reduction to equations with fixed moments of impulses. It should be emphasized that the results concerning the dependence of solutions on the initial conditions and on the right-side, and stability are presented in the full form for the first time in the literature. Also, for the first time conditions for the extension of solutions to the left are formulated. The main auxiliary concepts are: the topology in the set of discontinuous functions. A general nonlinear case is considered, and quasilinear systems are investigated.

The sixth chapter is concerned with differentiability properties of solutions of nonautonomous differential equations with variable moments of impulses with respect to the initial conditions and parameters. The subject is relatively new for discontinuous dynamics, especially for higher order derivatives. What makes this investigation possible is the implementation of the B-equivalence method. The same can be said about the issue of the analyticity of solutions. We propose a uniform approach so that not only solutions themselves but also their discontinuity moments can be differentiated.

The results on smoothness from the previous chapter are used in the seventh chapter to develop the method of small parameter for quasilinear systems. Both critical and noncritical cases for the existence of periodic solutions are discussed. Practically useful algorithms are derived.

Chapter 8 is the central part of the book. We obtain conditions sufficient to shape a motion that is very similar to the flow of an autonomous ordinary differential equation so that all the properties of the dynamical system – extension of all solutions on $\mathbb{R}$, continuous dependence on the initial value, the group property and uniqueness – are preserved. Differentiability in the initial value is considered. In fact, B-smooth discontinuous flows are obtained.

The last two chapters revolve around more specific topics. The ninth chapter develops the mechanisms for discovering the Hopf bifurcation of a discontinuous dynamical system. Additionally, the question of the persistence of focus and the problems of distinguishing the focus and the center in the critical case are discussed as preliminaries.

We consider complex behavior of a discontinuous dynamics in the tenth chapter. For a special initial value problem, where moments of impulses are generated through map iterations, analogs of all Devaney's ingredients, as well as the shadowing property, are studied. Examples illustrating the existence of the chaotic attractor and the intermittency phenomenon are provided.

Chapter 2
Description of the System with Fixed Moments of Impulses and Its Solutions

2.1 Spaces of Piecewise Continuous Functions

Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}$ be the sets of all real numbers, natural numbers, and integers, respectively. Denote by $\theta = \{\theta_i\}$ a strictly increasing sequence of real numbers such that the set $\mathcal{A}$ of indexes i is an interval in $\mathbb{Z}$.

Definition 2.1.1. θ is a B-sequence, if one of the following alternatives is valid:

(a) $\theta = \emptyset$;
(b) θ is a nonempty and finite set;
(c) θ is an infinite set such that $|\theta_i| \to \infty$ as $|i| \to \infty$.

Example 2.1.1. θ with $\mathcal{A} = \{-1, 1, 2\}$, and $\theta_{-1} = -5, \theta_1 = \pi, \theta_2 = 7$, satisfies condition (b).

Example 2.1.2. $\theta_i = i + \frac{1}{3}$, where $i \geq -102$, is a B-sequence of type (c).

Example 2.1.3. $\theta_i = -i + \frac{1}{5}$, where $i \in \mathbb{Z}$, is not a B-sequence.

It is obvious that any B-sequence has no finite limit points.

Example 2.1.4. $\theta_i = 1 - \frac{1}{i+3}$, where $i = 1, 2, 3, \ldots$, is not a B-sequence.

Denote by Θ the union of all B-sequences.

Fix a sequence $\theta \in \Theta$.

Definition 2.1.2. A function $\phi : \mathbb{R} \to \mathbb{R}^n, n \in \mathbb{N}$, is from the set $\mathcal{PC}(\mathbb{R}, \theta)$ if :

(i) it is left continuous;
(ii) it is continuous, except, possibly, points of θ, where it has discontinuities of the first kind.

The last definition means that if $\phi(t) \in \mathcal{PC}(\mathbb{R}, \theta)$, then the right limit $\phi(\theta_i+) = \lim_{t\to\theta_i+} \phi(t)$ exists and $\phi(\theta_i-) = \phi(\theta_i)$, where $\phi(\theta_i-) = \lim_{t\to\theta_i-} \phi(t)$, for each $\theta_i \in \theta$.

Let $\mathcal{PC}(\mathbb{R}) = \cup_{\theta\in\Theta}\mathcal{PC}(\mathbb{R}, \theta)$.

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